Strategy to transform continuous random variables.

• Consider a random variable $X$. Make sure you understand its p.d.f. $f_X(x)$ and its c.d.f. $F_X(x)$. These relations are important:

$$F_X'(x) = f_X(x)$$
$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt = \mathbb{P}(X < x)$$
$$F_X(b) - F_X(a) = \int_{a}^{b} f_X(t)dt = \mathbb{P}(a < X < b)$$

• Consider a function $g(x)$ and the transformed random variable $Y = g(X)$. Denote $y = g(x)$, which we can solve for $x$, to obtain $x = g^{-1}(y)$.

• For now, assume that $g$ is a monotone increasing function, and we understand the range of $Y$.

\[ \min(Y) = g(\min(X)) \leq Y \leq \max(Y) = g(\max(X)) \]

(1) The c.d.f. of $Y$ can be computed as

$$F_Y(y) = F_X(x) = F_X(g^{-1}(y))$$

(2) The p.d.f. of $Y$ can be computed as

$$f_Y(y) = F_Y'(y) = \frac{d}{dy}F_X(g^{-1}(y))$$

(3) The expected value $\mathbb{E}Y$ can be computed in two different ways as

$$\mathbb{E}Y = \int_{\min(Y)}^{\max(Y)} yf_Y(y)dy = \int_{\min(X)}^{\max(X)} g(x)f_X(x)dx$$

(4) To compute $\text{Var}(Y)$, use

$$\mathbb{E}Y^2 = \int_{\min(Y)}^{\max(Y)} y^2f_Y(y)dy = \int_{\min(X)}^{\max(X)} (g(x))^2f_X(x)dx$$

and then

$$\text{Var}(Y) = \mathbb{E}Y^2 - (\mathbb{E}Y)^2$$

• If $g$ is a monotone decreasing function, then some things are reversed

\[ \min(Y) = g(\max(X)) \leq Y \leq \max(Y) = g(\min(X)) \]

\[ F_Y(y) = 1 - F_X(x) = 1 - F_X(g^{-1}(y)) \]

• If $g$ is neither monotone decreasing nor monotone increasing, then we have to consider cases.