Sample Test 2 questions:

(1) Let X be a Poisson random variable with $\mathbb{E}X = 4$. Find the formula for $\mathbb{P}(2 \leqslant X \leqslant 5)$.

$$\sum_{k=2}^{5} e^{-4} \frac{4^k}{k!}$$

(2) Let X be a binomial random variable with $\mathbb{E}X = 4$ and $n = 10$. Find the formula for $\mathbb{P}(2 \leqslant X \leqslant 5)$.

$$p = \frac{EX}{10} = \frac{2}{5} \qquad \sum_{k=2}^{5} \binom{10}{k} \left(\frac{2}{5}\right)^k \left(\frac{3}{5}\right)^{10-k}$$

(3) Let X be the number of coin tosses until we have ~~at~~ one head. Find $\mathbb{E}X$ and $\mathrm{Var}X$. Find $\mathbb{P}(X \geqslant 3)$

geometric distribution from the table $Geo(p = \frac{1}{2})$

$$\mathbb{E}X = \frac{1}{p} = 2 \qquad Var(X) = \frac{\frac{1}{2}}{\frac{1}{2^2}} = 2 \qquad P(X \geqslant 3) = \sum_{k=3}^{\infty} 2^{-k} = \frac{2^{-3}}{1-\frac{1}{2}} = \frac{1}{4}$$

or can compute $P(X \geqslant 3) = 1 - P(X=1) - P(X=2) = 1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

(4) Suppose X has the following p.d.f.

$$f(x) = \begin{cases} ax^2 & \text{if } 1 \leqslant x \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

Find a, $\mathbb{E}X$, $\mathrm{Var}X$, and the cdf $F(x)$. Make a picture of the p.d.f. and c.d.f.

$a = \frac{3}{7}$ because $\int_1^2 x^2\,dx = \frac{7}{3}$

$$EX = \frac{3}{7}\int_1^2 x^3\,dx = \frac{3}{7}\left(\frac{16}{4} - \frac{1}{4}\right) = \frac{45}{28}$$

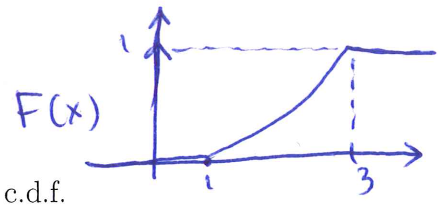
$$EX^2 = \frac{3}{7}\int_1^2 x^4\,dx = \frac{3}{7}\left(\frac{2^5}{5} - \frac{1}{5}\right)$$

$\mathrm{Var}\,X = (EX^{\cancel{4}\,2}) - (EX)^2$

do not simplify

(5) Suppose X has the following p.d.f.

$$f(x) = \begin{cases} \frac{a}{x^2} & \text{if } 1 \leqslant x \leqslant 3 \\ 0 & \text{otherwise} \end{cases}$$



Find a, $\mathbb{E}X$, $\mathrm{Var}X$, and the cdf $F(x)$. Make a picture of the p.d.f. and c.d.f.

$a = \frac{3}{2}$ because $\int_1^3 \frac{1}{x^2}dx = -\frac{1}{x}\Big|_1^3 = 1 - \frac{1}{3} = \frac{2}{3}$

$EX = \frac{3}{2}\int_1^3 \frac{1}{x}\,dx = \frac{3}{2}\log 3$ $\quad EX^2 = \frac{3}{2}\int_1^3 1\,dx = 3$

$\mathrm{Var}\,X = -\left(\frac{3}{2}\log 3\right)^2 + 3$ $\quad F(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ \frac{3}{2}\left(1 - \frac{1}{x}\right) & \text{if } 1 \leq x \leq 3 \\ 1 & \text{if } x \geqslant 3 \end{cases}$

(6) Find the numerical value for $\mathbb{P}(-2 \leq X \leq 3)$ if X is $\mathcal{N}(3, 4)$. Your answer should include Φ twice. After that, use the normal table attached in the end of the quiz find the approximate answer.

$X = 3 + 2Z$ $\quad P(-2 < 3 + 2Z < 3) =$

$= P\left(-\frac{5}{2} < Z < 0\right) = \Phi(0) - \Phi\left(-\frac{5}{2}\right) =$

$= \Phi(0) - \left(1 - \Phi\left(\frac{5}{2}\right)\right) = \Phi\left(\frac{5}{2}\right) - \frac{1}{2}$

(7) Find a formula for $\mathbb{P}(-2 \leq X \leq 3)$ if X is $\mathcal{N}(-3, 4)$. Your answer should include Φ twice. After that, use the normal table attached in the end of the quiz find the approximate answer.

$X = -3 + 2Z$ $\quad P(-2 < -3 + 2Z < 3) =$

$= \left(\frac{1}{2} < Z < 3\right) = \Phi(3) - \Phi\left(\frac{1}{2}\right)$

(8) If Z_1 and Z_1 are standard normal independent random variables, and $X = 3Z_1 + 4Z_2$, find $\mathbb{P}(-2 \leq X \leq 3)$ Your answer should include Φ twice. After that, use the normal table attached in the end of the quiz find the approximate answer.

$$\Phi(0.6) + \Phi(0.4) - 1$$

this was done in class

$\mathbb{E}X = 60$ $\quad n = 150$

(9) Let X be a binomial random variable with $\cancel{\mathbb{E}X = 12}$ and $\cancel{n = 30}$. Find the normal approximation for $\mathbb{P}(54 \leqslant X \leqslant 72)$

$$p = \frac{60}{150} = \frac{2}{5}$$

$$\cancel{\mu = \mathbb{E}X = 12 \quad \sigma^2 = np(1-p) = 30 \cdot \frac{2}{5} \cdot \frac{3}{5} = \frac{12}{5}}$$

$$\mu = \mathbb{E}X = 60 \qquad \sigma^2 = 150 \cdot \frac{2}{5} \cdot \frac{3}{5} = 36 \qquad \sigma = 6$$

$$P(54 \leq X \leq 72) \approx P(54 \leq 60 + 6Z \leq 72) = P(-1 < Z < 2)$$

$$= \Phi(2) - \Phi(-1) = \Phi(2) + \Phi(1) - 1$$

(10) Let X be the number of tails in 25 fair coin tosses. Find the best normal approximation for $\mathbb{P}(X \leqslant 15)$.

$$n = 25 \qquad p = \frac{1}{2} \qquad \mu = \frac{25}{2} \qquad \sigma^2 = 25 \cdot \frac{1}{4} \qquad \sigma = \frac{5}{2}$$

$$P(X \leq 15) \approx P\left(12.5 + \frac{5}{2}Z \leq 15.5\right) \approx P(Z \leq 1.2) = \Phi(1.2)$$

In the table $\Phi(1.2) \approx 0.88493$

(11) Let a fair dice be thrown 2000 times. Find the normal approximation of the probability that 6 appears at least 300 times.

$\Phi(2)$

This was done in class

(12) Let X be an exponential random variable with parameter $\lambda = 2$. Find
(a) Find $P(X > 3)$
(b) Find $P(X > 3 | \cancel{X} > 2)$
(c) Find $P(4 < X < 6 | \cancel{X} > 2)$

(a) e^{-6}

(b) e^{-2}

(c) $e^{-4} - e^{-8}$

use that exponential r.v. is memoryless

(13) Let X be a uniform random variable on $[1, 5]$. Find
(a) Find $P(X > 3)$
(b) Find $P(X > 3 | \cancel{X} > 2)$
(c) Find $P(4 < X < 6 | \cancel{X} > 2)$
(d) Find $\mathbb{E}e^{2X}$.
(e) Find $\mathbb{E}\dfrac{1}{X^2}$.

$P(X > 3) = \frac{1}{2}$

$P(X > 3 | P > 2) = \frac{2}{3}$

$P(4 < X < 6 | P > 2) = \frac{1}{3}$

$$\mathbb{E}\, e^{2X} = \int_1^5 \frac{1}{4} e^{2x}\, dx = \frac{1}{4}\frac{1}{2} e^{2x}\Big|_1^5 = \frac{1}{8}(e^{10} - e^{2})$$

$$\mathbb{E}\,\frac{1}{X^2} = \int_1^5 \frac{1}{4}\frac{1}{x^2}\, dx = \frac{1}{4}\left(-\frac{1}{x}\right)\Big|_1^5 = \frac{1}{4}\frac{4}{5} = \frac{1}{5}$$

(14) Consider random variables X and Y with the joint probability density function

$$f(x,y) = \begin{cases} axy^2 & \text{if } 0 \le x \le y \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find a
(b) Find the marginal p.d.f. $f_X(x)$.
(c) Find the marginal p.d.f. $f_Y(y)$.

$a = \frac{5}{16}$ because $\int_0^2 \int_0^y xy^2 \, dx \, dy = \frac{1}{2} \int_0^2 y^4 dy = \frac{2^5}{10}$

$f_X(x) = a \int_x^2 xy^2 dy = \frac{a}{3}(8 - x^3) x$ $\quad f_Y(y) = a \int_0^y xy^2 dx = \frac{a}{2} y^4$

(15) Consider random variables X and Y with the joint probability density function

$$f(x,y) = \begin{cases} axy^2 & \text{if } 0 \le y \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find a
(b) Find the marginal p.d.f. $f_X(x)$.
(c) Find the marginal p.d.f. $f_Y(y)$.

$a = \frac{15}{32}$ because $\int_0^2 \int_0^x xy^2 \, dy \, dx = \frac{1}{3} \int_0^2 x^4 dx = \frac{2^5}{15}$

$f_X(x) = a \int_0^x xy^2 dy = \frac{a}{3} x^4$ $\quad f_Y(y) = a \int_y^2 xy^2 dx = \frac{a}{2} y^2 (4 - y^2)$

(16) Consider random variables X and Y with the joint probability density function

$$f(x,y) = \begin{cases} axy^2 & \text{if } 0 \le x,\ 0 \le y,\ x+y < 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find a
(b) Find the marginal p.d.f. $f_X(x)$.
(c) Find the marginal p.d.f. $f_Y(y)$.

In class: $1 = \int_0^2 \int_0^{2-y} axy^2 \, dx \, dy =$

$= \int_0^2 \frac{a}{2} y^2 (y-2)^2 dy = \frac{a}{2} \left(\frac{y^5}{5} - \frac{4}{4} y^4 + \frac{4}{3} y^3 \right) \Big|_0^2$

$a = \dfrac{2}{\frac{1}{5} 2^5 - 2^4 + \frac{4}{3} 2^3}$

$f_X(x) = \frac{a}{3} x (2-x)^3$

$f_Y(y) = \frac{a}{2} y^2 (2-y)^2$

was done in class