(1) Suppose one has 7 indistinguishable balls. How many ways can one put them in 3 boxes? Explain your solution.

Answer: $\binom{9}{7} = \binom{9}{2} = \binom{8}{2} + 8 = 36$

(2) A pair of fair dice is rolled. What is the probability that the first die lands at least twice higher value as the second die?

Answer: $\frac{1}{4}$

(3) Suppose A is the event for which the probability is computed in the previous question, and B is the event "the second die is an even number". Are these events independent? Explain your solution.

Answer: Not independent

(4) The probability that an "accident prone" policy holder has an accident within a year is 0.8, while the probability that a "non-accident prone" policyholder has an accident within a year is 0.2. Assume that 30% of the policyholders are "accident prone". If a policyholder had an accident, what is the probability that this policyholder is not accident prone?

Answer: The probability that a random individual has an accident is $\mathbb{P}(A) = 0.3 \cdot .8 + 0.7 \cdot 0.2 = .38$
$\mathbb{P}(\text{not acccident prone}|A) = \dfrac{0.7 \cdot .2}{.38} = \dfrac{7}{19}$

(5) A pair of fair dice is rolled. What is the expected value of the difference between the higher and the lower value?

Answer: $\frac{35}{18}$

(6) Suppose X is a Poisson random variable with $P(X = 2)/P(X = 4) = \frac{1}{3}$. What is λ?

Answer: 6

(7) Suppose X has density

$$f(x) = \begin{cases} \dfrac{a}{x^2} & \text{if } 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}.$$

Find a, $\mathbb{E}X$, $\mathbb{E}X^2$, $\text{Var}X$.

Answer: $a = 2, \mathbb{E}X = 2\ln 2 \mathbb{E}X^2 = 2$, $\text{Var}X = 2 - 4(\ln(2))^2$

(8) Let $Z \in \mathcal{N}(0,1)$, that is, a standard normal random variable. Find the cumulative distribution function and the probability density function for $X = Z^2$.

Answer: if $x > 0$ then the functions are $F(x) = 2\Phi(\sqrt{x}) - 1$ and $f(x) = \exp(-x/2)/\sqrt{2\pi x}$, and both are 0 otherwise

(9) Suppose the dice is rolled 720 times. Use the normal approximation to estimate the probability that 3 occurred exactly 123 times.

Answer: $\Phi(0.35) - \Phi(0.25) \approx 0.13683 - 0.09871 = 0.03812$

(10) Suppose X is an exponential random variable with $\mathbb{E}X = 2$. Find the conditional probability that $3 < X < 4$ given that $X > 2$.

Answer: $e^{-1/2} - e^{-1}$

(11) Suppose that random variables X and Y are uniformly distributed in the region $y > 0$, $x > y$, $x + y < 4$. Find $\mathbb{P}(X < 3)$

Answer: $\mathbb{P}(X > 3) = 1/8$ by integration or geometry, and so $\mathbb{P}(X < 3) = 1 - \frac{1}{8} = \dfrac{7}{8}$

(12) Suppose again that random variables X and Y are uniformly distributed in the region $y > 0$, $x > y$, $x + y < 4$. Find $\mathbb{E}X$ and $\mathbb{E}Y$

Answer: $\mathbb{E}X = 2$ and $\mathbb{E}Y = \frac{2}{3}$

(13) Let $S_4 = X_1 + ... + X_4$ is the sum of 4 independent random variables, and each X_i is exponential with $\lambda = 3$. Find the moment generating function $m(t)$ for S_4. Also find $m'(0)$ and $m''(0)$.

Answer: $m(t) = \left(\dfrac{3}{3-t}\right)^4 \quad m'(0) = 4/3 \quad m''(0) = 20/9$

(14) Let $S_{16} = X_1 + ... + X_{16}$ is the sum of 16 independent random variables, and each X_i is exponential with $\lambda = 3$. Use the CLT to estimate the probability that $|S_{16} - 6| < 2$.

Answer: $\Phi(2) + \Phi(1) - 1$

(15) In the situation of questions (11) and (12), find $VarX$ and $VarY$

Answer:
$VarX = 56/12 - (2)^2 = 2/3$,
$VarY = 8/12 - (2/3)^2 = 2/9$,

because

$\int_0^2 \int_y^{4-y} 1dxdy = 4$
$\int_0^2 \int_y^{4-y} xdxdy = 8$
$\int_0^2 \int_y^{4-y} x^2dxdy = \frac{56}{3}$
$\int_0^2 \int_y^{4-y} ydxdy = \frac{8}{3}$
$\int_0^2 \int_y^{4-y} y^2dxdy = \frac{8}{3}$

Hint: one can find another solution by using problem (17) below.

(16) In the situation of questions (11), (12) and (15), find the correlation between X and Y. Are X and Y independent? Explain.

Answer: $\rho(X, Y) = (16/12 - 2(2/3))/\sqrt{(56/12 - (2)^2)(8/12 - (2/3)^2)} = 0$
because $\int_0^2 \int_y^{4-y} xydxdy = \frac{16}{3}$.
There is another solution that does not use integrals but the left-to-right symmetry.
No, X and Y not independent because the region of integration has a triangular shape.

(17) In the situation of questions (11), (12), (15) and (16), find the marginal densities of X and Y.

Answer: $f_X(x) = \begin{cases} \dfrac{x}{4} & \text{if } 0 \le x \le 2 \\ \dfrac{4-x}{4} & \text{if } 2 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$ and $f_Y(y) = \begin{cases} \dfrac{2-y}{2} & \text{if } 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$

(18) In the situation of questions (11), (12), (15), (16) and (17), find the conditional densities $f(X|Y)$, $f(Y|X)$.

Answer: $f(X|Y) = \begin{cases} \dfrac{1}{4-2y} & \text{if } 0 \le y \le 2,\ y < x < 4-y \\ 0 & \text{otherwise} \end{cases}$

$$f(Y|X) = \begin{cases} \dfrac{1}{x} & \text{if } 0 \le x \le 2, 0 < y < x \\ \dfrac{1}{4-x} & \text{if } 2 \le x \le 4, 0 < y < 4-x \\ 0 & \text{otherwise} \end{cases}$$

Hint: this can be obtained without integration.

(19) In the situation of questions (11), (12), (15), (16), (17) and (18), find the conditional expectations $\mathbb{E}(X|Y)$ and $\mathbb{E}(Y|X)$.

Answer: $f(X|Y) = \begin{cases} 2 & \text{if } \ 0 \le y \le 2, \ y < x < 4-y \\ 0 & \text{otherwise} \end{cases}$

$$f(Y|X) = \begin{cases} \dfrac{x}{2} & \text{if } \ 0 \le x \le 2, 0 < y < x \\ \dfrac{4-x}{2} & \text{if } \ 2 \le x \le 4, 0 < y < 4-x \\ 0 & \text{otherwise} \end{cases}$$

Hint: this can be obtained without integration using either geometry or the table.

(20) In the situation of questions (11), (12), (15), (16), (17), (18) and (19), find the conditional variances $\mathrm{Var}(X|Y)$ and $\mathrm{Var}(Y|X)$.

Answer: $\mathrm{Var}(X|Y) = \begin{cases} \dfrac{(4-2y)^2}{12} & \text{if } \ 0 \le y \le 2, \ y < x < 4-y \\ 0 & \text{otherwise} \end{cases}$

$$\mathrm{Var}(Y|X) = \begin{cases} \dfrac{x^2}{12} & \text{if } \ 0 \le x \le 2, 0 < y < x \\ \dfrac{(4-x)^2}{12} & \text{if } \ 2 \le x \le 4, 0 < y < 4-x \\ 0 & \text{otherwise} \end{cases}$$

Hint: this can be obtained without integration using the table.

(21) Suppose $\mathbb{E}X = \mu$, $\mathrm{Var}X = \sigma^2$ and a is a number. Find expression for $\mathbb{E}(X-a)^2$ that uses only μ, σ^2, a and numbers. For which a the expectation $\mathbb{E}(X-a)^2$ attains its minimal value, and what is this value?

Answer: $\mathbb{E}(X-a)^2 = \mathbb{E}(X-\mu+\mu-a)^2 = \mathbb{E}(X-\mu)^2 + 2\mathbb{E}(X-\mu)(\mu-a) + \mathbb{E}(\mu-a)^2 = \sigma^2 + (\mu-a)^2$
The minimal value is σ^2 when $a = \mu$.