

BV and Besov spaces on fractals with Dirichlet forms

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Joint work with

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- Besov class via heat semigroup on Dirichlet spaces I:
Sobolev type inequalities, arXiv:1811.04267
- Besov class via heat semigroup on Dirichlet spaces II:
BV functions and Gaussian heat kernel estimates, arXiv:1811.11010
- Besov class via heat semigroup on Dirichlet spaces III:
BV functions and sub-Gaussian heat kernel estimates, arXiv:1903.10078
 - ▶ New conjectures about Hölder continuity on the Sierpinski carpet
 - ▶ BV functions on nested fractals (research in progress).
 - ▶ Introduction (Fabrice Baudoin)
 - ▶ Introduction (Luke Rogers)

Abstract: we introduce heat semigroup-based Besov classes for general Dirichlet spaces, study quantitative regularization estimates for the heat semigroup in this scale of spaces, and obtain a far reaching L^p -analogue, $p \geq 1$, of the Sobolev inequality that was proved for $p = 2$ by N. Varopoulos under the assumption of ultracontractivity for the heat semigroup. The case $p = 1$ may yield isoperimetric type inequalities and Bounded Variation (BV) function spaces.

Motivation

Isoperimetric inequalities and BV functions, for \mathbb{R}^n and manifolds, were studied by **Caccioppoli, De Giorgi, Federer, Ledoux, Miranda** et al and more recently by **Ambrosio (related to Cheeger, Hajłasz, Heinonen, Koskela)** et al in non-smooth setting.

Besov and related spaces in DMMS and heat semi-group setting were studied by **Barlow, Bass, Hambly, Hinz, Hu, Jonsson, Grigoryan, Kumagai, Lau, Pietruska-Pałuba, Triebel, Wallin, Zähle**.

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Sobolev type inequalities for general Dirichlet forms

In \mathbb{R}^n , $n \geq 2$,

$$\|f\|_{L^q(\mathbb{R}^n)} \leq C \|\nabla f\|_{L^p(\mathbb{R}^n)}, \quad f \in C_0^\infty(\mathbb{R}^n) \quad (1)$$

$1 \leq p < n$, $q = \frac{np}{n-p}$, and C depends on n and p .

For a measure space (X, μ) with a symmetric Dirichlet form \mathcal{E} with domain \mathcal{F} , N. Varopoulos proved in 1985 that a heat kernel upper bound of the type $p_t(x, y) \leq \frac{C}{t^{n/2}}$, $n > 2$, implies the following Sobolev inequality

$$\|f\|_{L^q(X, \mu)} \leq C \sqrt{\mathcal{E}(f, f)}, \quad f \in \mathcal{F}, \quad (2)$$

where $q = \frac{2n}{n-2}$.

Note that condition

$$p_t(x, y) \leq C t^{-\beta}$$

is possible to verify in many classical and fractal examples because it is equivalent to the Nash inequality

$$\|f\|_{L^2(X, \mu)}^{2+2/\beta} \leq C \mathcal{E}(f, f) \|f\|_{L^1(X, \mu)}^{2/\beta} \quad (3)$$

by Carlen-Kusuoka-Stroock 1987.

Theorem (Strong Sobolev inequality):

Let $(X, \mu, \mathcal{E}, \mathcal{F})$ be a Dirichlet space. Let $\{P_t\}_{t \in [0, \infty)}$ denote the Markovian semigroup associated with $(X, \mu, \mathcal{E}, \mathcal{F})$. Let $p \geq 1$. Assume that P_t admits a measurable heat kernel $p_t(x, y)$ satisfying, for some $C > 0$ and $\beta > 0$, $p_t(x, y) \leq Ct^{-\beta}$ for $\mu \times \mu$ -a.e. $(x, y) \in X \times X$, and for each $t \in (0, +\infty)$. Assume that there exist $\alpha > 0$ and $C > 0$ such that for every $f \in L^p(X, \mu)$

$$\|f\|_{p, \alpha} \leq C \liminf_{t \rightarrow 0} t^{-\alpha} \left(\int_X \int_X |f(x) - f(y)|^p p_t(x, y) d\mu(x) d\mu(y) \right)^{1/p},$$

where

$$\|f\|_{p, \alpha} := \sup_{t > 0} t^{-\alpha} \left(\int_X \int_X |f(x) - f(y)|^p p_t(x, y) d\mu(x) d\mu(y) \right)^{1/p}.$$

Then, if $0 < \alpha < \beta$ and $p < \frac{\beta}{\alpha}$, there exists a constant $C_{p, \alpha, \beta} > 0$ such that for every $f \in L^p(X, \mu)$,

$$\|f\|_{L^q(X, \mu)} \leq C_{p, \alpha, \beta} \|f\|_{p, \alpha},$$

where $q = \frac{p\beta}{\beta - p\alpha}$.

Locality in time

Lemma

Let $p \geq 1$ and $\alpha \geq 0$. Then $\mathbf{B}^{p,\alpha}(\mathbf{X}) =$

$$\left\{ f \in L^p(\mathbf{X}, \mu) : \limsup_{t \rightarrow 0} t^{-\alpha} \left(\int_{\mathbf{X}} P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} < +\infty \right\}.$$

Moreover, if $\beta > \alpha$, then $\mathbf{B}^{p,\beta}(\mathbf{X}) \subset \mathbf{B}^{p,\alpha}(\mathbf{X})$. Furthermore, for $f \in \mathbf{B}^{p,\alpha}(\mathbf{X})$, and for every $t > 0$, we have

$$\|f\|_{p,\alpha} \leq \frac{2}{t^\alpha} \|f\|_{L^p(\mathbf{X}, \mu)} + \sup_{s \in (0, t]} s^{-\alpha} \left(\int_{\mathbf{X}} P_s(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p}.$$

Triviality of some of the spaces $\mathbf{B}^{p,\alpha}(\mathbf{X})$

$\mathcal{F} = \mathbf{B}^{2,1/2}(\mathbf{X})$ is dense in L^2 . Assume $\mathcal{E}(\mathbf{f}, \mathbf{f}) = 0$ only for constant \mathbf{f} . Then, any $\mathbf{f} \in \mathbf{B}^{p,\alpha}(\mathbf{X})$ with $1 \leq p \leq 2$ and $\alpha > 1/p$ is constant.

Theorem

Let $p > 2$. If $\mathbf{f} \in \mathbf{B}^{p,1/2}(\mathbf{X}) \cap \mathcal{F}$ then there is $\Gamma(\mathbf{f}) \in L^1(\mathbf{X}, \mu)$ such that for all $\mathbf{g} \in L^\infty(\mathbf{X}, \mu) \cap \mathcal{F}$,

$$\int_{\mathbf{X}} \mathbf{g} \Gamma(\mathbf{f}) d\mu = 2\mathcal{E}(\mathbf{g}\mathbf{f}, \mathbf{f}) - \mathcal{E}(\mathbf{f}^2, \mathbf{g}).$$

Corollary

If $\mathbf{B}^{p,1/2}(\mathbf{X}) \cap \mathcal{F}$ is dense in \mathcal{F} for some $p > 2$, then \mathcal{E} admits a carré du champ operator.

Suppose that for all $\mathbf{f} \in \mathcal{F}$ we have that \mathbf{f} is constant whenever $\mathcal{E}(\mathbf{f}, \mathbf{f}) = 0$. If \mathcal{E} is regular and the energy measure $\nu_{\mathbf{f}}$ is singular to μ for any non-constant $\mathbf{f} \in \mathcal{F}$. Then $\mathbf{B}^{p,1/2}(\mathbf{X})$ contains only constant functions when $p > 2$.

Continuity of P_t on the Besov spaces

Theorem

Let $1 < p \leq 2$. There exists a constant $C_p > 0$ such that for every $f \in L^p(X, \mu)$ and $t \geq 0$

$$\|P_t f\|_{p,1/2} \leq \frac{C_p}{t^{1/2}} \|f\|_{L^p(X, \mu)}.$$

In particular $P_t : L^p(X, \mu) \rightarrow B^{p,1/2}(X)$ is bounded for $t > 0$.

Corollary

Let $2 \leq p < +\infty$ and $\alpha > 1/2$. If $f \in B^{p,\alpha}(X)$ then $\mathcal{E}(f, f) = 0$.

Let $1 < p \leq 2$. Let L be the generator of \mathcal{E} and \mathcal{L}_p be the domain of L in $L^p(X, \mu)$. Then

$$\mathcal{L}_p \subset B^{p,1/2}(X)$$

and for every $f \in \mathcal{L}_p$, $\|f\|_{p,1/2}^2 \leq C \|Lf\|_{L^p(X, \mu)} \|f\|_{L^p(X, \mu)}$.

Besov spaces and critical exponents

$$\alpha_p^*(\mathbf{X}) = \sup\{\alpha > 0 : \mathbf{B}^{p,\alpha}(\mathbf{X}) \text{ is dense in } L^p(\mathbf{X}, \mu).\}$$

$$\alpha_p^\#(\mathbf{X}) = \sup\{\alpha > 0 : \mathbf{B}^{p,\alpha}(\mathbf{X}) \text{ contains non-constant functions}\}.$$

① Both $p \mapsto \alpha_p^*(\mathbf{X})$ and $p \mapsto \alpha_p^\#(\mathbf{X})$ are non-increasing;

② For $1 \leq p \leq 2$ we have $\alpha_p^\#(\mathbf{X}) \geq \alpha_p^*(\mathbf{X}) \geq \frac{1}{2}$;

If we assume that $\mathcal{E}(f, f) = 0$ implies f constant, then we have in addition

③ If $1 \leq p \leq 2$ then $\alpha_p^*(\mathbf{X}) \leq \alpha_p^\#(\mathbf{X}) \leq \frac{1}{p}$;

④ $\alpha_2^*(\mathbf{X}) = \alpha_2^\#(\mathbf{X}) = \frac{1}{2}$;

⑤ For $2 \leq p < \infty$ one has $\alpha_p^*(\mathbf{X}) \leq \alpha_p^\#(\mathbf{X}) \leq \frac{1}{2}$;

Furthermore if \mathcal{E} is regular and the energy measure ν_f for each non-constant $f \in \mathcal{F}$ is singular to μ (as is the case on some fractals) we obtain

⑥ For $p > 2$ one has $\alpha_p^*(\mathbf{X}) \leq \alpha_p^\#(\mathbf{X}) < \frac{1}{2}$.

Theorem

For Riemannian manifolds, sub-Riemannian manifolds, or metric graphs one has $\alpha_1^*(\mathbf{X}) = \alpha_1^\#(\mathbf{X}) = \frac{1}{2}$, but for nested fractals or their products

$$\alpha_1^*(\mathbf{X}) = \alpha_1^\#(\mathbf{X}) = \frac{d_H - d_{tH} + 1}{d_W}$$

Topological-Hausdorff dimension:

$$d_{tH} := 1 + \inf(d_H(\partial O))$$

where the **inf** is taken over collections of open sets O that form a base of the topology.

Compare to the topological dimension $d_t := 1 + \inf(d_t(\partial O))$ where the **inf** is taken over collections of open sets O that form a base of the topology.

Main examples

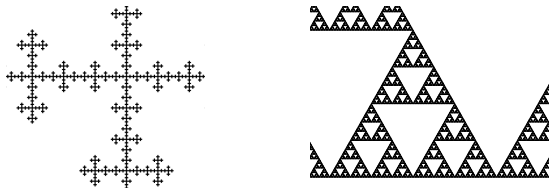


Figure: A part of an infinite, or unbounded, Vicsek set and a Sierpinski gasket.

$$d_H = \frac{\log N}{\log c}, \quad d_{tH} = 1, \quad \alpha_1^*(X) = \alpha_1^\#(X) = \frac{d_H - d_{tH} + 1}{d_W} = \frac{d_H}{d_W}$$

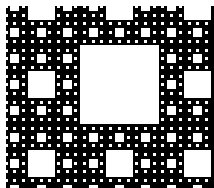


Figure: A part of an infinite, or unbounded, Sierpinski carpet, $d_{tH} = 1 = \frac{\log 2}{\log 3}$

Besov class via heat semigroup on Dirichlet spaces III: BV functions and sub-Gaussian heat kernel estimates.

Let (X, d, μ) be a locally compact complete metric Radon measure space and

$$\|f\|_{KS^{\lambda,p}(X)}^p := \limsup_{r \rightarrow 0^+} \int_X \int_{B(x,r)} \frac{|f(y) - f(x)|^p}{r^{\lambda p} \mu(B(x,r))} d\mu(y) d\mu(x) < +\infty.$$

The L^p -Korevaar-Schoen critical exponent is

$$\lambda_p^\# = \sup\{\lambda > 0 : KS^{\lambda,p}(X) \text{ contains non-constant functions}\}.$$

With a **1**-Poincaré inequality and doubling, one has $\lambda_p^\# = \mathbf{1}$ for every $p \geq \mathbf{1}$. Note that, at the critical exponent $\lambda_2^\# = \mathbf{1}$, one can construct a Dirichlet form

$$\mathcal{E}(f) \simeq \|f\|_{KS^{1,2}(X)}^2$$

with domain $KS^{1,2}(X)$ by using a choice of a Cheeger differential structure. This Dirichlet form is then strictly local and the intrinsic distance $d_{\mathcal{E}}$ associated to \mathcal{E} is bi-Lipschitz equivalent to the original metric d .

At the critical exponent $\lambda_1^\# = \mathbf{1}$, one has $KS^{1,1}(X) = BV(X)$ and

$$\mathbf{Var}(f) \simeq \|f\|_{KS^{1,1}(X)}.$$

$$\lambda_2^\# = \frac{d_W}{2}$$

$$\mathcal{E}(f) \simeq \|f\|_{KS^{\frac{d_W}{2}, 2}(X)}^2$$

For nested fractals

$$\lambda_1^\# = d_H$$

is the Hausdorff dimension.

For the Sierpinski carpet we prove that

$$\lambda_1^\# \geq d_H - d_{tH} + 1$$

and conjecture that in fact there is an equality. Here d_{tH} is the topological-Hausdorff dimension.

Weak Bakry-Émery nonnegative curvature condition

We say that the weak Bakry-Émery non-negative curvature condition $wBE(\kappa)$ is satisfied if there exist a constant $C > 0$ and a parameter $0 < \kappa < d_W$ such that for every $t > 0$, $g \in L^\infty(X, \mu)$ and $x, y \in X$,

$$|P_t g(x) - P_t g(y)| \leq C \frac{d(x, y)^\kappa}{t^{\kappa/d_W}} \|g\|_{L^\infty(X, \mu)}.$$

This inequality is proved by Barlow for resistance spaces under the sGHKE.

For nested fractals, $wBE(\kappa)$ is satisfied with $\kappa = d_W - d_H$. In that case the value $d_W - d_H$ is optimal.

For the Sierpinski carpet satisfies $wBE(\kappa)$ with $\kappa = d_W - d_H$, however we conjecture that in fact the Sierpinski carpet satisfies $wBE(\kappa)$ with $\kappa > d_W - d_H$.

It will be a subject of future work to investigate whether the Sierpinski carpet has

$$\kappa = d_W - d_H + d_{tH} - 1 = d_W - d_H + \frac{\log 2}{\log 3}$$

[Strongly supported by numerical results: L.Rogers et al]

Main results on BV functions under the $wBE(\kappa)$ condition

Define

$$BV(X) := KS^{\lambda_1^\#, 1}(X) = B^{1, \alpha_1^\#}(X)$$

and for $f \in BV(X)$,

$$\mathbf{Var}(f) := \liminf_{r \rightarrow 0^+} \int_X \int_{B(x, r)} \frac{|f(y) - f(x)|}{r^{\lambda_1^\#} \mu(B(x, r))} d\mu(y) d\mu(x).$$

We show that for nested fractals and their products, $BV(X)$ is dense in $L^1(X, \mu)$.

Locality property There is a constant $C > 0$ such that for every $f \in BV(X)$,

$$\sup_{r>0} \frac{1}{r^{d_H+d_W-\kappa}} \int_X \int_{B(y,r)} |f(x) - f(y)| d\mu(x) d\mu(y) \leq C \text{Var}(f).$$

Co-area estimate

There exist constants $c, C > 0$ such that for every non-negative $f \in BV(X)$,

$$c \int_0^\infty \text{Var}(1_{E_t(f)}) dt \leq \text{Var}(f) \leq C \int_0^\infty \text{Var}(1_{E_t(f)}) dt,$$

where $E_t(f) = \{x \in X : f(x) > t\}$. In particular, for $f \in BV(X)$ the sets $E_t(f) = \{x \in X : f(x) > t\}$ are of finite perimeter for almost every $t > 0$.

Minkowski There exists a constant $C > 0$ such that for every Borel set $E \subset X$,

$$P(E) \leq C C_{d_W-\kappa}^*(E),$$

where $C_{d_W-\kappa}^*(E)$ denotes the $(d_W - \kappa)$ -codimensional lower Minkowski content of E . In particular, any set whose measure-theoretic boundary has finite $(d_W - \kappa)$ -codimensional lower Minkowski content has finite perimeter

Sobolev inequality I Assume $d_W - \kappa < d_H$. Then $BV(X) \subset L^{1^*}(X, \mu)$ and there is $C > 0$ such that for every $f \in BV(X)$,

$$\|f\|_{L^{1^*}(X, \mu)} \leq C \text{Var}(f),$$

where the critical Sobolev exponent 1^* is given by the formula

$$\frac{1}{1^*} = 1 - \frac{d_W - \kappa}{d_H} \quad 1^* = \frac{d_H - d_W + \kappa}{d_H}.$$

Isoperimetric inequality

$$\mu(E)^{\frac{d_H - d_W + \kappa}{d_H}} \leq CP(E).$$

Sobolev inequality II

Assume $\kappa = d_W - d_H > 0$. Then $BV(X) \subset L^\infty(X, \mu)$ and there exists a constant $C > 0$ such that for every $f \in BV(X)$, and a.e. $x, y \in X$

$$|f(x) - f(y)| \leq C \text{Var}(f).$$

For nested fractals we do have $\kappa = d_W - d_H > 0$. Moreover, a set has finite perimeter if and only if it has finite boundary, $P(E) \sim \#(\partial E)$.

Theorem (research in progress)

$f \in \mathbf{BV}$ iff ∇f is a “vector valued Radon measure”.

This is understood in the distributional sense (Hinz, Rogers, Strichartz et al)

Corollary

- 1 on the Vicsek set, any BV function is \mathbb{R}^1 -BV along each geodesic path.
- 2 on the Sierpiński gasket, any BV function is discontinuous.

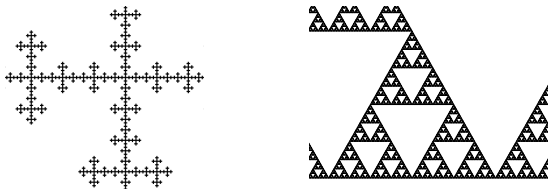


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