Spectral analysis and waves on fractals

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outline:

- Introduction and motivation: Spectral analysis on fractals.
- Weak Uncertainty Principle (Okoudjou, Saloff-Coste, Strichartz, T., 2008)
- Bohr asymptotics on infinite Sierpinski gasket (with Chen, Molchanov, 2015).
- Singularly continuous spectrum of a self-similar Laplacian on the half-line (with Chen, 2016).
- Laplacians on fractals with spectral gaps gaps have nicer Fourier series (Strichartz, 2005)
- Dynamical systems: canonical diffusions on the pattern spaces of aperiodic Delone sets (with Alonso-Ruiz, Hinz, Treviño, 2018).

This is a part of the broader program to develop **probabilistic, spectral and vector analysis on singular spaces** by **carefully building approximations by graphs or manifolds**.

Asymptotic aspects of Schreier graphs and Hanoi Towers groups

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Presented by Étienne Ghys

Abstract

We present relations between growth, growth of diameters and the rate of vanishing of the spectral gap in Schreier graphs of automaton groups. In particular, we introduce a series of examples, called Hanoi Towers groups since they model the well known Hanoi Towers Problem, that illustrate some of the possible types of behavior. *To cite this article: R. Grigorchuk, Z. Šunić, C. R. Acad. Sci. Paris, Ser. I 344 (2006).*

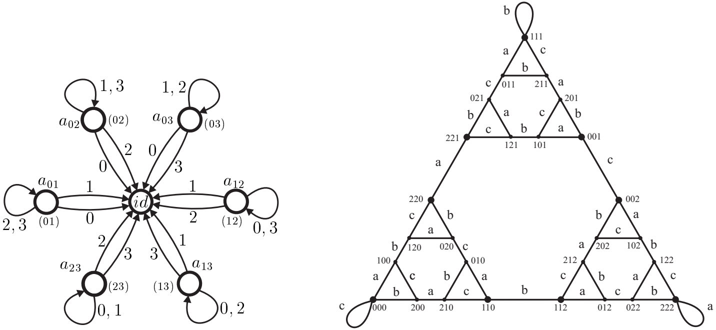
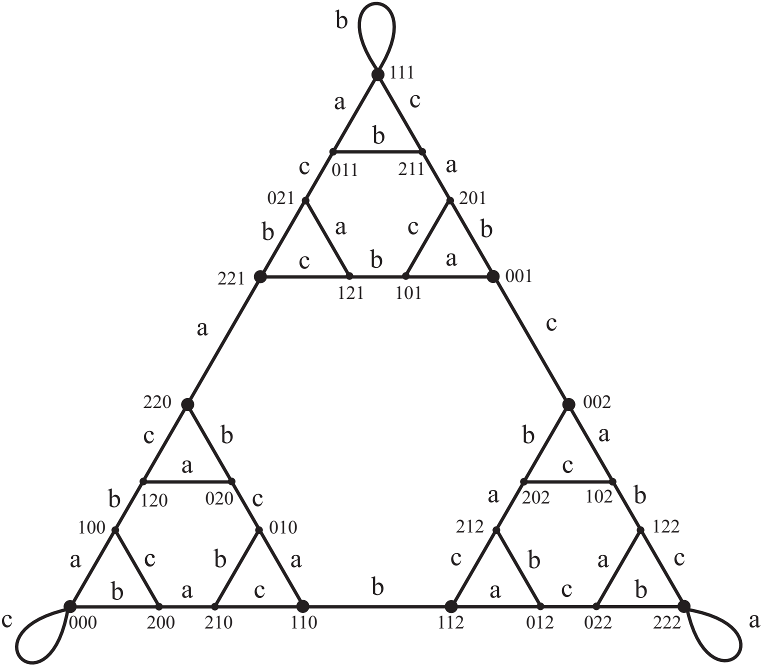


Figure 1. The automaton generating $H^{(4)}$ and the Schreier graph of $H^{(3)}$ at level 3 / L'automate engendrant $H^{(4)}$ et le graphe de Schreier de $H^{(3)}$ au niveau 3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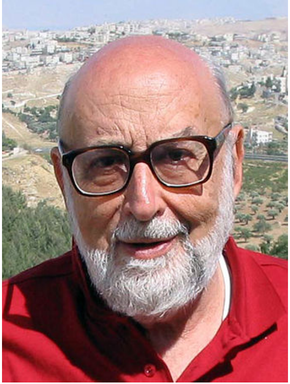
François Englert

From Wikipedia, the free encyclopedia

François Baron Englert (French: [ɑ̃glɛʁ]; born 6 November 1932) is a Belgian theoretical physicist and 2013 Nobel prize laureate (shared with Peter Higgs). He is Professor emeritus at the Université libre de Bruxelles (ULB) where he is member of the Service de Physique Théorique. He is also a Sackler Professor by Special Appointment in the School of Physics and Astronomy at Tel Aviv University and a member of the Institute for Quantum Studies at Chapman University in California. He was awarded the 2010 J. J. Sakurai Prize for Theoretical Particle Physics (with Gerry Guralnik, C. R. Hagen, Tom Kibble, Peter Higgs, and Robert Brout), the Wolf Prize in Physics in 2004 (with Brout and Higgs) and the High Energy and Particle Prize of the European Physical Society (with Brout and Higgs) in 1997 for the mechanism which unifies short and long range interactions by generating massive gauge vector bosons. He has made contributions in statistical physics, quantum field theory, cosmology, string theory and supergravity.[4] He is the recipient of the 2013 Prince of Asturias Award in technical and scientific research,

François Englert



François Englert in Israel, 2007

Nuclear Physics B280 [FS 18] (1987) 147–180
North-Holland, Amsterdam

METRIC SPACE-TIME AS FIXED POINT OF THE RENORMALIZATION GROUP EQUATIONS ON FRACTAL STRUCTURES

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Received 19 February 1986

We take a model of foamy space-time structure described by self-similar fractals. We study the propagation of a scalar field on such a background and we show that for almost any initial conditions the renormalization group equations lead to an effective highly symmetric metric at large scale.

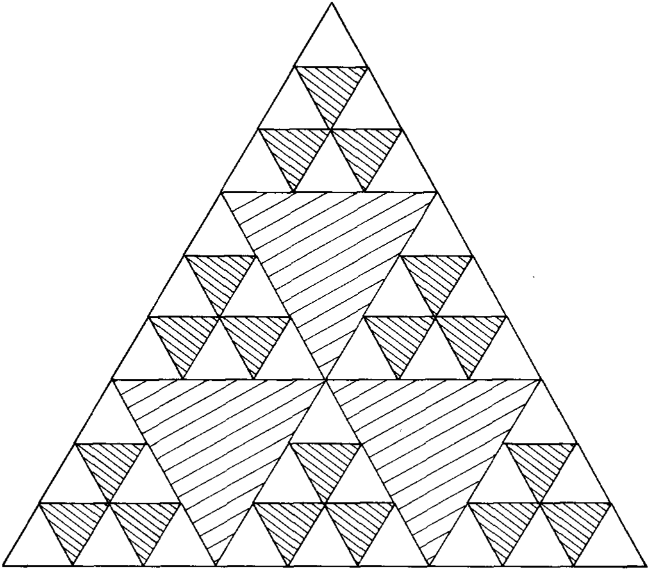

Fig. 1. The first two iterations of a 2-dimensional 3-fractal.

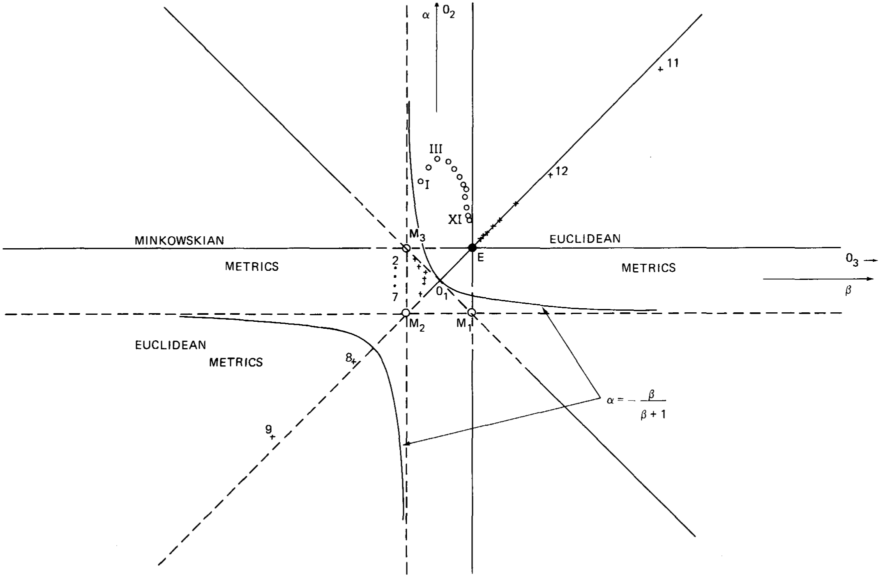

Fig. 5. The plane of 2-parameter homogeneous metrics on the Sierpinski gasket. The hyperbole $\alpha = -\beta/(\beta+1)$ separates the domain of euclidean metrics from minkowskian metrics and corresponds – except at the origin – to 1-dimensional metrics. M_1, M_2, M_3 denote unstable minkowskian fixed geometries while E corresponds to the stable euclidean fixed point. The unstable fixed points 0_1, 0_2 and 0_3 associated to 0-dimensional geometries are located at the origin and at infinity on the (α, β) coordinates axis. The six straight lines are subsets invariant with respect to the recursion relation but repulsive in the region where they are dashed. The first points of two sequences of iterations are drawn. Note that for one of them the 10th point ($\alpha = -56.4$, $\beta = -52.5$) is outside the frame of the figure.

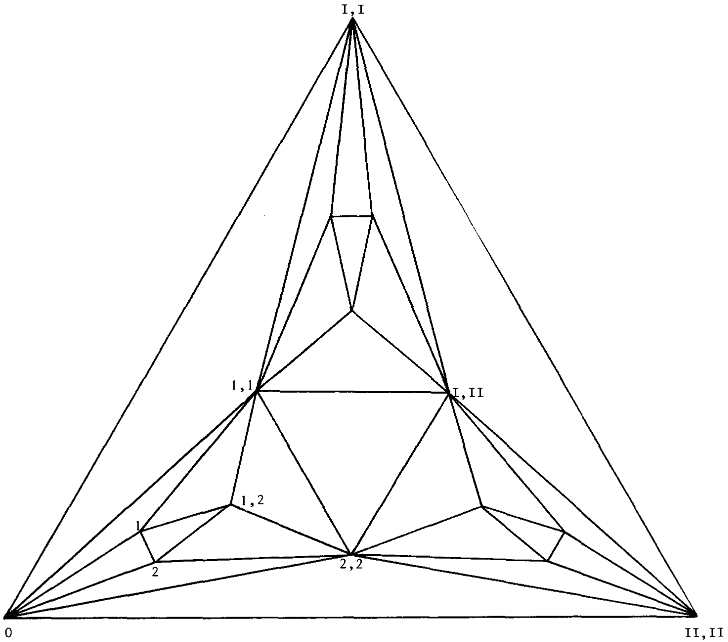

Fig. 10. A metrical representation of the two first iterations of a 2-dimensional 2-fractal corresponding to the euclidean fixed point. Vertices are labelled according to fig. 4.

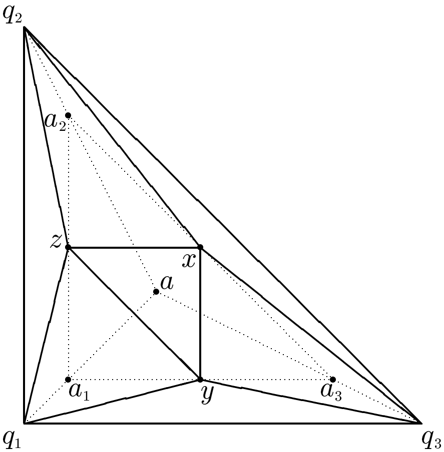

Figure 6.4. Geometric interpretation of Proposition 6.1.

PRL **95,** 171301 (2005) PHYSICAL REVIEW LETTERS week ending 21 OCTOBER 2005

The Spectral Dimension of the Universe is Scale Dependent

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(Received 13 May 2005; published 20 October 2005)

We measure the spectral dimension of universes emerging from nonperturbative quantum gravity, defined through state sums of causal triangulated geometries. While four dimensional on large scales, the quantum universe appears two dimensional at short distances. We conclude that quantum gravity may be "self-renormalizing" at the Planck scale, by virtue of a mechanism of dynamical dimensional reduction.

DOI: 10.1103/PhysRevLett.95.171301 PACS numbers: 04.60.Gw, 04.60.Nc, 98.80.Qc

Quantum gravity as an ultraviolet regulator?—A shared hope of researchers in otherwise disparate approaches to quantum gravity is that the microstructure of space and time may provide a physical regulator for the ultraviolet infinities encountered in perturbative quantum field theory.

tral dimension, a diffeomorphism-invariant quantity obtained from studying diffusion on the quantum ensemble of geometries. On large scales and within measuring accuracy, it is equal to four, in agreement with earlier measurements of the large-scale dimensionality based on the other hand, the “short-distance spectral dimension,” obtained by extrapolating Eq. (12) to $\sigma \longrightarrow 0$ is given by

$$D_S(\sigma = 0) = 1.80 \pm 0.25, \tag{15}$$

and thus is compatible with the integer value two.

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Fractal space-times under the microscope: a renormalization group view on Monte Carlo data

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Abstract: The emergence of fractal features in the microscopic structure of space-time is a common theme in many approaches to quantum gravity. In this work we carry out a detailed renormalization group study of the spectral dimension d_s and walk dimension d_w associated with the effective space-times of asymptotically safe Quantum Einstein Gravity (QEG). We discover three scaling regimes where these generalized dimensions are approximately constant for an extended range of length scales: a classical regime where $d_s = d, d_w = 2$, a semi-classical regime where $d_s = 2d/(2+d), d_w = 2+d$, and the UV-fixed point regime where $d_s = d/2, d_w = 4$. On the length scales covered by three-dimensional Monte Carlo simulations, the resulting spectral dimension is shown to be in very good agreement with the data. This comparison also provides a natural explanation for the apparent puzzle between the short distance behavior of the spectral dimension reported from Causal Dynamical Triangulations (CDT), Euclidean Dynamical Triangulations (EDT), and Asymptotic Safety.

Keywords: Models of Quantum Gravity, Renormalization Group, Lattice Models of Gravity, Nonperturbative Effects

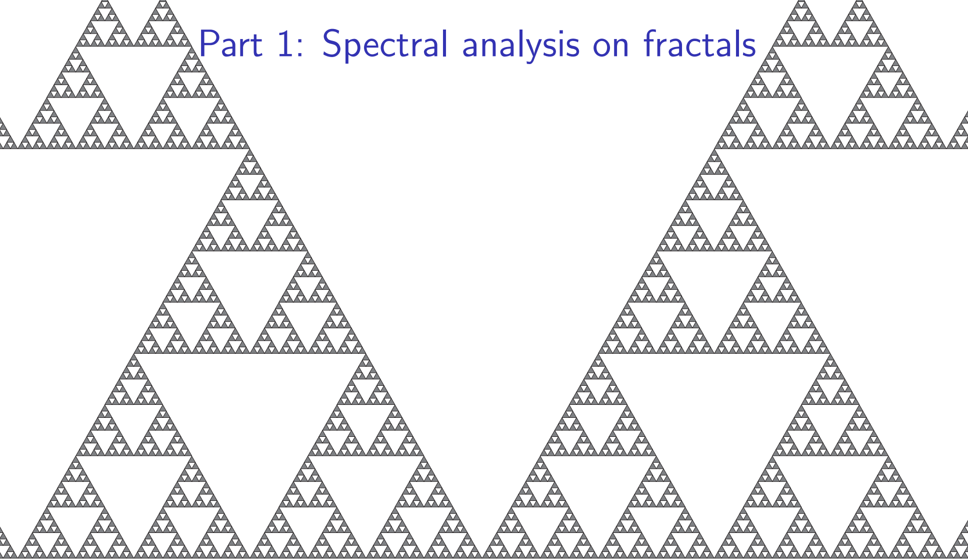
JHEP12(2011)012

Fractal space-times under the microscope: A Renormalization Group view on Monte Carlo data

Martin Reuter and Frank Saueressig

a classical regime where $d_s = d, d_w = 2$, a semi-classical regime where $d_s = 2d/(2+d), d_w = 2+d$, and the UV-fixed point regime where $d_s = d/2, d_w = 4$. On the length scales covered

Part 1: Spectral analysis on fractals



A part of an infinite Sierpiński gasket.

Weak Uncertainty Principle (Kasso Okoudjou, Laurent Saloff-Coste, T., 2008)

The $\mathbb{R}^1$ Heisenberg Uncertainty Principle is equivalent, if $\|f\|_{L^2} = 1$, to

$$\left(\int_{\mathbb{R}}\int_{\mathbb{R}} |x-y|^2 |f(x)|^2\,|f(y)|^2\,dx\,dy\right)\cdot\left(\int_{\mathbb{R}} |f'(x)|^2\,dx\right) \geqslant \frac{1}{8}$$

On a metric measure space (K, d, μ) with an energy form $\mathcal{E}$

a weak uncertainty principle

$$\boxed{Var_\gamma(u)\,\mathcal{E}(u,u) \geqslant C} \tag{1}$$

holds for $u \in L^2(K)\bigcap Dom(\mathcal{E})$

$$Var_\gamma(u) = \iint_{K\times K} d(x,y)^\gamma |u(x)|^2\,|u(y)|^2\,d\mu(x)\,d\mu(y). \tag{2}$$

provided either that d is the effective resistance metric, or some of the suitable Poincare inequalities are satisfied.

Bohr asymptotics

For 1D Schödinger operator

$$H\psi = -\psi'' + V(x)\psi, \qquad x \geq 0 \tag{3}$$

if $V(x) \to +\infty$ as $x \to +\infty$ then (H. Weyl), the spectrum of H in $L^2([0,\infty), dx)$ is discrete and, under some technical conditions,

$$N(\lambda, V) := \#\{\lambda_i(H) \leq \lambda\} \sim \frac{1}{\pi} \int_0^\infty \sqrt{(\lambda - V(x))_+}\, dx. \tag{4}$$

This is known as the Bohr's formula. It can be generalized for $\mathbb{R}^n$.

Theorem (Fractal Bohr's formula (Joe Chen, Stanislav Molchanov, T., J. Phys. A: Math. Theor. (2015)))

On infinite Sierpinski-type fractafolds, under mild assumptions,

$$\lim_{\lambda \to \infty} \frac{N(V,\lambda)}{g(V,\lambda)} = 1, \tag{5}$$

where

$$g(V,\lambda) := \int\limits_{K_\infty} \left[(\lambda - V(x))_+\right]^{d_s/2} G\left(\frac{1}{2}\log(\lambda - V(x))_+\right) \mu_\infty(dx), \tag{6}$$

where G is the Kigami-Lapidus periodic function, obtained via a renewal theorem.

Half-line example

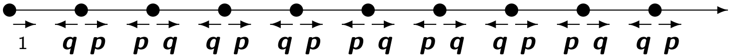


Figure: Transition probabilities in the pq random walk. Here $p \in (0,1)$ and $q = 1 - p$.

$$(\Delta_p f)(x) = \begin{cases} f(0) - f(1), & \text{if } x = 0 \\ f(x) - qf(x-1) - pf(x+1), & \text{if } 3^{-m(x)}x \equiv 1 \pmod 3 \\ f(x) - pf(x-1) - qf(x+1), & \text{if } 3^{-m(x)}x \equiv 2 \pmod 3 \end{cases}$$

Theorem (J.P.Chen, T., 2016)

If $p \neq \frac{1}{2}$, the Laplacian Δ_p on $\ell^2(\mathbb{Z}_+)$ has **purely singularly continuous spectrum**.

The spectrum is the Julia set of the polynomial $R(z) = \frac{z(z^2-3z+(2+pq))}{pq}$, which is a **topological Cantor set of Lebesgue measure zero**.

Laplacians on fractals with spectral gaps gaps have nicer Fourier series (Robert Strichartz, 2005)

If the Laplacian has an infinite sequence of exponentially large spectral gaps and the heat kernel satisfies sub-Gaussian estimates, then the partial sums of Fourier series (spectral expansions of the Laplacian) converge **uniformly** along certain special subsequences.

U.Andrews, J.P.Chen, G.Bonik, R.W.Martin, T.,
Wave equation on one-dimensional fractals with spectral decimation.
J. Fourier Anal. Appl. 23 (2017)
`http://teplyaev.math.uconn.edu/fractalwave/`

An introduction given in 2007:
`http://www.math.uconn.edu/~teplyaev/gregynog/AT.pdf`

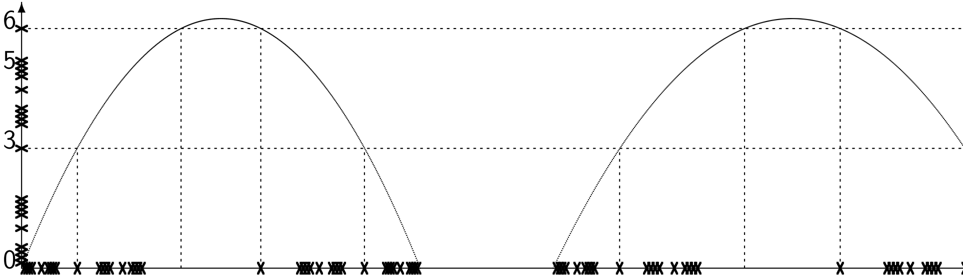


Figure: An illustration to the computation of the spectrum on the infinite Sierpiński gasket. The curved lines show the graph of the function $\mathfrak{R}(\cdot)$.

Theorem (Rammal, Toulouse 1983, Béllissard 1988, Fukushima, Shima 1991, T. 1998, Quint 2009)

On the infinite Sierpiński gasket the spectrum of the Laplacian consists of a **dense set of eigenvalues** $\mathfrak{R}^{-1}(\Sigma_0)$ **of infinite multiplicity** *and a* **singularly continuous component of spectral multiplicity one supported on** $\mathfrak{R}^{-1}(\mathcal{J}_R)$.

PHYSICAL REVIEW B VOLUME 31, NUMBER 3 1 FEBRUARY 1985

Energy spectrum for a fractal lattice in a magnetic field

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(Received 10 September 1984)

To simulate a kind of magnetic field in a fractal environment we study the tight-binding Schrödinger equation on a Sierpinski gasket. The magnetic field is represented by the introduction of a phase onto each hopping matrix element. The energy levels can then be determined by either direct diagonalization or recursive methods. The introduction of a phase breaks all the degeneracies which exist in and dominate the zero-field solution. The spectrum in the field may be viewed as considerably broader than the spectrum with no field. A novel feature of the recursion relations is that it leads to a power-law behavior of the escape rate. Green's-function arguments suggest that a majority of the eigenstates are truly extended despite the finite order of ramification of the fractal lattice.

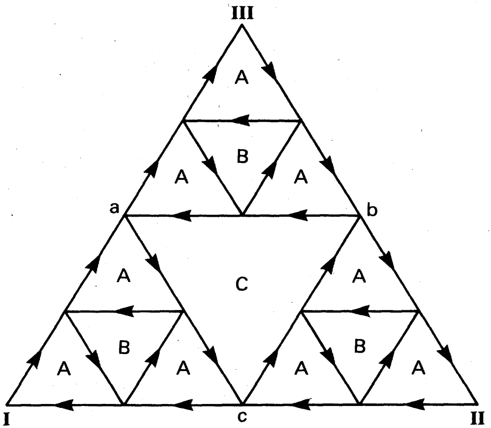

FIG. 1. Fragment of the Sierpinski gasket. The phase of the hopping matrix is equal to ϕ in the direction of the arrow and $-\phi$ otherwise.

Solid State Communications, Vol. 64, No. 10, pp. 1291–1294, 1987.
Printed in Great Britain.

0038–1098/87 $3.00 + .00

BAND SPECTRUM FOR AN ELECTRON ON A SIERPINSKI GASKET IN A MAGNETIC FIELD

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(*Received* 20 *July* 1987 *by S. Alexander*)

We consider a quantum charged particle on a fractal lattice given by a Sierpinski gasket, submitted to a uniform magnetic field, in a tight binding approximation. Its band spectrum is numerically computed and exhibits a fractal structure. The groundstate energy is also compared to the superconductor transition curve measured for a Sierpinski lattice of superconducting material.

choose the gauge in such a way that H depends only upon α and α' in a periodic way with period one. We will denote by $H(\alpha, \alpha')$ this operator from now on.

We also introduce the dilation operator D defined as:

$$D\varphi(m) = \varphi(2m). \tag{2}$$

The scaling properties of this system are expressed in the following Renormalization Group Equation (RGE) [23]:

$$E\{E1 - H(\alpha, \alpha')\}^{-1}D = G\{E^*1 - H(\alpha,^*, \alpha'^*)\}^{-1}, \tag{3}$$

where [7, 16]:

(i) $$G = \{E^3 - 3E - 2(XU + YV)\}/(S^2 + C^2)^{1/2},$$

(ii) $$E^* = \{E^4 - 7E^2 - [2(XU + YV) + 4X]E + 4(1 - U)\}/(S^2 + C^2)^{1/2}, \tag{4}$$

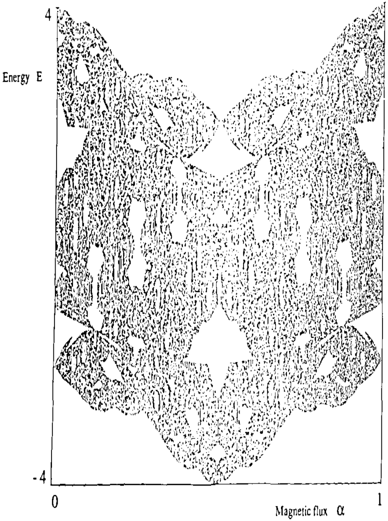

Fig. 2. Spectrum of $H(\alpha)$, computed by 10 iterations of F. α is the horizontal variable, ranging from 0 to 1. E is the vertical variable, ranging from -4 to 4.

These results have been compared with an experiment performed on an array of superconducting Al-wires shaped like a Sierpinski gasket with six levels of hierarchy. A description of this pattern generated by e-beam lithography has been given in [20]. More details will be published in a separate paper [21]. The transition curve in the parameter space (T, B), where observes experimentally the periodicity in the parameter α and also the scaling properties predicted by the RGE (equation 3). The plot in Fig. 4 shows the comparison between the experimental curve in log-log scale together with the theoretical results for the edge

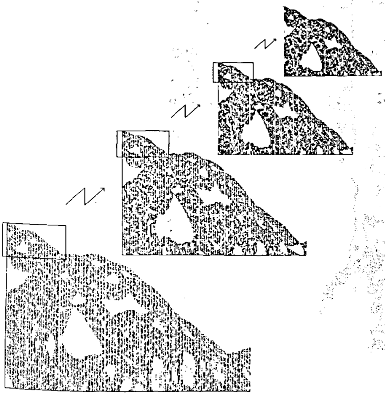

Fig. 3. Four enlargements of the upper left corner of Fig. 2, showing the fractal nature of the spectrum, with the approximate scaling law (7). α is the horizontal variable, ranging from 0 to 2^{-k}, $k = 2, 4, 6, 8$. E is the vertical variable, ranging from E_0 to 4, $E_0 = 2.4$, 3.68, 3.936, 3.9872.

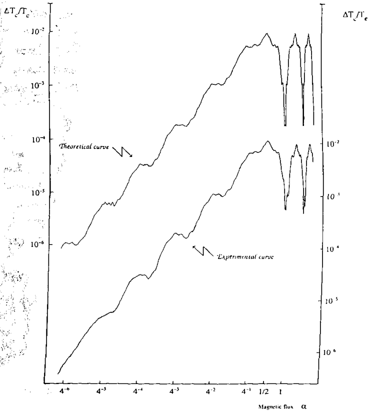

Fig. 4. Comparison between the calculated edge of the spectrum (left scale) with the experimental result (right scale) on the critical temperature of a superconducting gasket: $\Delta T_c/T_c$ vs α in log–log plot, where $\alpha = \Phi/\Phi_0$ is the reduced magnetic flux in the elementary triangle of the gasket: equation 8 has been used to calculate the theoretical curve using the best fit parameters as explained in the text. The two curves have been shifted for clarity.

Renormalization Group Analysis and Quasicrystals

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Table of Content

1.INTRODUCTION

Several quantal systems involving scale invariant properties, have been studied during the last few years by means of a Renormalization Group (RG) method. The most useful type of models is probably the hamiltonian describing the motion of a particle, phonon or

electron, in a quasicrystal. The first quantity to be calculated is the energy spectrum, from which we usually get others like the density of states (DOS), thermodynamical information, like the heat capacity or the magnetic susceptibility, or even various transport coefficients, like the conductivity. Using the spatial macroscopic symmetries, translations and scale invariance, it is possible to get equations satisfied by the model which happen to be sufficient to compute the spectrum in many cases. In particular the scale invariance will produce fractal spectra and scaling laws for the physical quantities.
The main difficulty is that unlike the 1D case for which the calculation can usually be performed by means of the transfer matrix method, the higher dimensional cases are far from being under control yet. In this short paper we want to give an account of a new strategy using operator algebras which should permit to extend the analysis to higher dimension. Eventhough the method is not yet completely developed, it has already given a certain number of convincing results, and we believe it should be the most efficient way of studying these problems. In this paper we compare it with the transfer matrix formulation for 1D chain and we show that both point of view are equivalent. We will only give an insight of what happens for higher dimensional quasicrystals, for this part of the work is still under progress.

2.JACOBI MATRIX OF A JULIA SET

2.1 The Julia Set of a Polynomial

The simplest model was designed in 1982 [Bellissard(82)], to get a new class of hamiltonians with Cantor spectra. It is the Jacobi matrix associated to a Julia set. Let $P(z) = z^N+p_1z^{N-1}+\ldots+p_{N-1}z+p_N$ be a polynomial with real coefficients. We then consider the dynamical system on the complex plane defined by $z_{n+1}=P(z_n)$. Clearly the point at infinity is fixed by P , and it is attractive, for there is $R>0$ big enough, such that whenever $|z| \geq R$, then $|P(z)| \geq R^N/2$. Let ζ be a fixpoint, namely a solution of $P(\zeta) = \zeta$, and let $D(\zeta)$ be the "domain of attraction of ζ", namely the open set of points z_0 such that $z_n \to \zeta$ as $n \to \infty$. The Julia set $J(P)$ of P is the complement of the union of the attraction domain of all fixpoints. Since the point at infinity is always attractive, $J(P)$ is always compact. A famous theorem by Julia and Fatou [Julia(18), Fatou(19), Douady(82)] asserts that $J(P)$ is completely disconnected whenever all critical points of P are attracted by the point at infinity.

3.SIERPINSKY LATTICE IN A MAGNETIC FIELD

3.1 The 2D Sierpinsky Lattice [Alexander(83,84), Rammal(84)]

The Sierpinsky lattice S in 2D is usually constructed according to the fig.1 below. Namely, let e_1, e_2 be two unit vectors making an angle of 60°. Then S is contained in the set Ne_1+Ne_2. Let then S_k be the subset of points $x \in S$ with $x = me_1+ne_2$ and $0 < m+n \leq 2^k$. S_k is recursively constructed as $S_1 = \{me_1+ne_2\,;\, 0 \leq m+n \leq 2\}$, $S_{k+1} = S_k \cup \{S_k+2^k e_1\} \cup \{S_k+2^k e_2\}$ for $k \geq 1$, and $S = \cup_{k\geq 1} S_k$.

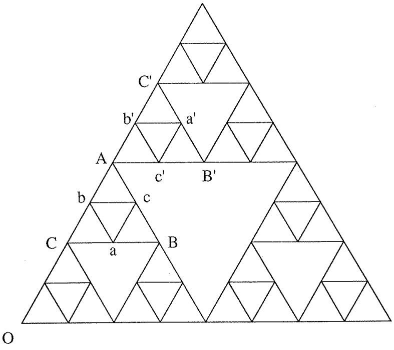

Fig.1- The subset S_3 of the Sierpinsky lattice in 2D -

From this construction it follows that $2S$ is included in S. A site in $2S$ is called "even", the others "odd". Any odd site admits the decomposition $2x+y$ where $x \in S$ and $y \in T = \{e_1, e_2, e_1+e_2\}$. The subsets $T(x) = T+2x$ are called "blocks". If $x \in S$, its nearest neigbours are all points in S within a distance 1 of x.

3.2 The Laplacean on S

The Laplace operators Δ_+ and Δ_- are defined on the Hilbert spaces $l^2(S)$ and $l^2(S\backslash\{0\})$ respectively by:

$$\Delta_+\phi\,(0) = \sqrt{2}\sum_{x';|x'|=1}\phi(x')\,, \qquad \Delta_+\phi\,(x) = \sqrt{2}\psi(0)+\sum_{x'\neq 0;|x'-x|=1}\phi(x')\,,$$

if $|x| = 1$, and

$$\Delta_+\phi\,(x) = \sum_{x';|x'-x|=1}\phi(x')\,, \qquad \text{if } |x|>1,\ \phi\in l^2(S)\,, \tag{11a}$$

$$\Delta_-\psi\,(x) = \sum_{x';|x'-x|=1}\psi(x')\,, \qquad x\in S\backslash\{0\}\,,\ \psi\in l^2(S\backslash\{0\})\,. \tag{11b}$$

Our goal is to compute the spectrum of Δ. In order to do so we will use the scale invariance of the Sierpinsky lattice. The main result is the following [Rammal(84), Bellissard(85)]

Theorem 3: The spectrum of $\Delta_\pm$ is made of two infinite sequences of eigenvalues of infinite multiplicity accumulating on the Julia set of the polynomial $P(z) = z(z-3)$. The first sequence consists of one isolated eigenvalue in each gap of $J(P)$, whereas the other consists of one edge of each gap of $J(P)$. ◊

Proof: Let us introduce the dilation operator D defined by

$$D\psi\,(x) = \psi(2x) \qquad x\in S, \qquad \psi\in l^2(S) \tag{12}$$

It is a partial isometry such that $DD^* = 1$. Then we claim that $\Delta_\pm$ are solutions of the following RG equation [Bellissard(85)]:

$$D\{z\mathbf{1}-\Delta\}^{-1}D^* = (z-2)(z+1)/(z+2)\,\{P(z)\mathbf{1}-\Delta\}^{-1}\,, \qquad P(z) = z(z-3) \tag{13}$$

$$E = \{z^4 - 7z^2 - [2(XU+YV)+ 4X]z + 4(1-U)\}/\{S^2 + C^2\}^{1/2} , \tag{17}$$

$$\beta + \beta' = 4(\alpha + \alpha') \qquad \beta - \beta' = 2(\alpha - \alpha') - 3/\pi \text{ Arctan } (S/C)$$

with:

$$S = [(XV + YU) + 2Y]z + 2(V + XY) , \quad C = z^2 + [(XY - YV) + 2X]z + 2(U - Y^2), \tag{18}$$

$$X = \cos 2\pi\alpha \ , Y = \sin 2\pi\alpha \ , \qquad U = \cos 2\pi(\alpha+\alpha') \qquad V = \sin 2\pi\ (\alpha+\alpha')$$

Following the intuition provided by the last section, the "dynamical spectrum" is defined as the invariant set of the map $F(z,\alpha,\alpha') = (E,\beta,\beta')$ of $\boldsymbol{R}x\boldsymbol{T}^2$. Since $\beta + \beta' = 4(\alpha + \alpha')$ in (17), only one of the two normalized fluxes is actually relevant, leading to an effective 2D map. *Is the dynamical spectrum equal to the actual spectrum of the original operator ? This is a question with no answer yet.* Nevertheless the numerical calculation of the dynamical spectrum given in fig.2 below [Ghez(87)], shows that it should be.
One should point out there that this calculation has been compared to an experiment performed in Grenoble, on a superconducting network designed according to fig. 1. Landau-Ginzburg's theory [de Gennes(81), Alexander(83)] shows that the transition between the normal metal and the super conducting phases occurs in the (T,B) plane (where T is the temperature) on a curve which is simply related to the edge of the dynamical spectrum as calculated above [Ghez(87)]. The comparison between theory and experiment is actually very accurate as shown in fig.3 below.

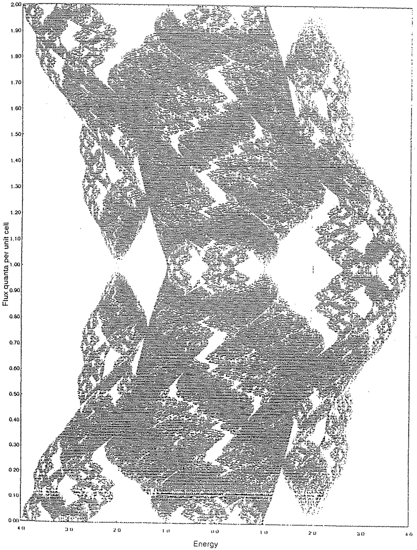

Fig.2- The dynamical spectrum of the Sierpinsky Laplacean in a magnetic field. The magnetic flux is represented on the horizontal axis, whereas the energy is represented on the vertical one. (Picture designed by C.Kreft) -

Following the intuition provided by the last section, the "dynamical spectrum" is defined as the invariant set of the map $F(z,\alpha,\alpha') = (E,\beta,\beta')$ of $\mathbf{R}xT^2$. Since $\beta + \beta' = 4(\alpha + \alpha')$ in (17), only one of the two normalized fluxes is actually relevant, leading to an effective 2D map. *Is the dynamical spectrum equal to the actual spectrum of the original operator ? This is a question with no answer yet.* Nevertheless the numerical calculation of the dynamical spectrum given in fig.2 below [Ghez(87)], shows that it should be.

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Recent refs related to Dirac operators on fractals

D. Guido, T. Isola, *Spectral triples for nested fractals*. J. Noncommut. Geom. 11 (2017)

M. Hinz, L. Rogers, *Magnetic fields on resistance spaces*. J. Fractal Geom. 3 (2016)

D. Kelleher, M. Hinz, A. Teplyaev, *Metrics and spectral triples for Dirichlet and resistance forms*. J. Noncommut. Geom. 9 (2015) arXiv:1309.5937

M. Hinz, A. Teplyaev, Dirac and magnetic *Schrodinger operators on fractals*. J. Funct. Anal. 265 (2013), arXiv:1207.3077

M. Ionescu, L. G. Rogers, A. Teplyaev, *Derivations and Dirichlet forms on fractals*. Journal of Functional Analysis, 263 (2012) arXiv:1106.1450
note especially Theorem 5.24

V. Nekrashevych and A. Teplyaev, *Groups and analysis on fractals*. Analysis on Graphs and its Applications, Proc. Symposia Pure Math., AMS, 77 (2008)
note especially Theorem 3.3

Canonical diffusions on the pattern spaces of aperiodic Delone sets (Patricia Alonso-Ruiz, Michael Hinz, T., Rodrigo Treviño)

A subset $\Lambda \subset \mathbb{R}^d$ is a **Delone set** if it is **uniformly discrete**:

$$\exists \varepsilon > 0 : \ |\vec{x} - \vec{y}| > \varepsilon \quad \forall \vec{x}, \vec{y} \in \Lambda$$

and relatively dense:

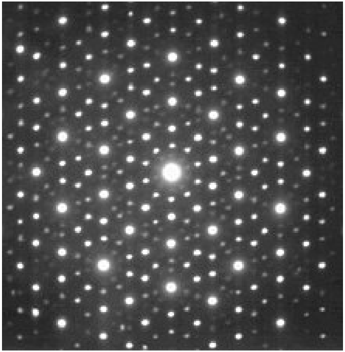
$$\exists R > 0 : \ \Lambda \cap B_R(\vec{x}) \neq \varnothing \quad \forall \vec{x} \in \mathbb{R}^d.$$

A Delone set has **finite local complexity** if $\forall R > 0 \exists$ finitely many clusters $P_1, \ldots, P_{n_R}$ such that for any $\vec{x} \in \mathbb{R}^d$ there is an i such that the set $B_R(\vec{x}) \cap \Lambda$ is translation-equivalent to P_i.

A Delone set Λ is **aperiodic** if $\Lambda - \vec{t} = \Lambda$ implies $\vec{t} = \vec{0}$. It is **repetitive** if for any cluster $P \subset \Lambda$ there exists $R_P > 0$ such that for any $\vec{x} \in \mathbb{R}^d$ the cluster $B_{R_P}(\vec{x}) \cap \Lambda$ contains a cluster which is translation-equivalent to P.

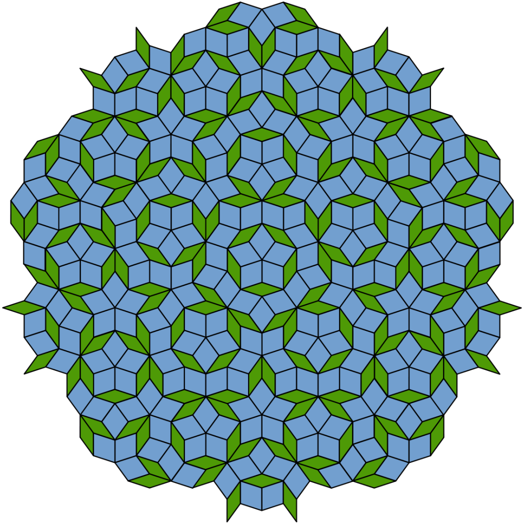
These sets have applications in crystallography ($\approx$ **1920**), coding theory, approximation algorithms, and the theory of quasicrystals.

Electron diffraction picture of a Zn-Mg-Ho quasicrystal



Aperiodic tilings were discovered by mathematicians in the early 1960s, and, some twenty years later, they were found to apply to the study of natural quasicrystals (1982 Dan Shechtman, 2011 Nobel Prize in Chemistry).

Penrose tiling



pattern space of a Delone set

Let $\Lambda_0 \subset \mathbb{R}^d$ be a **Delone set**. The **pattern space (hull)** of Λ_0 is the closure of the set of translates of Λ_0 with respect to the metric ϱ, i.e.

$$\Omega_{\Lambda_0} = \overline{\left\{\varphi_{\vec{t}}\,(\Lambda_0) : \vec{t} \in \mathbb{R}^d\right\}}.$$

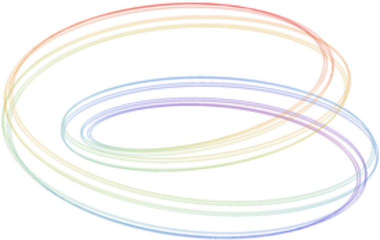
Definition

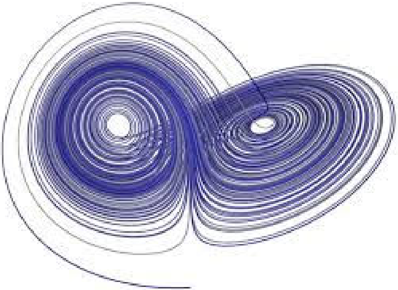
Let $\Lambda_0 \subset \mathbb{R}^d$ be a Delone set and denote by $\varphi_{\vec{t}}\,(\Lambda_0) = \Lambda_0 - \vec{t}$ its translation by the vector $\vec{t} \in \mathbb{R}^d$. For any two translates Λ_1 and Λ_2 of Λ_0 define $\varrho(\Lambda_1, \Lambda_2) = \inf\{\varepsilon > 0 : \exists\, \vec{s}, \vec{t} \in B_\varepsilon(\vec{0}) : B_{\frac{1}{\varepsilon}}(\vec{0}) \cap \varphi_{\vec{s}}(\Lambda_1) = B_{\frac{1}{\varepsilon}}(\vec{0}) \cap \varphi_{\vec{t}}(\Lambda_2)\} \wedge 2^{-1/2}$

Assumption

The action of $\mathbb{R}^d$ on Ω is uniquely ergodic:
Ω is a compact metric space with the unique $\mathbb{R}^d$-invariant probability measure μ.

Topological solenoids
(similar topological features as the pattern space $\boldsymbol{\Omega}$):





Theorem

(i) *If* $\vec{W} = (\vec{W}_t)_{t \geq 0}$ *is the standard Gaussian Brownian motion on* $\mathbb{R}^d$*, then for any* $\Lambda \in \Omega$ *the process* $X_t^\Lambda := \varphi_{\vec{W}_t}(\Lambda) = \Lambda - \vec{W}_t$ *is a conservative Feller diffusion on* (Ω, ϱ)*.*

(ii) *The semigroup* $P_t f(\Lambda) = \mathbb{E}[f(X_t^\Lambda)]$ *is*

$$\boxed{\text{self-adjoint on } L^2_\mu\text{, Feller but not strong Feller.}}$$

Its associated Dirichlet form is regular, strongly local, irreducible, recurrent, and has Kusuoka-Hino dimension d*.*

(iii) *The semigroup* $(P_t)_{t>0}$ **does not admit heat kernels with respect to** μ*. It does have Gaussian heat kernel with respect to the not-*σ*-finite (no Radon-Nykodim theorem) pushforward measure* λ^d_Ω

$$p_\Omega(t, \Lambda_1, \Lambda_2) = \begin{cases} p_{\mathbb{R}^d}(t, h_{\Lambda_1}^{-1}(\Lambda_2)) & \text{if } \Lambda_2 \in \mathbf{orb}(\Lambda_1), \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

(iv) **There are no semi-bounded or** L^1 **harmonic functions ("Liouville-type")**.

no classical inequalities

Useful versions of the Poincare, Nash, Sobolev, Harnack inequalities DO NOT HOLD, except in orbit-wise sense.

spectral properties

Theorem

The unitary **Koopman operators** $U_{\vec{t}}$ *on* $L^2(\Omega, \mu)$ *defined by* $U_{\vec{t}} f = f \circ \varphi_{\vec{t}}$ *commute with the heat semigroup*

$$\boxed{U_{\vec{t}} P_t = P_t U_{\vec{t}}}$$

hence commute with the Laplacian $\boldsymbol{\Delta}$*, and all spectral operators, such as the unitary Schrödinger semigroup.*

... hence we may have continuous spectrum (no eigenvalues) under some assumptions even though μ is a probability measure on the compact set Ω.

Under special conditions P_t is connected to the evolution of a **Phason**: "Phason is a quasiparticle existing in quasicrystals due to their specific, quasiperiodic lattice structure. Similar to phonon, phason is associated with atomic motion. However, whereas phonons are related to translation of atoms, phasons are associated with atomic rearrangements. As a result of these rearrangements, waves, describing the position of atoms in crystal, change phase, thus the term "phason" (from the wikipedia)".

Phason evolution

Corollary

The unitary **Koopman operators** $U_{\vec{t}}$ *on* $L^2(\Omega, \mu)$ *defined by* $U_{\vec{t}} f = f \circ \varphi_{\vec{t}}$ *commute with the heat semigroup*

$$\boxed{U_{\vec{t}} P_t = P_t U_{\vec{t}}}$$

hence commute with the Laplacian $\boldsymbol{\Delta}$*, and all spectral operators, including the unitary* **Schrödinger semigroup** $e^{i\Delta t}$

$$\boxed{U_{\vec{t}} e^{i\Delta t} = e^{i\Delta t} U_{\vec{t}}}$$

Recent physics work on phason ("accounts for the freedom to choose the origin"):
Topological Properties of Quasiperiodic Tilings
(Yaroslav Don, Dor Gitelman, Eli Levy and Eric Akkermans
Technion Department of Physics)
`https://phsites.technion.ac.il/eric/talks/`
J. Bellissard, A. Bovier, and J.-M. Chez, Rev. Math. Phys. 04, 1 (1992).

Topological Properties of Quasiperiodic Tilings

Yaroslav Don, Eli Levy and Eric Akkermans

Department of Physics, Technion – Israel Institute of Technology, Haifa, Israel

Abstract

Topological properties of finite quasiperiodic tilings are examined. We study two specific physical quantities: (a) the structure factor related to the Fourier transform of the structure; (b) spectral properties (using scattering matrix formalism) of the corresponding quasiperiodic Hamiltonian. We show that both quantities involve a phase, whose windings describe topological numbers. We link these two phases, thus establishing a "Bloch theorem" for specific types of quasiperiodic tilings.

Substitution Rules – 1D Tilings

Define a *binary* substitution rule by

$$\begin{aligned}\sigma(a) &= a^\alpha b^\beta \\ \sigma(b) &= a^\gamma b^\delta\end{aligned} \iff \begin{aligned} a &\mapsto a^\alpha b^\beta \\ b &\mapsto a^\gamma b^\delta.\end{aligned}$$

Associate occurrence matrix: $M = \left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)$ Consider only primitive matrices:

- Largest eigenvalue $\lambda_1 > 1$ (Perron-Frobenius)
- Left and right first eigenvectors are strictly positive

Distribution of letters underlies distribution of atoms:

$$\overset{x_0}{\bullet}\overset{a}{\text{———}}\overset{x_1}{\bullet}\overset{b}{\text{—}}\overset{x_2}{\bullet}\overset{b}{\text{—}}\overset{x_3}{\bullet}\overset{a}{\text{———}}\overset{x_4}{\bullet}\overset{b}{\text{—}}\overset{x_5}{\bullet}\overset{a}{\text{———}}\overset{x_6}{\bullet}\overset{b}{\text{—}}\overset{x_7}{\bullet}\ldots$$

Define atomic density

$$\rho(x) = \sum\nolimits_k \delta(x - x_k),$$

with distances for a and b given by $\delta_k = x_{k+1} - x_k = d_{a,b}$.
Let $\bar{d}$ be the mean distance and u_k the deviations from the mean. Define

$$x_k = \bar{d}\, k + \delta\, u_k, \quad \delta \equiv d_a - d_b.$$

Let $g(\xi) = \sum_k e^{-i\xi x_k}$ be the diffraction pattern, and $S(\xi) = |g(\xi)|^2$ the structure factor. Using $\xi_0 = 2\pi/\bar{d}$, the Bragg peaks are located at [1]

$$\xi_{m,N} = m\,\lambda_1^{-N}\,\xi_0, \quad m, N \in \mathbb{Z}.$$

We consider the following families:

Pisot. The second eigenvalue $|\lambda_2|<1$.

Non-Pisot. The second eigenvalue $|\lambda_2| \geq 1$.
Fluctuations u_k are unbounded [2]; there are no Bragg peaks [3].

Examine the following examples:

Fibonacci. $a \mapsto ab$, $b \mapsto a$. It is Pisot, $M = \left(\begin{smallmatrix}1 & 1 \\ 1 & 0\end{smallmatrix}\right)$, $\lambda_1 = (\sqrt{5}+1)/2 \equiv \tau$ the golden ratio and $\lambda_2 = -\tau^{-1}$. Bragg peaks are located at $\xi_{p,q} = (p + q/\tau)\,\xi_0$.
In C&P language, $s = 1/\tau$ and

$$\boxed{\xi_{p,q}/\xi_0 = p + qs} \quad p, q \in \mathbb{Z}.$$

Thue–Morse. $a \mapsto ab$, $b \mapsto ba$. Here it is Pisot, $M = \left(\begin{smallmatrix}1 & 1 \\ 1 & 1\end{smallmatrix}\right)$,
$\lambda_1 = 2$ and $\lambda_2 = 0$. Bragg peaks: $\xi_{m,N} = m\,2^{-N}\,\xi_0$, $m, N \in \mathbb{Z}$.

Spectral Properties of Tilings

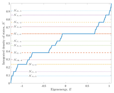
Consider a 1D tight-binding equation,

$$-(\psi_{k+1} + \psi_{k-1}) + V_k\psi_k = 2E\psi_k.$$

The gaps in the integrated density of states are given by the gap labeling theorem [4],

$$\mathcal{N}_{m,N} = \frac{1}{c} m\,\lambda_1^{-N} \pmod 1, \quad m, N \in \mathbb{Z}.$$

Here, c is the gcd of λ_1 and its corresponding eigenvectors in both M and the collared M_2.



In C&P sequences,

$$\boxed{\mathcal{N}_{p,q} = p + qs \pmod 1} \quad p, q \in \mathbb{Z}.$$

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We thank C. Schochet for useful discussions.

Structural Phase – Phason as a Gauge Field

Another way to define a tiling is by using a characteristic function. We consider the following choice [5, 6]:

$$\chi(n, \phi) = \operatorname{sign}[\cos(2\pi n\, s + \phi) - \cos(\pi s)].$$

with $s \to s_N = c_N/d_N$ the slope of the C&P scheme, and $n = 0 \ldots d_N - 1$.

The phase ϕ, called a phason, accounts for the gauge freedom to choose the origin. It is taken discretely as $\phi \to \phi_\ell = 2\pi\ell/d_N$.

Let $s_0(n) = \chi(n, 0)$. Let $\mathcal{T}[s_0(n)] = s_0(n+1)$ be the translation operator. Define

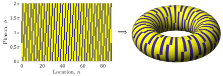
$$\Sigma_0 = \begin{pmatrix} s_0 \\ \mathcal{T}[s_0] \\ \cdots \\ \mathcal{T}^{d_N - 1}[s_0]\end{pmatrix} \implies \Sigma_0(n, \ell) = \mathcal{T}^\ell[s_0(n)].$$

Consider now the row permuted Σ_1

$$\Sigma_1(n, \ell) = \mathcal{T}^{m(\ell)}[s_0(n)], \qquad m(\ell) = \ell\, c_N^{-1} \pmod{d_N}.$$

Lemma. For $\phi_\ell = 2\pi\ell/d_N$ with $n, \ell = 0 \ldots d_N - 1$ one has $\chi(n, \phi_\ell) = \Sigma_1(n, \ell)$.
This defines a *discrete* phason ϕ_ℓ for the structure.

The structure of Σ_1 ($d_N \times d_N$) is that of a torus:



The discrete Fourier transform of Σ_1 about n reads

$$G(\xi, \ell) \equiv \sum\nolimits_{n=0}^{d_N - 1} \omega^{-\xi n}\,\Sigma_1(n, \ell) = \omega^{m(\ell)\xi}\, \varsigma_0(\xi).$$

- The structure factor $S(\xi, \phi) = |\varsigma_0(\xi)|^2$ is ϕ-independent.
- The phase of $G(\xi, \ell)$ reads

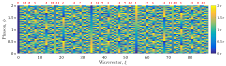
$$\boxed{\Theta(\xi, \ell) \equiv \arg\omega^{m(\ell)\xi} = \phi_\ell\,\xi/c_N \pmod{2\pi}.}$$

Corollary. For any diffraction peak (discrete Bragg peak) $\xi_{p,q} = qc_N$ one has the (discrete) winding number at $\xi_{p,q}$,

$$\Theta(\xi_{p,q}) = \frac{2\pi}{d_N}\ell\, q.$$

hence

$$\boxed{W_{\xi_{p,q}} = \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial\Theta(\xi = \xi_{p,q}, \phi)}{\partial\phi}\,d\phi = q.}$$



Here we used the Fibonacci sequence ($s = 1/\tau$) with $d_N = 89$ sites. The winding numbers are indicated by the red numbers above.

Remark. The analysis above for the winding numbers is done for rational approximations $s_N = c_N/d_N$. It holds by construction for the irrational case $s_N \to s$.

Spectral Phase: Scattering Matrix Approach

Spectral properties are also accessible from the continuous wave equation,

$$-\psi''(x) - k_0^2\, v(x)\,\psi(x) = k_0^2\,\psi(x),$$

with scattering boundary conditions.

The scattering $\mathcal{S}$-matrix is defined by $\left(\begin{smallmatrix}\overrightarrow{o} \\ \overleftarrow{o}\end{smallmatrix}\right) = \left(\begin{smallmatrix}\overrightarrow{\tau}(k) & \overleftarrow{r}(k) \\ \overrightarrow{r}(k) & \overleftarrow{\tau}(k)\end{smallmatrix}\right)\left(\begin{smallmatrix}\overrightarrow{\iota} \\ \overleftarrow{\iota}\end{smallmatrix}\right) \equiv \mathcal{S}\left(\begin{smallmatrix}\overrightarrow{\iota} \\ \overleftarrow{\iota}\end{smallmatrix}\right)$, with $\overrightarrow{\tau} = \overrightarrow{R}\,e^{i\overrightarrow{\theta}}$ and $\overleftarrow{\tau} = \overleftarrow{R}\,e^{i\overleftarrow{\theta}}$. It is unitary and can be diagonalized to $\mathcal{S} \mapsto \left(\begin{smallmatrix}e^{i\gamma_1} & 0 \\ 0 & e^{i\gamma_2}\end{smallmatrix}\right)$ so that $\det\mathcal{S} = e^{2i\delta(k)}$ with the total phase shift $\delta(k) = (\gamma_1(k) + \gamma_2(k))/2$ independent of ϕ. We are interested in the chiral phase,

$$\boxed{\alpha(k, \phi) = \overrightarrow{\theta}(k, \phi) - \overleftarrow{\theta}(k, \phi).}$$

Using the Krein-Schwinger formula [7] allows to relate the change of density of states to the scattering data,

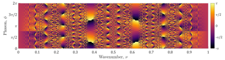
$$\varrho(k) - \varrho_0(k) = \frac{1}{2\pi}\operatorname{Im}\frac{d}{dk}\ln\det\mathcal{S}(k).$$

So that the integrated density of states is

$$\mathcal{N}(k) - \mathcal{N}_0(k) = \delta(k)/\pi.$$

The total phase shift $\delta(k)$ is independent of the phason ϕ unlike the *chiral* phase $\alpha(k, \phi)$, whose winding for values of k inside the gaps is given by [8],

$$\boxed{W_{\alpha_q} = \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial\alpha(k = k_{p,q}, \phi)}{\partial\phi}\,d\phi = 2q.}$$



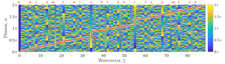
Here we used the Fibonacci sequence ($s = 1/\tau$) with $d_N = 233$ sites.

Relation between both Phases: a "Bloch Theorem"

In 1D C&P structures, The locations of Bragg peaks for a diffraction spectrum correspond to the spectral density of states,

$$\boxed{\xi_{p,q}/d_N = p + qs = \mathcal{N}_{p,q}} \quad p, q \in \mathbb{Z}.$$

Drawing the integrated density of states $\mathcal{N}$ (red line) on top the structural phase $\Theta(\xi, \phi)$, shows numerically the relation between $\xi_{p,q}$ and $\mathcal{N}_{p,q}$.



Both $\xi_{p,q}$ and $\mathcal{N}_{p,q}$ are isomorphic to $\mathbb{Z} \oplus s\mathbb{Z}$. The second $\mathbb{Z}$, corresponding to q, can be derived independently from the windings of both the structural phase $\Theta(\phi)$ and the chiral phase $\alpha(\phi)$. Since these phases account for windings, they are isomorphic

$$\boxed{\Theta(\phi) \cong \alpha(\phi).}$$

The winding gives a topological interpretation to these phases. This result can be viewed as a Bloch theorem for quasiperiodic tilings [9].

Cut and Project Scheme

An alternative method to build quasiperiodic tilings is the Cut & Project scheme. The procedure is as follows [10].

Cut.

1. Start with an n-dimensional space $R = \mathbb{R}^n$.
2. Insert "atoms" on the integer lattice $Z = \mathbb{Z}^n$.
3. Divide R into the *physical space* E and the *internal space* $E_\perp$ such that $E \oplus E_\perp = R$ and $E \cap E_\perp = \emptyset$.
4. To resolve ambiguity for E, choose an initial location $\boldsymbol{c} \in R$ such that E passes through $\boldsymbol{c}$. There is no such requirement for $E_\perp$.

Project.

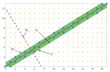
1. Inspect the hypercube $\mathbb{I}_n = [-0.5, 0.5)^n$.
2. The *window* is its projection on the internal space $W = \pi_\perp(\mathbb{I}_n)$.
3. The *strip* is the product with the physical space $S = W \otimes E$.
4. Choose only the points inside the strip $S \cap Z$, and project them onto the physical space, $Y = \pi(S \cap Z)$.
5. The *atomic density* is given by $\rho(x) \equiv \rho_c(x) = \sum_{y \in Y}\delta(x - y)$ with $x \in E$. Note the implicit dependency of Y on $\boldsymbol{c}$.

For 1D systems, define the phason

$$\phi = 2\pi\, b/W \qquad b \in E_\perp,$$

where W is the window above.
The slope s is given by

$$1/s = 1 + \cot\beta.$$



Useful Tools

In periodic structures, topological numbers are described as Chern numbers. This does not happen in quasiperiodic tilings, since there exists no notion of a Brillouin zone. But alternative tools exist to describe topological properties of quasiperiodic tilings. We now enumerate some of them.

- Tiling space $\mathcal{T}$ (dependent on λ_1 or s) and its hull $\Omega_\mathcal{T}$.
- Čech cohomology $\check{H}^1(\Omega_\mathcal{T})$, simplicial cohomology $H^1(\Gamma_n)$ and Bratteli graphs [11, 12].
- K-theory, $K_0(\Omega_\mathcal{T})$ group and the abstract gap labeling theorem [4, 13].
- The Bloch theorem described before can be given an interpolation for 1D C&P tilings (for an irrational slope $s \notin \mathbb{Q}$) by means of the "commutative diagram":

$$\begin{tikzcd}
\Theta(\phi) \arrow[r, "\cong"] & \alpha(\phi) \\
\mathbb{Z}^2 \cong \check{H}^1(\Omega_{C\&P}) \arrow[u] \arrow[r, "\psi"] \arrow[d, "C_{p(s)}"] & K_0(\Omega_{C\&P}) \cong \mathbb{Z}\oplus\mathbb{Z} \arrow[u] \arrow[d, "\tau_*"] \\
\mathbb{Z}\oplus s\mathbb{Z} \arrow[r, "\cong"] & \mathbb{Z}\oplus s\mathbb{Z}
\end{tikzcd}$$

The topological features are contained in $\check{H}^1$ or K_0 groups.

Conclusions

- We have defined two types of phases—a structural and spectral one—whose windings unveil topological features of quasiperiodic tilings.
- We found a relation between these two phases, which can be interpreted as a Bloch-like theorem.
- We have considered here a subset of tilings, which are known as Sturmian (C&P) words. Our results can be extended to a broader families of tilings in one dimension, and to tiles in higher dimensions ($D > 1$).
- All these features have been observed experimentally [5, 6].

References

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Topological Properties of Qu

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Ve study two
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two phases,
ic tilings.

The Phason – Structural Phase

Another way to define a tiling is by using a characteristic function. We consider the following choice [4, 5]:

$$\chi(n,\phi) = \operatorname{sign}\left[\cos\left(2\pi n\,\lambda_1^{-1} + \phi\right) - \cos\left(\pi\lambda_1^{-1}\right)\right]$$

with $n = 0 \ldots F_N - 1$ and $[0, 2\pi] \ni \phi \to \phi_\ell = 2\pi F_N^{-1}\,\ell$. The phase ϕ—called a phason—accounts for the freedom to choose the origin.
Let $s_0(n) = \chi(n, 0)$. Let $\mathcal{T}[s_0(n)] = s_0(n+1)$ be the translation operator. Define

$$\Sigma_0 = \begin{pmatrix} s_0 \\ \mathcal{T}[s_0] \\ \cdots \\ \mathcal{T}^{F_N-1}[s_0] \end{pmatrix} \implies \Sigma_0(n,\ell) = \mathcal{T}^{\ell}[s_0(n)]$$

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$\vec{r} = \vec{R}\,e^{i\,\vec{\vartheta}}$

Helmholtz, Hodge and de Rham

Theorem

Assume $\boldsymbol{d} = \mathbf{1}$. *Then the space* $\boldsymbol{L}^2(\boldsymbol{\Omega}, \mu, \mathbb{R}^1)$ *admits the orthogonal decomposition*

$$\boldsymbol{L}^2(\boldsymbol{\Omega}, \mu, \mathbb{R}^1) = \mathbf{Im}\,\boldsymbol{\nabla} \oplus \mathbb{R}(\boldsymbol{dx}). \tag{8}$$

In other words, the $\boldsymbol{L}^2$-cohomology is 1-dimensional, which is surprising because the **de Rham cohomology is not one dimensional**.

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end of the talk :-)

Thank you!