Dirichlet forms and vector analysis on fractal spaces

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ICIAM 2019

Valencia

Motivation

- Long term goals:
  - **understanding magnetic properties of fractal structures** (partially based on experimental physics) [Akkermans, Geim, Peeters et al]
  - **understanding spaces appearing in quantum gravity** (partially based on theoretical physics and numerical experiments) [Loll, Reuter et al]

Non-quantized penetration of magnetic field in the vortex state of superconductors

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As first pointed out by Bardeen and Ginzburg in the early sixties[1,2], the amount of magnetic flux carried by vortices in superconducting materials depends on their distance from the sample edge, and can be smaller than one flux quantum, $\phi_0 = h/2e$ (where h is Planck's constant and e is the electronic charge). In bulk superconductors, this reduction of flux becomes negligible at submicrometre distances from the edge, but in thin films the effect may survive much farther into the material[3,4]. But the effect has

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January 2007

EPL, **77** (2007) 27005
doi: 10.1209/0295-5075/77/27005

www.epljournal.org

Superconducting disk with magnetic coating: Re-entrant Meissner phase, novel critical and vortex phenomena

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received 28 September 2006; accepted in final form 22 November 2006
published online 22 January 2007

PACS 74.78.Na – Mesoscopic and nanoscale systems
PACS 74.25.Op – Mixed state, critical fields, and surface sheaths

Abstract – Within the Ginzburg-Landau formalism, we study the mixed state of a superconducting disk surrounded by a magnetic ring. The stray field of the magnet, concentrated at the rim of the superconducting disk, favors ring-like arrangement of induced vortices, to the point that even a *single vortex state exhibits asymmetry.* A novel route for the destruction of superconductivity with increasing magnetization of the magnetic coating is found: first all vortices leave the sample, and are replaced by a *re-entered Meissner phase* with a full depression of the order-parameter at the sample edge; subsequently, superconductivity is then gradually suppressed from the edge inwards, *contrary to the well-known surface superconductivity.* When exposed to an additional homogeneous magnetic field, we find a *field-polarity–dependent vortex structure* in our sample —for all vorticities, only giant- *or* multi-vortex states are found for given polarity of the external field. In large samples, the *number of vortex shells and number of flux quanta in each of them can be controlled* by the parameters of the magnetic coating.

Novel vortex phenomena in a superconducting disk with magnetic coating

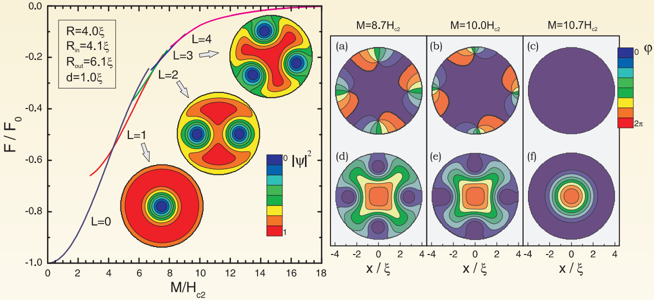

Fig. 2: The free energy of the states with different vorticity L as a function of the magnetization of the magnetic coating. Insets show the Cooper-pair density contourplots of the corresponding states. (a-c) Superconducting phase and (d-f) $|\psi|^2$-density plots, illustrate simultaneous vortex exit and suppression of superconductivity at the rim of the superconducting disk for high magnetization.

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In our theoretical treatment of this system, we use the non-linear Ginzburg-Landau (GL) formalism, combined with Neumann boundary conditions (zero current perpetrating the boundary). To investigate the superconducting state of a sample with volume V, we minimize, with respect to the order parameter ψ, the GL free energy

$$\mathcal{F} = \int \frac{\mathrm{d}v}{V} \left(|(-i\vec{\nabla} - \vec{A}_H - \vec{A}_m)\psi|^2 - |\psi|^2 + \frac{1}{2}|\psi|^4 \right), \tag{2}$$

Minimization of eq. (2) leads to equations for the order parameter and superconducting current

$$(-i\vec{\nabla} - \vec{A})^2\psi = (1 - |\psi|^2)\psi, \tag{3}$$

$$\vec{j} = \Im(\psi^*\vec{\nabla}\psi) - |\psi|^2\vec{A}, \tag{4}$$

which we solve following a numerical approach proposed by Schweigert *et al.* (see ref. [2]) on a uniform Cartesian grid with typically 10 points/ξ in each direction. We then start from randomly generated initial distribution of ψ, increase/decrease the magnetization of the magnet or change the value of the applied external field, and let eq. (3) relax to its steady-state solution. In addition, we always recalculate the vortex structure starting from the pure Meissner state[1] ($\psi = 1$) or the normal state ($\psi \approx 0$) as initial condition. All stable states are then collected and their energies are compared to find the ground state configuration.

M. V. Milošević *et al.*

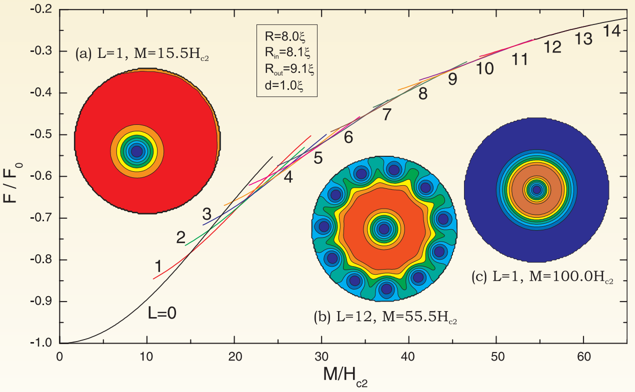

Fig. 3: Free energy diagram for a large superconducting disk with thin magnetic coating. Insets show the $|\psi|^2$-density plots of distinct vortex states.

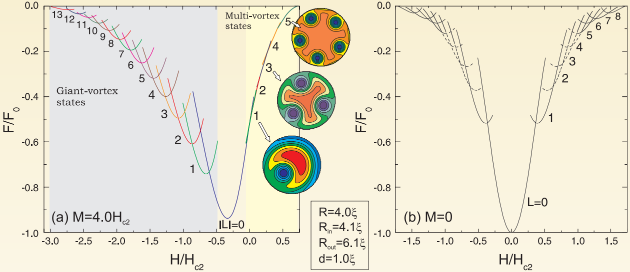

Fig. 4: (a) Free energy of a superconducting disk with magnetic coating as a function of applied homogeneous magnetic field. Insets show the Cooper-pair density plots for indicated states. (b) Same as (a), but for demagnetized coating. In (b), dashed lines denote multi-vortex and solid lines giant-vortex configurations.

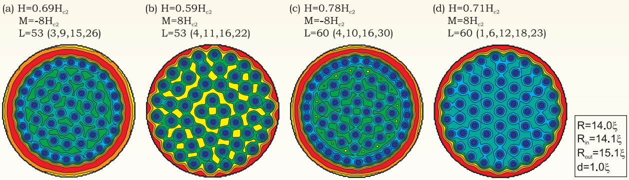

Fig. 5: The $|\psi|^2$-density plots illustrating the arrangement of vortex shells in a large superconducting disk for $L = 53$ and $L = 60$, with magnetic coating with (a,c) negative ($M = -8H_{c2}$), or (b,d) positive ($M = 8H_{c2}$) magnetization.

GEOMETRICAL DESCRIPTION OF VORTICES IN GINZBURG-LANDAU BILLIARDS

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4.2 The Bogomol'nyi identities

For the special value $\kappa = \frac{1}{\sqrt{2}}$, the equations for ψ and $\vec{A}$ can be reduced to first order differential equations. This special point was first used by Sarma [41] in his discussion of type-I *vs.* type-II superconductors and then identified by Bogomol'nyi [40] in the more general context of stability and integrability of classical solutions of some quantum field theories. This special point is also called a duality point. We first review some properties of the Ginzburg-Landau free energy at the duality point. We use the following identity true for two dimensional systems

$$|(\vec{\nabla} - i\vec{A})\psi|^2 = |\mathcal{D}\psi|^2 + \vec{\nabla} \times \vec{\jmath} + B|\psi|^2 \tag{64}$$

where $\vec{\jmath} = \mathrm{Im}(\psi^* \vec{\nabla}\psi) - |\psi|^2 \vec{A}$ is the current density and the operator $\mathcal{D}$ is defined as $\mathcal{D} = \partial_x + i\partial_y - i(A_x + iA_y)$. This relation is a relative of the Weitzenböck formula (61). At the duality point $\kappa = \frac{1}{\sqrt{2}}$ the expression (63) for $\mathcal{F}$ can be rewritten using (64) as

$$\mathcal{F} = \int_\Omega \left(\frac{1}{2}|B - 1 + |\psi|^2|^2 + |\mathcal{D}\psi|^2 \right) + \oint_{\partial\Omega} (\vec{\jmath} + \vec{A}).\vec{\mathrm{d}l} \tag{65}$$

- Ginzburg/Landau: replace linear eqs. in infinite dimensions by nonlinear in finite dimensions
- Geim et al: can confine system to bounded domain
- topological behaviour of solutions matter

Can we study related questions, theoretically and/or numerically, in fractal snowflake domains ?

Some recent related math papers (vector analysis):

- M. Hinz, M. Röckner, A. Teplyaev, *Vector analysis for local Dirichlet forms and quasilinear PDE and SPDE on fractals*, Stoch. Proc. Appl. 123 (2013), 4373–4406.
- M. Hinz, A. Teplyaev, *Dirac and magnetic Schrödinger operators on fractals*, J. Funct. Anal. 265 (2013), 2830–2854.
- M. Hinz, A. Teplyaev, *Local Dirichlet forms, Hodge theory, and the Navier-Stokes equations on topologically one-dimensional fractals*, Trans. Amer. Math. Soc. 367 (2015), 1347–1380.
- M. Hinz, L. Rogers, *Magnetic fields on resistance spaces*, to appear in J. Fractal Geometry (2015+).
- M. Hinz, *Magnetic energies and Feynman-Kac-Itô formulas for symmetric Markov processes*, Stoch. Anal. Appl. 33 (6) (2015), 1020-1049.

- M. Hinz, *Sup-norm-closable bilinear forms and Lagrangians*, to appear in Ann. Mat. Pura Appl. (2015+).

Review paper:

- M. Hinz, A. Teplyaev, *Finite energy coordinates and vector analysis on fractals*, Progress in Probab. 70, Fractal Geometry and Stochastics V, Springer, 2015, pp. 209-227.

Related Physics papers:

- E. Akkermans, G. Dunne, A. T. *Physical Consequences of Complex Dimensions of Fractals*, Europhys. Lett. **88**, 40007 (2009).
- E. Akkermans, G. Dunne, A. T. *Thermodynamics of photons on fractals*, Phys. Rev. Lett. **105**(23):230407, 2010.

Language: Dirichlet forms

X loc. compact sep. metric space, m finite Radon measure on X, dense support, $(\mathcal{E}, \mathcal{F})$ be a regular symmetric Dirichlet form on $L_2(X, m)$ with core $\mathcal{C} := \mathcal{F} \cap C_0(X)$.

Endowed with the norm $\|f\|_{\mathcal{F}_b} := \mathcal{E}(f)^{1/2} + \sup_X |f|$ the space $\mathcal{C}$ is algebra

$$\mathcal{E}(fg)^{1/2} \leq \|f\|_{\mathcal{F}_b} \|g\|_{\mathcal{F}_b}, \quad f, g \in \mathcal{C}.$$

Can define a first order derivation ∂ from algebra $\mathcal{C}$ into a Hilbert space $\mathcal{H}$ of L^2-vector fields (or 1-forms) (Cipriani/Sauvageot '03 and many related earlier papers)

Examples:

(i) *Dirichlet forms on Euclidean domains*. Let $X = \Omega$ be a domain in $\mathbb{R}^n$ and $\mathcal{E}(f,g) = \int_\Omega \nabla f \nabla g \, dx, \quad f,g \in H^1_0(\Omega)$.
$(\mathcal{E}, H^1_0(\Omega))$ is a local regular Dirichlet form on $L_2(\Omega)$,
$\mathcal{H} = L^2(\Omega, \mathbb{R}^n)$ and $\partial f = \nabla f$ in L^2-sense.

(ii) *Dirichlet forms on Riemannian manifolds*. Let $X = M$ be a compact Riemannian manifold and

$$\mathcal{E}(f,g) = \int_M \langle \nabla f, \nabla g \rangle_{TM} \, dvol, \quad f,g \in H^1(M)$$

where *dvol* is the Riemannian volume measure. Then
$\mathcal{H} = L^2(M, TM, dvol)$ and $\partial f = \nabla f$ is usual gradient in L^2-sense.

(iii) *Dirichlet forms induced by resistance forms on self-similar p.c.f. fractals* (gradients with respect to Kusuoka-Kigami measures).

This language allows study of equations involving vector terms:

$$\operatorname{div}(a(\nabla u)) = f$$

$$\Delta u + b(\nabla u) = f$$

$$i\frac{\partial u}{\partial t} = (-i\nabla - A)^2 u + Vu.$$

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u - \Delta u + \nabla p = 0, \\ \operatorname{div} u = 0, \end{cases} \tag{1}$$

Some examples of results:

$$\mathcal{E}^{a,V}(f,g) = \langle(-i\partial - a)f, (-i\partial - a)g\rangle_{\mathcal{H}} + \langle fV, g\rangle_{L_2(X,m)}, \quad f,g \in \mathcal{F}_{\mathbb{C}},$$

Theorem (H./T. '13)

Let $a \in \mathcal{H}_\infty$ and $V \in L_\infty(X,m)$.

(i) *The quadratic form $(\mathcal{E}^{a,V}, \mathcal{F}_{\mathbb{C}})$ is closed.*

(ii) *The self-adjoint non-negative definite operator on $L_{2,\mathbb{C}}(X,m)$ uniquely associated with $(\mathcal{E}^{a,V}, \mathcal{F}_{\mathbb{C}})$ is given by*

$$H^{a,V} = (-i\partial - a)^*(-i\partial - a) + V.$$

Related Dirac operator is well defined and self-adjoint

$$D = \begin{pmatrix} 0 & -i\partial^* \\ -i\partial & 0 \end{pmatrix}$$

(Feynman-Kac-Itô formula)

Theorem (H.'15)

Suppose $a \in \mathcal{H}_a$ is real valued and v is a real valued Borel function. For $t > 0$ and bounded Borel f set

$$P_t^{a,v} f(x) := \mathbb{E}_x[e^{i\int_{Y([0,t])} a - \int_0^t v(Y_s)ds} f(Y_t)], \quad x \in X$$

where $\int_{Y([0,t])} a$ is a properly defined Stratonovich line integral.

(Hodge theorem and Navier-Stokes equations in topo dimension 1)

Theorem (H./T. '15)

If the space X is compact, connected and topologically 1-dimensional of arbitrarily large Hausdorff and spectral dimensions, then a 1-form $\omega \in \mathcal{H}$ is harmonic if and only if it is in $(Im\,\partial)^{\perp}$: $\operatorname{div} \omega = 0$.

By the classical identity $\frac{1}{2}\nabla|u|^2 = (u \cdot \nabla)u + u \times \operatorname{curl} u$ equation (1) becomes $\frac{\partial u}{\partial t} + \frac{1}{2}\partial\Gamma_{\mathcal{H}}(u) - \Delta_1 u + \partial p = 0, \ \ \operatorname{div} u = \partial^* u = 0$. Therefore any weak solution u of (1) is unique, harmonic and stationary (i.e. $u_t = u_0$ is independent of $t \in [0, \infty)$) for any divergence free initial condition u_0.

Goals

- General goal:
  Linear and non-linear equations on rough domains and fractals involving
  - Gradient terms
  - Magnetic potentials
- Here in this talk:
  Domain with fractal snowflake boundary
  - Tangential derivative along boundary and *pointwise interpretation*
  - Model problem: Parabolic Ventsell problem with drifts in interior domain and on fractal boundary
  - Lipschitz and C^1-extensions of functions on the boundary *(preparation for Lipschitz coordinates and studies of magnetic fields)*

Ventsell problem

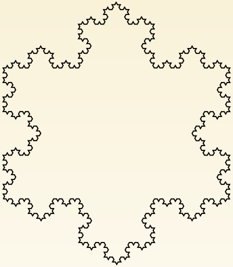


Figure: Fractal (closed) snowflake domain Ω.

- Consider a 'mixed'/interacting diffusion in the interior and on the boundary

- Central tool: bilinear form

$$\mathcal{E}_{\mathcal{A}}(f,f) = \int_{\Omega} (A(x)\cdot\nabla f(x))\cdot\nabla f(x)\mathcal{L}^2(dx) + c_0\mathcal{E}_{\partial\Omega}(f,f)$$

- $\mathcal{A}$ bounded measurable uniformly elliptic coefficient matrix,
  $\mathcal{L}^2$ two-dim. Lebesgue,
  $c_0 > 0$ fixed constant,
  $\mathcal{E}_{\partial\Omega}$ Kusuoka-Kigami Dirichlet form on snowflake boundary $\partial\Omega$
- $\mathcal{E}_{\mathcal{A}}$, equipped with suitable domain, becomes Dirichlet form

Consider the parabolic Ventsell problem

$$u_t(t,x) - L_{\mathcal{A}} u(t,x) - \vec{b}(x) \cdot \nabla u(t,x) = f(t,x)$$

in $(0,T] \times \Omega$

$$u_t(t,x) - c_0 \Delta_{\partial\Omega} u(t,x) - b_{\partial\Omega}(x) D_{\partial\Omega} u(t,x) + c(x) u(t,x) = -\frac{\partial u(t,x)}{\partial n_{\mathcal{A}}} + f(t,x)$$

in $(0,T] \times \partial\Omega$

$$u(0,x) = u_0(x)$$

in Ω.

(generalizes a problem of Lancia/Vernole '14)

Here

- $L_{\mathcal{A}}u(t,\cdot) = \text{div}(\mathcal{A}\cdot\nabla u(t,\cdot))$
- $\Delta_{\partial\Omega}$ Kusuoka-Kigami Laplacian on $\partial\Omega$
- c stationary scalar potential on $\partial\Omega$, u_0 given initial condition
- $\frac{\partial u(t,\cdot)}{\partial n_{\mathcal{A}}}$ is the co-normal derivative of $u(t,\cdot)$ across $\partial\Omega$
- $\vec{b}$ stationary drift vector field in Ω, $b_{\partial\Omega}$ is a drift vector field on $\partial\Omega$ (not necessarily related)
- $D_{\partial\Omega}u(t,\cdot)$ 'tangential derivative' of $u(t,\cdot)$ along $\partial\Omega$

Energy and tangential gradients

- Discrete energy forms on Koch curve:

$$\mathcal{E}_K^{(n)}(u) := \frac{1}{2}\, 4^n \sum_{p \in V_n(K)} \sum_{q \sim_n p} (u(p) - u(q))^2$$

- V_n 'dyadic points of level n' (images of boundary points $V_0 = \{0, 1\}$ under similarity maps)
- non-decreasing in n with non-trivial limit $\mathcal{E}_K(u) := \lim_{n\to\infty} \mathcal{E}_K^{(n)}(u)$
- obtain regular resistance form $(\mathcal{E}_K, \mathcal{D}(\mathcal{E}_K))$ in the sense of Kigami
- $\mathcal{D}(\mathcal{E}_K) \subset C(K)$
- resistance metric

$$d_R(p,q) := \sup\left\{ \frac{|u(p) - u(q)|^2}{\mathcal{E}_K(u)} : \quad u \in \mathcal{D}(\mathcal{E}_K),\ \mathcal{E}_K(u) > 0 \right\}$$

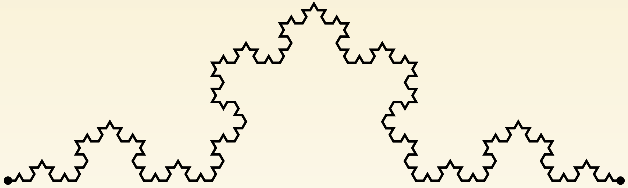


Figure: Koch curve with boundary $V_0 = \{0, 1\}$

- h unique harmonic function on K with boundary values $h(0) = 0$ and $h(1) = 1$ (coordinate function)
- $h : K \to [0, 1]$ homeomorphism onto $[0, 1]$
- with $\delta = \dim_H K = \frac{\ln 4}{\ln 3}$ and quasi-distance

$$d_\delta(x, y) := |x - y|^\delta,$$

K is variational fractal in the sense of Mosco '97, and for any $p, q \in V_n(K)$ with $p \sim_n q$, have

$$|h(p) - h(q)| = d_\delta(p, q) = |p - q|^\delta$$

- energy measure μ_K of h,

$$\mathcal{E}_K(h, uh) - \frac{1}{2}\mathcal{E}_K(u, h^2) = \int_K u \, d\mu_K,$$

  equals δ-dim. Hausdorff measure (up to constant)
- subspace

$$\mathcal{S}_K := \left\{ F \circ h : F \in C^1(\mathbb{R}) \right\}$$

  is dense in $\mathcal{D}(\mathcal{E}_K)$
- $\mathcal{E}_K(u) = \int_K F'(h(x))^2 \mu_K(dx)$ for any $u = F \circ h$ from $\mathcal{S}_K$
- $(\mathcal{E}_K, \mathcal{D}(\mathcal{E}_K))$ is a strongly local regular Dirichlet form on $L^2(K, \mu_K)$ (Kigami-Kusuoka energy)

- Koch snowflake $\partial\Omega$ = union of three congruent copies K_1, K_2 and K_3 resp. K_4, K_5 and K_6 of K

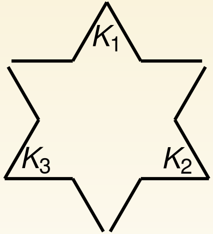


Figure: Using the copies K_1, K_2 and K_3.

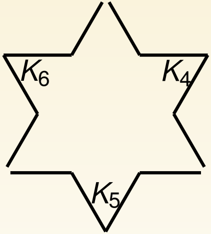


Figure: Using the copies K_4, K_5 and K_6.

- φ_i Euclidean motion that maps K_i into K
- $\mathcal{D}(\mathcal{E}_{\partial\Omega}) := \left\{ u : \partial\Omega \to \mathbb{R} : u|_{K_i} \circ \varphi_i^{-1} \in \mathcal{D}(\mathcal{E}_K),\ i = 1, ..., 6 \right\}$
- $\mathcal{E}_{\partial\Omega}(u) := \mathcal{E}_{K_1}(u|_{K_1}) + \mathcal{E}_{K_2}(u|_{K_2}) + \mathcal{E}_{K_3}(u|_{K_3}),\ \ u \in \mathcal{D}(\mathcal{E}_{\partial\Omega})$.
- identify $u|_{K_i}$ with $u|_{K_i} \circ \varphi^{-1}$, identify μ_{K_i} with image measures of μ_K and so on

- equip $\partial\Omega$ with measure $\mu := \mu_{K_1} + \mu_{K_2} + \mu_{K_3}$
- $(\mathcal{E}_{\partial\Omega}, \mathcal{D}(\mathcal{E}_{\partial\Omega}))$ strongly local Dirichlet form on $L^2(\partial\Omega, \mu)$ ... Kusuoka-Kigami energy form on $\partial\Omega$ (see Freiberg/Lancia '04)
- assoc. Laplacian in variational sense is

$$\langle \Delta_{\partial\Omega} f, g \rangle_{((\mathcal{D}(\mathcal{E}_{\partial\Omega}))', \mathcal{D}(\mathcal{E}_{\partial\Omega}))} = -\mathcal{E}_{\partial\Omega}(f, g),\ \ f, g \in \mathcal{D}(\mathcal{E}_{\partial\Omega})$$

Set

$$\mathcal{S}_{\partial\Omega} := \left\{ u : \partial\Omega \to \mathbb{R} : u|_{K_i} = F_i \circ h_i \in \mathcal{S}_{K_i}, i = 1, ..., 6 \right\}$$

with $\mathcal{S}_{K_i}$ similarly as before.

Lemma

The space $\mathcal{S}_{\partial\Omega}$ is dense in $\mathcal{D}(\mathcal{E}_{\partial\Omega})$.

Write $V_*(\partial\Omega) := \bigcup_{i=1}^{6} V_*(K_i)$, where for any K_i the set $V_*(K_i)$ is 'set of all dyadic points' (images of bd. points).

Lemma

Let $u \in \mathcal{S}_{\partial\Omega}$. Then for any $p \in V_*(\partial\Omega)$ the limit

$$D_{\partial\Omega}u(p) := \lim_{V_*(\partial\Omega)\ni q\to p} \frac{u(p)-u(q)}{|p-q|^{\delta}},$$

exists uniformly in $p \in V_*(\partial\Omega)$. If $p \in K_i$ and $u|_{K_i} = F_i \circ h_i$, then $D_{\partial\Omega}u(p) = F_i'(h_i(p))$.
Moreover, if $p \in K_i \cap K_j \cap V_*(\partial\Omega)$ and $u|_{K_j} = F_j \circ h_j$ then $F_i'(h_i(p)) = F_j'(h_j(p))$.

(concept of limit goes back to Mosco)

Pointwise definition of tangential gradient of composed functions:

Definition

Given $x \in \mathring{K}_i$ and $u \in \mathcal{S}_{\partial\Omega}$ such that $u|_{K_i} = F_i \circ h_i$, we set

$$D_{\partial\Omega} u(x) := F_i'(h_i(x)). \tag{2}$$

We refer to $D_{\partial\Omega} u(x)$ as the *tangential gradient* of u along $\partial\Omega$ in $x \in \partial\Omega$.

Pointwise definition ('new') useful for numerical schemes.
Connections/applications to research of Rozanova-Pierrat (acoustics, heat content) and Sapoval.

Corollary

The definition (2) is independent of the particular choice of h_i and F_i.

Interpreted as a linear operator:

Corollary

The tangential gradient $D_{\partial\Omega}$ defines a bounded linear operator $D_{\partial\Omega} : \mathcal{S}_{\partial\Omega} \to L^2(\partial\Omega, \mu)$, more precisely, we have $\|D_{\partial\Omega} u\|^2_{L^2(\partial\Omega,\mu)} = \mathcal{E}_{\partial\Omega}(u)$ for all $u \in \mathcal{S}_{\partial\Omega}$.

The closedness of $(\mathcal{E}_{\partial\Omega}, \mathcal{D}(\mathcal{E}_{\partial\Omega}))$ implies:

Proposition

$D_{\partial\Omega}$ extends to a closed unbounded operator $D_{\partial\Omega} : L^2(\partial\Omega, \mu) \to L^2(\partial\Omega, \mu)$ with domain $\mathcal{D}(\mathcal{E}_{\partial\Omega})$. Moreover, for any $u \in \mathcal{D}(\mathcal{E}_{\partial\Omega})$ we have $\mathcal{E}_{\partial\Omega}(u) = \|D_{\partial\Omega} u\|^2_{L^2(\partial\Omega,\mu)}$ and the energy measure ν_u of u is absolutely continuous with respect to μ with density $\Gamma_{\partial\Omega}(u) = (D_{\partial\Omega} u)^2$ μ-a.e.

- $D_{\partial\Omega}$ version of first order derivation ∂ (abstract gradient) in the sense of Cipriani/Sauvageot '03
- special situation here (on curve): gradients themselves are scalar valued L^2-functions

Cauchy problem and interpretation

- consider the function space
  $V(\Omega,\partial\Omega) := \{u \in H^1(\Omega) : u|_{\partial\Omega} \in \mathcal{D}(\mathcal{E}_{\partial\Omega})\}$ equipped with

  $$\langle u,v\rangle_{V(\Omega,\partial\Omega)} = \langle u,v\rangle_{H^1(\Omega)} + \mathcal{E}_{\partial\Omega}(u|_{\partial\Omega}, v|_{\partial\Omega}) + \langle u|_{\partial\Omega}, v|_{\partial\Omega}\rangle_{L^2(\partial\Omega,\mu)}$$

  Hilbert.
- consider $m := \mathcal{L}^2|_\Omega + \mu$ measure on $\overline{\Omega}$
- then

  $$\mathcal{E}_0(u,v) := \int_\Omega \nabla u \cdot \nabla v \, d\mathcal{L}^2 + c_0 \mathcal{E}_{\partial\Omega}(u|_{\partial\Omega}, v|_{\partial\Omega})$$

  defines Dirichlet form $(\mathcal{E}_0, V(\Omega,\partial\Omega))$ on $L^2(\overline{\Omega}, m)$ (Lancia '02/'03, Lancia/Vernole '14)

Assume we are given the following data:

(i) $\mathcal{A} = (a_{ij})_{i,j=1}^2$, $a_{ji} = a_{ij}$ bounded,

$$\sum_{i,j=1}^{2} a_{ij}\xi_i\xi_j \geq \lambda \sum_{i=1}^{2} |\xi_i|^2, \quad \xi = (\xi_1, \xi_2) \in \Omega$$

(ii) a vector field $\vec{b} = (b_1, b_2) \in L^2(\Omega, \mathbb{R}^2)$ such that

$$\int_\Omega u^2 (\vec{b}\cdot\vec{b})\, d\mathcal{L}^2 \leq \gamma_1 \int_\Omega (\nabla u)^2 d\mathcal{L}^2 + \gamma_2 \|u\|^2_{L^2(\Omega)}, \quad u \in V(\Omega, \partial\Omega),$$

with positive constants γ_1 and γ_2 such that $\sqrt{2\gamma_1} < \lambda$,

(iii) a 'boundary vector field' $b_{\partial\Omega} \in L^2(\partial\Omega, \mu)$ such that

$$\int_{\partial\Omega} u^2 b_{\partial\Omega}^2\, d\mu \leq \delta_1 \mathcal{E}_{\partial\Omega}(u|_{\partial\Omega}) + \delta_2 \|u\|^2_{L^2(\partial\Omega,\mu)}, \quad u \in V(\Omega, \partial\Omega),$$

with positive constants δ_1 and δ_2 such that $\sqrt{2\delta_1} < c_0$,

(iv) a continuous function c on $\partial\Omega$.

Consider

$$\mathcal{E}(u,v) = \int_\Omega (\mathcal{A}(x)\cdot\nabla u(x))\cdot\nabla v(x)\mathcal{L}^2(dx) - \int_\Omega (\vec{b}(x)\cdot\nabla u(x))v(x)\mathcal{L}^2(dx)$$

$$+c_0\mathcal{E}_{\partial\Omega}(u,v) - \int_{\partial\Omega} b_{\partial\Omega}(x)D_{\partial\Omega}u(x)v(x)\mu(dx) + \int_{\partial\Omega} c(x)u(x)v(x)m(dx),$$

$u, v \in V(\Omega, \partial\Omega)$.

Given $\alpha > 0$ write

$$\mathcal{E}_\alpha(u,v) := \mathcal{E}(u,v) + \alpha\left\langle u, v\right\rangle_{L^2(\overline{\Omega},m)}.$$

Proposition

Let $\mathcal{A}$, $\vec{b}$, $b_{\partial\Omega}$ and c be as above. Then $(\mathcal{E}, V(\Omega, \partial\Omega))$ is a closed coercive form in the wide sense, i.e. there is some $\alpha > 0$ such that $u \mapsto \mathcal{E}_\alpha(u, u)$ defines a positive definite closed quadratic form on $L^2(\overline{\Omega}, m)$ with domain $V(\Omega, \partial\Omega)$ and

$$|\mathcal{E}_{\alpha+1}(u, v)| \leq K\, \mathcal{E}_{\alpha+1}(u)^{1/2} \mathcal{E}_{\alpha+1}(v)^{1/2}, \quad u, v \in V(\Omega, \partial\Omega),$$

with a universal constant $K > 0$.

- $(\mathcal{E}, V(\Omega, \partial\Omega))$ generates an analytic semigroup $(T_t)_{t\geq 0}$ on $L^2(\overline{\Omega}, m)$ with generator $(A, \mathcal{D}(A))$

Given $f : [0, T] \to L^2(\overline{\Omega}, m)$ and $u_0 \in L^2(\overline{\Omega}, m)$, consider abstract Cauchy problem

$$\begin{cases} \frac{du(t)}{dt} = Au(t) + f(t), \quad 0 < t \leq T, \\ u(0) = u_0 \end{cases} \tag{3}$$

in $L^2(\overline{\Omega}, m)$.

$u \in C([0,T], L^2(\overline{\Omega}, m))$ *classical solution* if
$u \in C^1((0,T], L^2(\overline{\Omega}, m)) \cap C((0,T], \mathcal{D}(A))$ and u satisfies (3).

Theorem

Suppose $0 < \theta < 1$ *and* $f \in C^\theta((0,T], L^2(\overline{\Omega}, m))$ *and that assumptions above are satisfied. Then (3) has a unique classical solution u, given by*

$$u(t) = T_t u_0 + \int_0^t T_{t-s} f(s) ds, \quad 0 \leq t \leq T.$$

Interpretation of abstract Cauchy solution:

- consider
$$V(\Omega) = \left\{ g \in H^1(\Omega) : L_{\mathcal{A}} g \in L^2(\Omega) \right\},$$
where $L_{\mathcal{A}} g = \mathrm{div}(\mathcal{A} \nabla g)$, with norm
$$\|g\|_{V(\Omega)} := \|L_{\mathcal{A}} g\|_{L^2(\Omega)} + \|\nabla g\|_{L^2(\Omega,\mathbb{R}^2)} + \|g\|_{L^2(\Omega)} .$$
- similarly as in Lancia '02 and Lancia/Vernole '06, for any $g \in V(\Omega)$ we can define a distribution $l_g \in (H^1(\Omega))'$ by
$$l_g(v) := \int_\Omega (\mathcal{A} \cdot \nabla g) \cdot \nabla v \, d\mathcal{L}^2 + \int_\Omega v L_{\mathcal{A}} g \, d\mathcal{L}^2, \quad v \in H^1(\Omega).$$

- can view each l_g as distribution $\frac{\partial g}{\partial n_{\mathcal{A}}}$ in $(B^{2,2}_{\delta/2}(\partial\Omega))'$ and write

$$\left\langle \frac{\partial g}{\partial n_{\mathcal{A}}}, v|_{\partial\Omega} \right\rangle_{(B^{2,2}_{\delta/2}(\partial\Omega))', B^{2,2}_{\delta/2}(\partial\Omega))} = l_g(v), \quad v \in H^1(\Omega)$$

  (dual pairing)
- can show classical solution u is in $C((0,T], V(\Omega))$
- by embedding of $\mathcal{D}(\mathcal{E}_{\partial\Omega})$ into $B^{2,2}_{\delta/2}(\partial\Omega)$, may view $\frac{\partial u}{\partial n_{\mathcal{A}}}(t)$ as an element of $(\mathcal{D}(\mathcal{E}_{\partial\Omega}))'$

Arrive at rigorous version of parabolic Ventsell problem:

$$u_t(u) - L_{\mathcal{A}} u(t) - \vec{b} \cdot \nabla u(t) = f(t)$$

in $L^2(\Omega)$, $t \in (0, T]$,

$$u_t(t) - c_0 \Delta_{\partial\Omega} u(t)|_{\partial\Omega} - b_{\partial\Omega} D_{\partial\Omega} u(t)|_{\partial\Omega} + cu(t) = -\frac{\partial u}{\partial n_{\mathcal{A}}}(t) + f(t)$$

in $(\mathcal{D}(\mathcal{E}_{\partial\Omega}))'$, $t \in (0, T]$,

$$u(0) = u_0$$

in $L^2(\overline{\Omega}, m)$.

A core of Lipschitz functions

- Difficulty for vector analysis on $\overline{\Omega}$: Ω no Lipschitz domain
- Idea here: consider intrinsic Lipschitz functions on $\partial\Omega$:

$$d_{\mathcal{E}_{\partial\Omega}}(x,y) := \sup\left\{u(x)-u(y) : u \in \mathcal{D}(\mathcal{E}_{\partial\Omega}) \text{ with } \Gamma_{\partial\Omega}(u) \leq 1 \ \mu\text{-a.e.}\right\},$$

$x, y \in \partial\Omega$.
Here $\Gamma_{\partial\Omega}(u) = (D_{\partial\Omega}u)^2$ energy density of $u \in \mathcal{D}(\mathcal{E}_{\partial\Omega})$.

Locally, the intrinsic metric on $\partial\Omega$ can be expressed via coordinates.

Corollary

For any i and any $x, y \in \mathring{K}_i$ we have $d_{\mathcal{E}_{\partial\Omega}}(x,y) = |h_i(x) - h_i(y)|$.

Note that also $d_R(x,y) = d_{\mathcal{E}_{\partial\Omega}}(x,y)$, $x, y \in \mathring{K}_i$.

The corollary allows for the following extension principle.

Theorem

Any $\mathcal{E}_{\partial\Omega}$-intrinsically Lipschitz function on $\partial\Omega$ has an Euclidean-Lipschitz extension to $\overline{\Omega}$.

Recall the notation

$$\mathcal{E}_0(u,v) := \int_\Omega \nabla u \cdot \nabla v \, d\mathcal{L}^2 + c_0 \mathcal{E}_{\partial\Omega}(u|_{\partial\Omega}, v|_{\partial\Omega}), \quad u, v \in V(\Omega, \partial\Omega),$$

$u, v \in V(\Omega, \partial\Omega)$.

Corollary

(i) *The Dirichlet form $\mathcal{E}_0$ has a core consisting of functions in $C(\overline{\Omega})$ whose restriction to Ω is in $C^1(\Omega)$, and whose restriction to $\partial\Omega$ is a $\mathcal{E}_{\partial\Omega}$-intrinsic C^1-function. For a core function u the gradients ∇u and $D_{\partial\Omega} u$ are well defined pointwise in Ω and on $\partial\Omega$, respectively.*

(ii) *There are two coordinate functions y_1, y_2 which are contained in the core and separate the points of $\overline{\Omega}$.*

- For a core function u have $D_{\partial\Omega} u$ continuous on $\partial\Omega$ and ∇u continuous on Ω, and usual calculus rules.
- ∇u not necessarily continuous on $\overline{\Omega}$, a typical function in Euclidean $C^1(\overline{\Omega})$ is not in the domain of $\mathcal{E}$.
- Can be used to obtain a coordinate representation for the gradients.

Have two 'constructios / proofs' of the Theorem, suitable to create numerical schemes. Assume that Ω is embedded in $\mathbb{C}$ and inscribed in the circle of radius $\sqrt{3}$

- the six most outward points of $\overline{\Omega}$ are $\sqrt{3}e^{ik\pi/3+i\pi/6}$, $k = 0, 1, ..., 5$;
- the inscribed circle is the unit circle
- the six most inward points of $\partial\Omega$ are the 6th roots of unity $e^{ik\pi/3}$, $k = 0, 1, ..., 5$
- most left and right most inward points of $\partial\Omega$ are $-1, 1 \in \mathbb{C}$ respectively

Slight change of notation: now points p and q from $V_{n+1}(\partial\Omega)$ that are neighbors, $p \sim_{n+1} q$, have Euclidean distance 3^{-n}

Assume f be an $\mathcal{E}_{\partial\Omega}$-intrinsically Lipschitz on $\partial\Omega$, like to construct a Euclidean-Lipschitz extension g of f to $\overline{\Omega}$,.

Use natural "approximate" triangulations of Ω.

First method is reminiscent of Lapidus/Pearse '06 and Evans '12, '12 and uses weakly self-similar triangulation, *which is nicely separated from the boundary*.

Consider lattice $L_n = 3^{-n}\mathbb{Z}\{e^{ik\pi/3} | k = 0, 1, ..., 5\}$. For each $x \in L_n \cap \Omega$ there are finitely many points of $L_n \cap \partial\Omega = V_{n+1}(\partial\Omega)$ that closest to x On $L_n \cap \Omega$ we obtain a function g_n by defining $g_n(x)$ to be the average of f at these points.

Assume that points of each lattice L_n are the vertices of closed equilateral triangles $T_{n,m}$ of sides 3^{-n}, and so each lattice L_n defines a triangulation of $\mathbb{R}^2 = \cup_m T_{n,m}$ into such triangles $T_{n,m}$.

Define $\mathcal{T}_n := \bigcup_{m: T_{n,m} \subset \Omega} T_{n,m}$ as the union of triangles $T_{n,m}$ that lie inside Ω.

$\mathcal{T}_n$ compact set contained in Ω, and $\mathcal{T}_n$ is separated from $\partial\Omega$. Denote boundary of $\mathcal{T}_n$ by $\partial\mathcal{T}_n$.

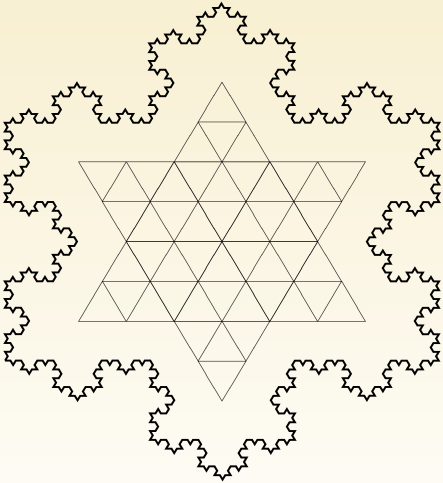


Figure: The set $\mathcal{T}_1 \subset \Omega$, triangulated by triangles of side length $\frac{1}{3}$.

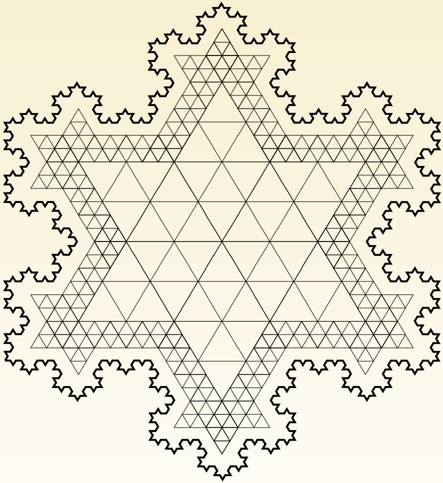


Figure: The set $\mathcal{T}_2 \subset \Omega$, with $\mathcal{T}_1 \subset \mathcal{T}_2$ triangulated by triangles of side length $\frac{1}{3}$ and $\mathcal{T}_2 \setminus \mathring{\mathcal{T}}_1$ triangulated by triangles of side length $\frac{1}{9}$.

From the functions g_n on $L_n \cap \Omega$ now inductively define g:

- on $\mathcal{T}_1$ define g as piece-wise linear interpolation of g_1 on the triangulation by triangles $T_{1,m} \subset \mathcal{T}_1$.
- extend g from $\mathcal{T}_1$ to $\mathcal{T}_2$ using piece-wise linear interpolation of g_2 on the triangles $T_{2,m}$ contained in $\mathcal{T}_2$ but not in $\mathcal{T}_1$.
- if g is already defined on $\mathcal{T}_{n-1}$, we extend it to $\mathcal{T}_n$ using piece-wise linear interpolation of g_n on the triangles $T_{n,m}$ contained in $\mathcal{T}_n$ but not in $\mathcal{T}_{n-1}$

To show g is Lipschitz use:

For any $p \in V_n(\partial\Omega)$ we have

$$L_{\mathcal{E}_{\partial\Omega}}(f) \geq \sup_{q \in V_n(\partial\Omega), q \sim_n p} \frac{|f(p) - f(q)|}{d_{\mathcal{E}_{\partial\Omega}}(p,q)} = \left(\frac{4}{3}\right)^n \sup_{q \in V_n(\partial\Omega), q \sim_n p} \frac{|f(p) - f(q)|}{|p - q|},$$

where $L_{\mathcal{E}_{\partial\Omega}}(f)$ denotes the $\mathcal{E}_{\partial\Omega}$-intrinsic Lipschitz constant of f.

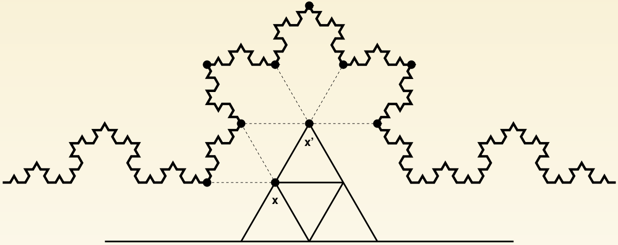


Figure: Chain of 8 consecutive points from $V_{n+1}(\partial\Omega)$ that contains all points considered when computing the values $g_n(x)$ and $g_n(x')$ for neighbours x and x' on the shell $\partial\mathcal{T}_n$.

Second method of construction / proof: weakly self-similar triangulation *which is not nicely separated from the boundary* $\partial\Omega$.

- define $g(0)$ to be the average of f at the 6th roots of unity $e^{ik\pi/3}$, $k = 0, 1, ..., 5$
- at each 6th root of unity $e^{ik\pi/3}$, $k = 0, 1, ..., 5$, define $g(e^{ik\pi/3}) = f(e^{ik\pi/3})$
- interpolate g linearly in each unit equilateral triangle with vertices $0, e^{ik\pi/3}, e^{i(k+1)\pi/3}$

This defines g in the regular closed convex unit hexagon $\overline{\Omega}_1$ which is the convex span of the 6th roots of unity.

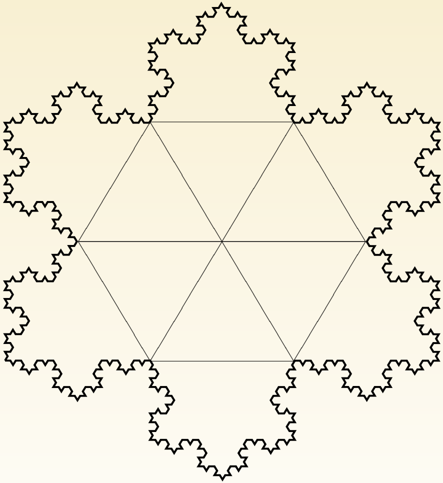


Figure: The triangulated hexagon $\overline{\Omega}_1$.

- $\Omega \setminus \overline{\Omega}_1$ consists of six disjoint isometric open sets
- closure of each of these contains a regular closed convex hexagon with sides $\frac{1}{3}$. On the joint facet with $\overline{\Omega}_1$ the function g is already defined (and on incident vertices)
- remaining four vertices of this hexagon lie on $\partial\Omega$, and so we define $g(x) = f(x)$ at these vertices
- define g at center of hexagon as the average at the vertices
- interpolate g linearly in each of these triangle

Have defined g in the closed set which is the union of the unit regular hexagon $\overline{\Omega}_1$ and the six adjacent closed hexagons with sides $\frac{1}{3}$, denote this union by $\overline{\Omega}_2$.

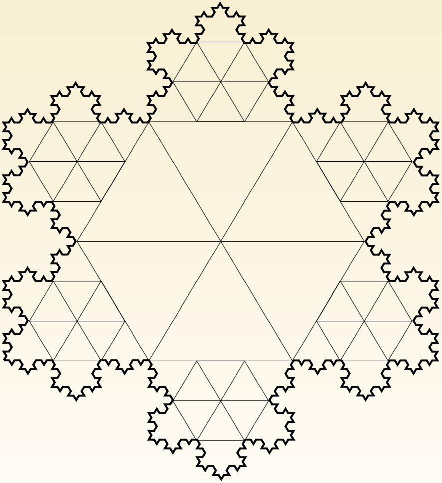


Figure: The triangulated set $\overline{\Omega}_2$.

- $\Omega \setminus \overline{\Omega_2}$ consists of 30 open components of two shapes:
- 18 components are $\frac{1}{3}$ in size in comparison to the components considered before
- 12 component are of the same scale, but have only one fractal side
- closure of each of these 30 components has an inscribed regular closed convex hexagon with sides $\frac{1}{9}$.

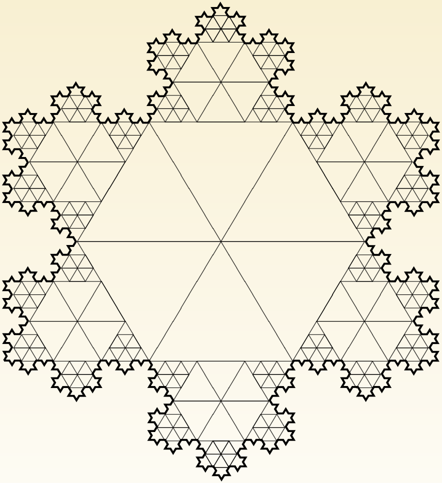


Figure: The triangulated set $\overline{\Omega}_3$.

Generalization

Assume $\Omega \subset \mathbb{R}^2$ domain with boundary $\partial\Omega$ homeo to circle.

Assume that locally $R = R(x, y)$ is a (geodesic) distance along the boundary, assume that locally

$$R(x, y) \leq c|x - y|.$$

(prevents near-intersections)

Assume that Dirichlet form $\mathcal{E}$ on interior Ω satisfies $\mathcal{E}(f) \leq c\,\|f\|_{\text{Lip}}$.

Consider resistance form $\mathcal{E}_{\partial\Omega}$ on $\partial\Omega$ associated with R.

Corollary

The form $\mathcal{E} + \mathcal{E}_{\partial\Omega}$ is closable on a Lipschitz core, closure is Dirichlet form on $\overline{\Omega}$ (with sum of measures).

Theorem

Any R-Lipschitz function on $\partial\Omega$ has an Euclidean-Lipschitz extension to $\overline{\Omega}$.

Do no longer need boundary to be d-set / self-similar (usual trace results omitted).

Can again define tangential derivative, now by

$$D_{\partial\Omega}f(x) := \lim_{y\to x}\frac{f(x)-f(y)}{R(x,y)}$$

(or with other sign, depending on orientation)

Example:

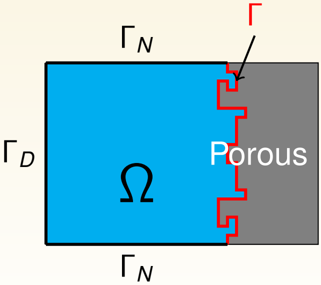
Assume $\partial\Omega$ has Hausdorff-dim. δ, then set

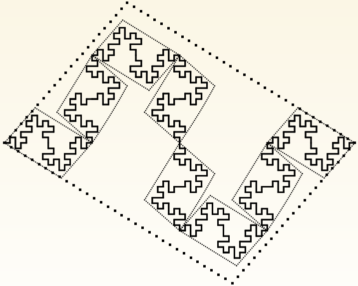
$$R(x,y) = H^{\delta}([x,y]),$$

where $[x,y]$ is the piece of $\partial\Omega$ joining two points x and y and H^{δ} δ-dim. Hausdorff measure on $\partial\Omega$ (parametrization by Hausdorff measure).

Research in progress

- **Mixed boundary valued problem for linear and nonlinear wave equations in domains with fractal boundaries**
  Adrien Dekkers, Anna Rozanova-Pierrat, A.T.
- **Fractal shape optimization in linear acoustics**,
  Michael Hinz, Anna Rozanova-Pierrat, and A.T.





DISCRETIZATION OF THE KOCH SNOWFLAKE DOMAIN WITH BOUNDARY AND INTERIOR ENERGIES

MALCOLM GABBARD, CARLOS LIMA, GAMAL MOGRABY, LUKE G. ROGERS, AND ALEXANDER TEPLYAEV

Abstract. We show how to discretize the energy form on the Koch snowflake domain so that both the fractal boundary and the Euclidean interior can support positive energy. We compute eigenvalues and eigenfunctions, and demonstrate the high energy localization of the eigenfunctions on the boundary, following a modified Filoche-Mayboroda arguments.

Contents

Our work is part of the long term study that aims to provide robust computational tools to address, in a fractal setting, a number of linear and nonlinear problems arising from physics [GDG+00, BCP04, AM99, Akk13, Dun12, ADT10]. The physics problems involving magnetic fields and vector equations are particularly challenging on fractal spaces. Also, the discretization is excepted to be essential to study quantum walks [ABM10, Ord83, APSS12, ADZ93, Kem03]. On the mathematical side, our work is related to [AST06, AST07, HKM18, CAV14, CLV+18, BCH+17, RT10, HR16, HT13, SST13, ADS13, FS12]. Our paper follows results on diffusion problems involving fractal membranes [Lan02, CL14, LV14]. Further investigations have been addressed in a series of papers, see for instance [FLV95, Lap91, LP95, SW19]. We also study a discretized version of the eigenvalue problem with a zero-Dirichlet boundary condition. We compare our computations for the Dirichlet Laplacian mainly to the results in Lapidus et. al. Snowflake harmonics [LNRG96]. Physical experiments on mechanical vibrations of fractal drums motivated Lapidus work.

In particular, we follow [HLTV18] where Dirichlet forms on the Koch snowflake domain were investigated. They represent a central tool in this paper and we consider the following concrete version for the computations,

$$\mathscr{E}(u) := \int_{\Omega} (\nabla u)^2 d\mathcal{L}^2 + \mathscr{E}_{\partial\Omega}(u|_{\partial\Omega}), \tag{1.1}$$

together with a suitable domain of definition, where $\mathcal{L}^2$ is the usual Lebesgue measure on $\mathbb{R}^2$ and $\mathscr{E}_{\partial\Omega}$ denotes the Kusuoka-Kigami Dirichlet form on the Koch snowflake boundary $\partial\Omega$.

We use a graph (triangular grid) to approximate the Koch snowflake domain and define a corresponding sequence of graph energies $\{\mathscr{E}_n\}_{n\in\mathbb{N}}$ to approximate $\mathscr{E}$ in 1.1 in a sense explained in section 2. Then we identify a discrete Laplcian as the generator of $\mathscr{E}_n$ and denote it by L_n. The Dirichlet Laplacian is obtained by deleting the rows and columns in L_n corresponding to boundary vertices.

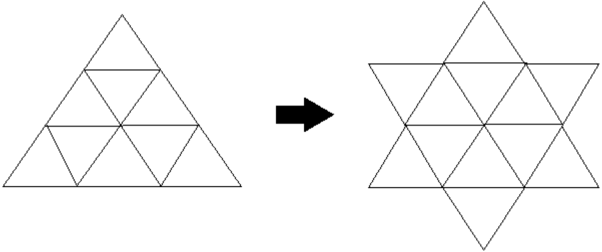


FIGURE 1. Mesh construction through scaled equilateral triangles.

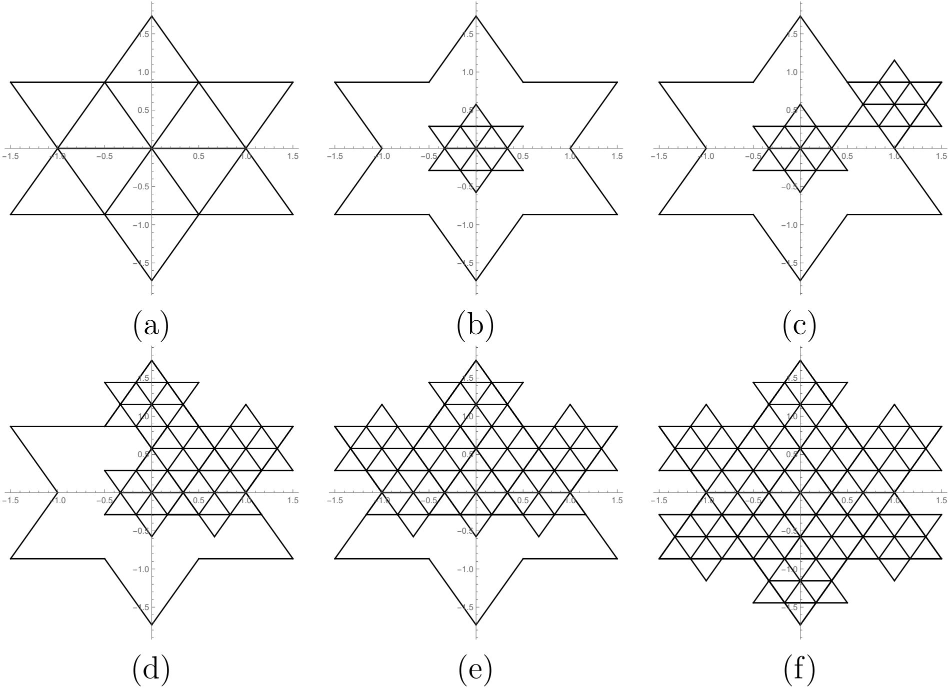


FIGURE 2. Algorithm to generate the vertices of the graph Γ_n

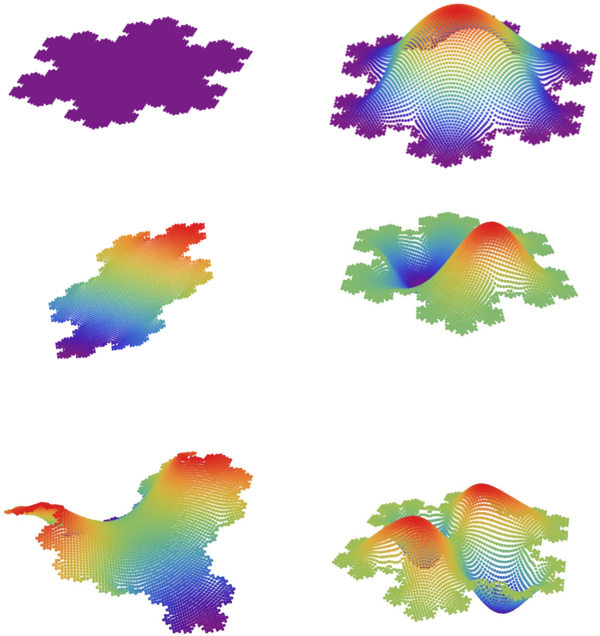


FIGURE 3. Eigenvectors of L_n (left) compared with Dirichlet eigenvectors (right). (a) 1st eigenvector of L_n, eigenvalue 0. (b) 1st Dirichlet eigenvector, eigenvalue 118.8. (c) 2nd eigenvector of L_n, eigenvalue 15.1. (d) 2nd Dirichlet eigenvector, eigenvalue 294.5. (e) 4th eigenvector of L_n, eigenvalue 48.1. (f) 4th Dirichlet eigenvector, eigenvalue 499.8.

FIGURE 4. Eigenvector of L_n (left) compared with Dirichlet eigenvectors (right). (a) 34th eigenvector of L_n, eigenvalue 1098.6. (b) 13th Dirichlet eigenvector, eigenvalue 1084.6. (Level 4 graph approximation)

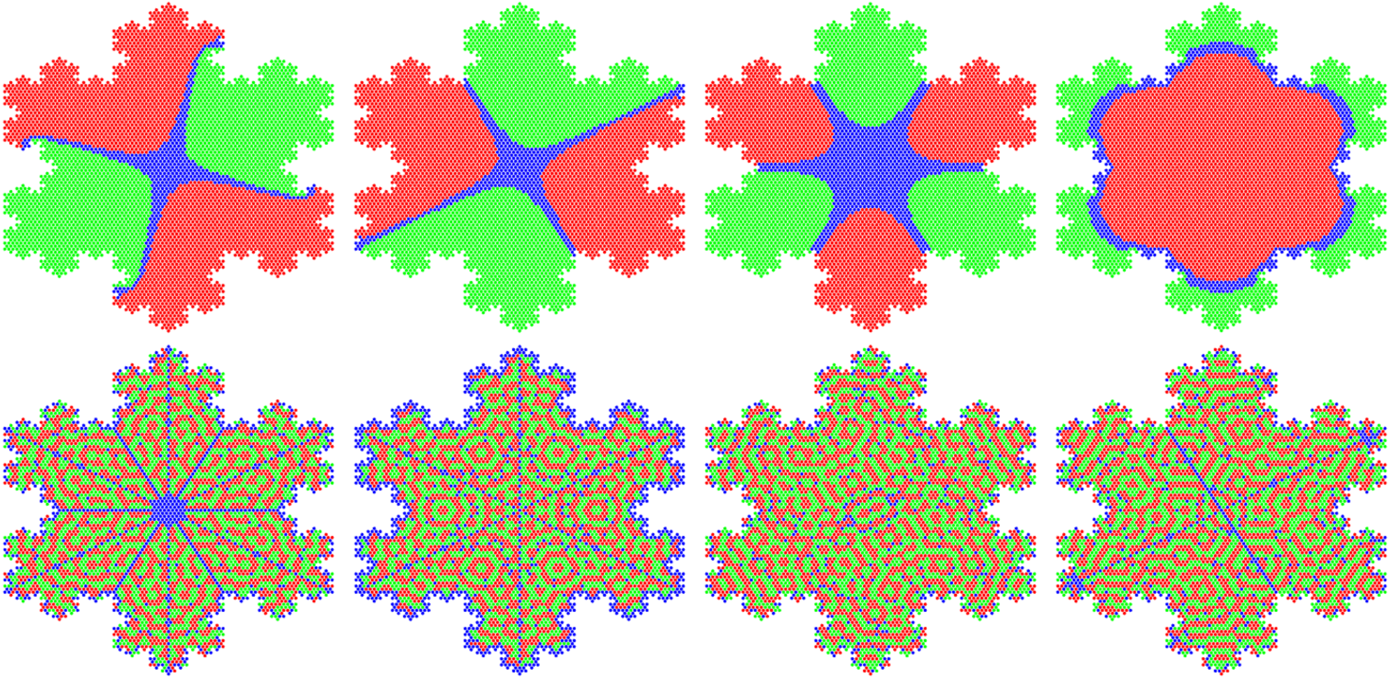


FIGURE 5. Contour Plots of the Eigenvectors of L_n corresponding to eigenvalues λ: (a) 4th eigenvector, $\lambda = 48.1$. (b) 5th eigenvector, $\lambda = 48.1$. (c) 6th eigenvector, $\lambda = 85.1$. (d) 8th eigenvector $\lambda = 125.4$. (e) 1153rd eigenvector $\lambda = 49965.7$. (f) 1157th eigenvector $\lambda = 50156.6$. (g) 1161st eigenvector, $\lambda = 50188.8$ and (h) 1162nd eigenvector, $\lambda = 50188.83$. Blue regions indicate the values of an eigenvector in $(-\epsilon, \epsilon)$, red regions in (ϵ, ∞) and green regions in $(-\infty, -\epsilon)$, where $\epsilon = 0.01$. (Level 4 graph approximation)

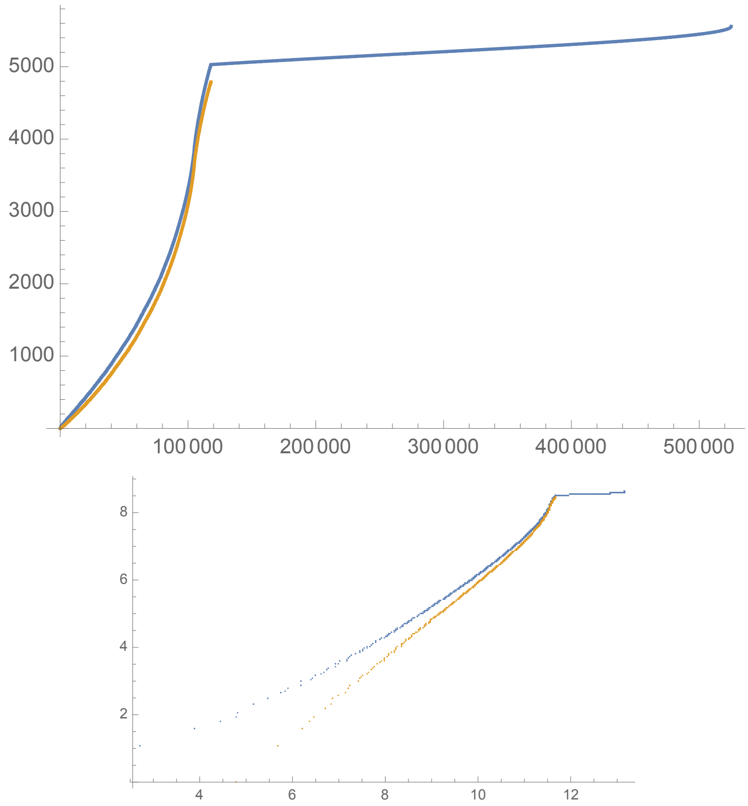


FIGURE 6. (Upper) Eigenvalue counting functions of Dirichlet Laplacian (orange) and L_n (blue). (Lower) Log-Log plot of the eigenvalue counting functions of Dirichlet Laplacian (orange) and L_n (blue) (Level 4 graph approximation).

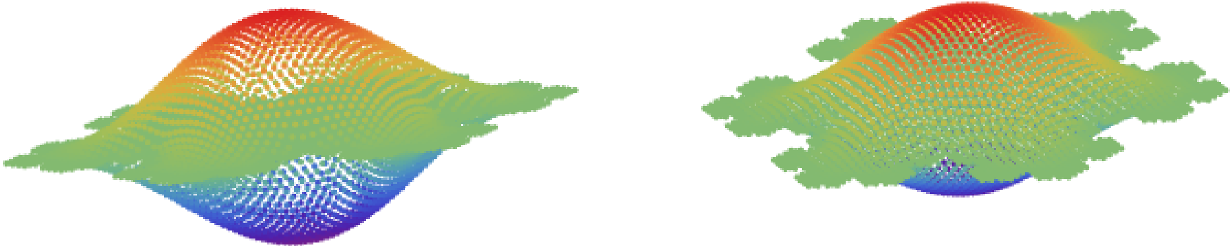


FIGURE 7. (a) The $5,028$th eigenvector of L_n, $\lambda = 118038.02$. (b) The last Dirichlet eigenvector, $\lambda = 118039.37$. The oval-shaped graph is due to a high oscillation of both eigenvectors

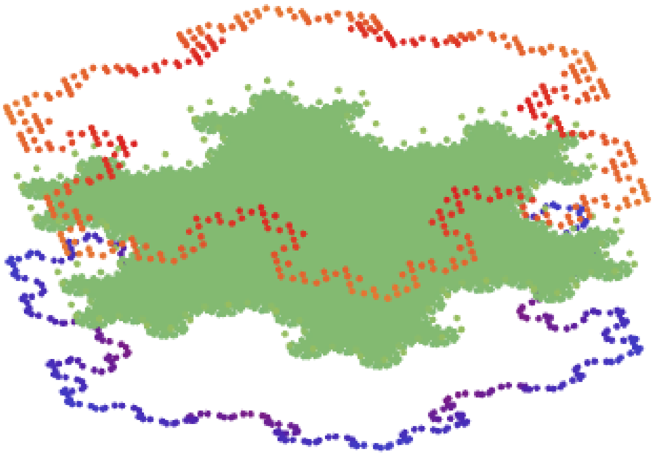


FIGURE 8. The last L_n eigenvector, $\lambda = 524999.69$. The graph splits into two parts, above and below the Koch snowflake domain due to a high oscillation (Level 4 graph approximation).

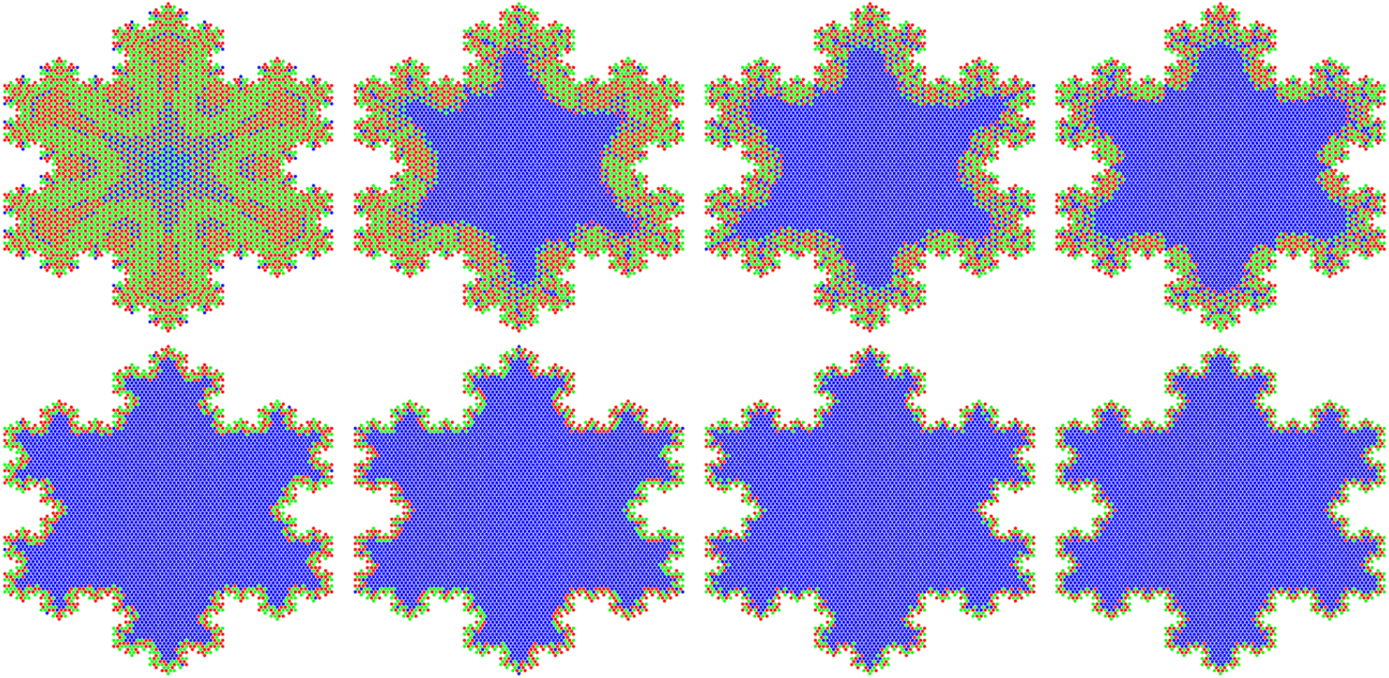
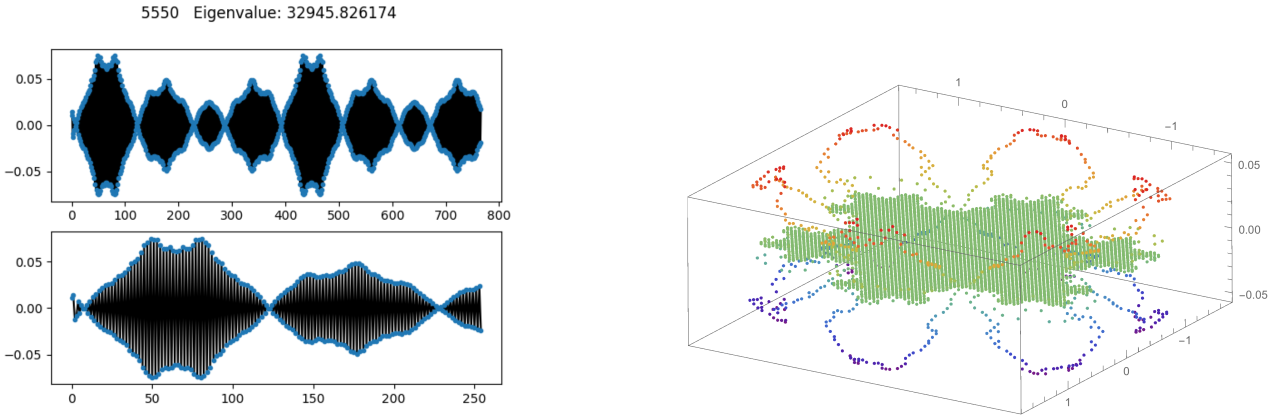


FIGURE 9. L_n eigenvectors localization with eigenvalues λ: (a) 5030th eigenvector, $\lambda = 118048.66$. (b) 5031th eigenvector, $\lambda = 119678.65$. (c) 5032th eigenvector, $\lambda = 119678.65$. (d) 5033th eigenvector, $\lambda = 121460.72$. (e) 5100th eigenvector, $\lambda = 185367.41$. (f) 5200th eigenvector, $\lambda = 291364.38$. (g) 5300th eigenvector, $\lambda = 392584.97$. (h) 5557th eigenvector, $\lambda = 524999.69$. Blue regions indicate the values of an eigenvector in $(-\epsilon, \epsilon)$, red regions in (ϵ, ∞) and green regions in $(-\infty, -\epsilon)$, where $\epsilon = 0.01$ (Level 4 graph approximation).


5550 Eigenvalue: 32945.826174
0.05
0.00
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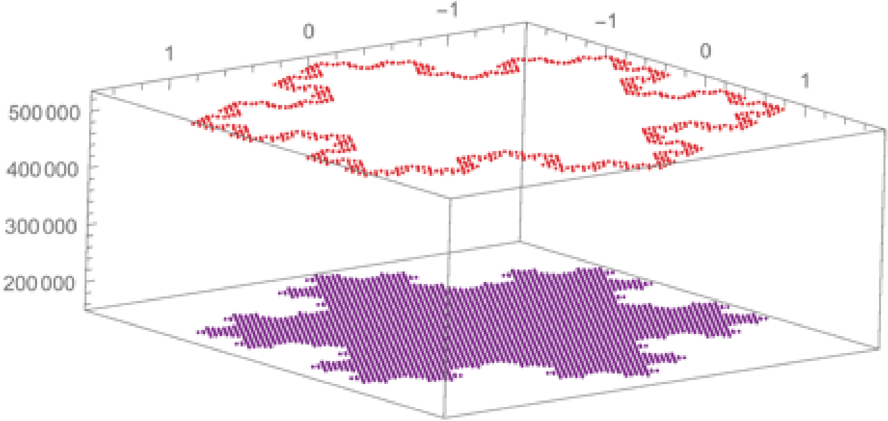

FIGURE 10. The high frequency landscape vector attains just the following two values the boundary vertices 527360 and 524288. It is constant on the interior vertices with the value 157464. (Level 4 graph approximation)