Dirichlet forms and vector analysis on fractal spaces

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Valencia
Motivation

**Long term goals:**

- **Understanding magnetic properties of fractal structures**
  (partially based on experimental physics) [Akkermans, Geim, Peeters et al]

- **Understanding spaces appearing in quantum gravity**
  (partially based on theoretical physics and numerical experiments) [Loll, Reuter et al]
Non-quantized penetration of magnetic field in the vortex state of superconductors

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As first pointed out by Bardeen and Ginzburg in the early sixties⁵, the amount of magnetic flux carried by vortices in superconducting materials depends on their distance from the sample edge, and can be smaller than one flux quantum, Φ₀ = ℏ/2e (where ℏ is Planck’s constant and e is the electronic charge). In bulk superconductors, this reduction of flux becomes negligible at submicrometre distances from the edge, but in thin films the effect may survive much farther into the material³⁴. But the effect has...
Superconducting disk with magnetic coating: Re-entrant Meissner phase, novel critical and vortex phenomena

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Abstract – Within the Ginzburg-Landau formalism, we study the mixed state of a superconducting disk surrounded by a magnetic ring. The stray field of the magnet, concentrated at the rim of the superconducting disk, favors ring-like arrangement of induced vortices, to the point that even a single vortex state exhibits asymmetry. A novel route for the destruction of superconductivity with increasing magnetization of the magnetic coating is found: first all vortices leave the sample, and are replaced by a re-entered Meissner phase with a full depression of the order-parameter at the sample edge; subsequently, superconductivity is then gradually suppressed from the edge inwards, contrary to the well-known surface superconductivity. When exposed to an additional homogeneous magnetic field, we find a field-polarity-dependent vortex structure in our sample—for all vorticities, only giant- or multi-vortex states are found for given polarity of the external field. In large samples, the number of vortex shells and number of flux quanta in each of them can be controlled by the parameters of the magnetic coating.
Novel vortex phenomena in a superconducting disk with magnetic coating

Fig. 2: The free energy of the states with different vorticity $L$ as a function of the magnetization of the magnetic coating. Insets show the Cooper-pair density contour plots of the corresponding states. (a-c) Superconducting phase and (d-f) $|\psi|^2$-density plots, illustrate simultaneous vortex exit and suppression of superconductivity at the rim of the superconducting disk for high magnetization.

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In our theoretical treatment of this system, we use the non-linear Ginzburg-Landau (GL) formalism, combined with Neumann boundary conditions (zero current penetrating the boundary). To investigate the superconducting state of a sample with volume \( V \), we minimize, with respect to the order parameter \( \psi \), the GL free energy

\[
F = \int \frac{dv}{V} \left( |(-i\nabla - \vec{A}_H - \vec{A}_m)\psi|^2 - |\psi|^2 + \frac{1}{2}|\psi|^4 \right),
\]

(2)
Minimization of eq. (2) leads to equations for the order parameter and superconducting current

\[ (-i\vec{\nabla} - \vec{A})^{2}\psi = (1 - |\psi|^{2})\psi, \]  
\[ \vec{j} = \Im(\psi^{*}\vec{\nabla}\psi) - |\psi|^{2}\vec{A}, \]  

which we solve following a numerical approach proposed by Schweigert et al. (see ref. [2]) on a uniform Cartesian grid with typically 10 points/\(\xi\) in each direction. We then start from randomly generated initial distribution of \(\psi\), increase/decrease the magnetization of the magnet or change the value of the applied external field, and let eq. (3) relax to its steady-state solution. In addition, we always recalculate the vortex structure starting from the pure Meissner state\(^1\) (\(\psi = 1\)) or the normal state (\(\psi \approx 0\)) as initial condition. All stable states are then collected and their energies are compared to find the ground state configuration.
Free energy diagram for a large superconducting disk with thin magnetic coating. Insets show the $|\psi|^2$-density plots of distinct vortex states.

Fig. 3: Free energy diagram for a large superconducting disk with thin magnetic coating. Insets show the $|\psi|^2$-density plots of distinct vortex states.
Novel vortex phenomenon in a superconducting disk with magnetic coating

Fig. 4: (a) Free energy of a superconducting disk with magnetic coating as a function of applied homogeneous magnetic field. Insets show the Cooper-pair density plots for indicated states. (b) Same as (a), but for demagnetized coating. In (b), dashed lines denote multi-vortex and solid lines giant-vortex configurations.

Fig. 5: The $|\psi|^2$-density plots illustrating the arrangement of vortex shells in a large superconducting disk for $L=53$ and $L=60$, with magnetic coating with (a,c) negative ($M=-8H_{c2}$), or (b,d) positive ($M=8H_{c2}$) magnetization.
GEOMETRICAL DESCRIPTION OF VORTICES IN
GINZBURG-LANDAU BILLIARDS

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4.2 The Bogomol’nyi identities

For the special value $\kappa = \frac{1}{\sqrt{2}}$, the equations for $\psi$ and $\vec{A}$ can be reduced to first order differential equations. This special point was first used by Sarma [41] in his discussion of type-I vs. type-II superconductors and then identified by Bogomol’nyi [40] in the more general context of stability and integrability of classical solutions of some quantum field theories. This special point is also called a duality point. We first review some properties of the Ginzburg-Landau free energy at the duality point. We use the following identity true for two dimensional systems

$$|(\nabla - i\vec{A})\psi|^2 = |D\psi|^2 + \nabla \times \vec{j} + B|\psi|^2$$

(64)

where $\vec{j} = \text{Im}(\psi^* \nabla \psi) - |\psi|^2 \vec{A}$ is the current density and the operator $D$ is defined as $D = \partial_x + i\partial_y - i(A_x + iA_y)$. This relation is a relative of the Weitzenböck formula (61). At the duality point $\kappa = \frac{1}{\sqrt{2}}$ the expression (63) for $\mathcal{F}$ can be rewritten using (64) as

$$\mathcal{F} = \int_{\Omega} \left( \frac{1}{2} |B - 1 + |\psi|^2|^2 + |D\psi|^2 \right) + \int_{\partial\Omega} (\vec{j} + \vec{A}) \cdot d\vec{l}$$

(65)
Ginzburg/Landau: replace linear eqs. in infinite dimensions by nonlinear in finite dimensions

Geim et al: can confine system to bounded domain

Topological behaviour of solutions matter

Can we study related questions, theoretically and/or numerically, in fractal snowflake domains?
Some recent related math papers (vector analysis):


Review paper:


Related Physics papers:


Language: Dirichlet forms

$X$ loc. compact sep. metric space, $m$ finite Radon measure on $X$, dense support, $(\mathcal{E}, \mathcal{F})$ be a regular symmetric Dirichlet form on $L_2(X, m)$ with core $\mathcal{C} := \mathcal{F} \cap C_0(X)$.

Endowed with the norm $\|f\|_{\mathcal{F}_b} := \mathcal{E}(f)^{1/2} + \sup_X |f|$ the space $\mathcal{C}$ is algebra

$$\mathcal{E}(fg)^{1/2} \leq \|f\|_{\mathcal{F}_b} \|g\|_{\mathcal{F}_b}, \quad f, g \in \mathcal{C}.$$ 

Can define a first order derivation $\partial$ from algebra $\mathcal{C}$ into a Hilbert space $\mathcal{H}$ of $L^2$-vector fields (or 1-forms) (Cipriani/Sauvageot ’03 and many related earlier papers)
Examples:

(i) **Dirichlet forms on Euclidean domains.** Let $X = \Omega$ be a domain in $\mathbb{R}^n$ and $\mathcal{E}(f, g) = \int_{\Omega} \nabla f \cdot \nabla g \, dx$, $f, g \in H^1_0(\Omega)$. 

$(\mathcal{E}, H^1_0(\Omega))$ is a local regular Dirichlet form on $L^2(\Omega)$, $\mathcal{H} = L^2(\Omega, \mathbb{R}^n)$ and $\partial f = \nabla f$ in $L^2$-sense.

(ii) **Dirichlet forms on Riemannian manifolds.** Let $X = M$ be a compact Riemannian manifold and

$$\mathcal{E}(f, g) = \int_{M} \langle \nabla f, \nabla g \rangle_{TM} \, dvol, \quad f, g \in H^1(M)$$

where $dvol$ is the Riemannian volume measure. Then $\mathcal{H} = L^2(M, TM, dvol)$ and $\partial f = \nabla f$ is usual gradient in $L^2$-sense.

(iii) **Dirichlet forms induced by resistance forms on self-similar p.c.f. fractals** (gradients with respect to Kusuoka-Kigami measures).
This language allows study of equations involving vector terms:

\[ \text{div}(a(\nabla u)) = f \]

\[ \Delta u + b(\nabla u) = f \]

\[ i \frac{\partial u}{\partial t} = (-i\nabla - A)^2 u + Vu. \]

\[
\begin{aligned}
\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \Delta u + \nabla p &= 0, \\
\text{div} u &= 0,
\end{aligned}
\]

Some examples of results:
\[ \mathcal{E}^{a,V}(f,g) = \langle (-i\partial - a)f, (-i\partial - a)g \rangle_{\mathcal{H}} + \langle fV, g \rangle_{L_2(X,m)}, \quad f, g \in \mathcal{F}_\mathbb{C}, \]

**Theorem (H./T. '13)**

Let \( a \in \mathcal{H}_\infty \) and \( V \in L_\infty(X, m) \).

(i) The quadratic form \((\mathcal{E}^{a,V}, \mathcal{F}_\mathbb{C})\) is closed.

(ii) The self-adjoint non-negative definite operator on \( L_2(X, m) \) uniquely associated with \((\mathcal{E}^{a,V}, \mathcal{F}_\mathbb{C})\) is given by

\[ H^{a,V} = (-i\partial - a)^* (-i\partial - a) + V. \]

Related Dirac operator is well defined and self-adjoint

\[ D = \begin{pmatrix} 0 & -i\partial^* \\ -i\partial & 0 \end{pmatrix} \]
(Feynman-Kac-Itô formula)

**Theorem (H.’15)**

Suppose $a \in \mathcal{H}_a$ is real valued and $v$ is a real valued Borel function. For $t > 0$ and bounded Borel $f$ set

$$P_t^{a,v} f(x) := \mathbb{E}_x \left[ e^{i \int_{Y([0,t])} a - \int_0^t v(Y_s) ds} f(Y_t) \right], \quad x \in X$$

where $\int_{Y([0,t])} a$ is a properly defined Stratonovich line integral.
(Hodge theorem and Navier-Stokes equations in topo dimension 1)

**Theorem (H./T. ’15)**

*If the space X is compact, connected and topologically 1-dimensional of arbitrarily large Hausdorff and spectral dimensions, then a 1-form $\omega \in \mathcal{H}$ is harmonic if and only if it is in $(\text{Im } \partial)^\perp$: div $\omega = 0$.*

By the classical identity $\frac{1}{2} \nabla |u|^2 = (u \cdot \nabla)u + u \times \text{curl } u$ equation (1) becomes $\frac{\partial u}{\partial t} + \frac{1}{2} \partial \Gamma_{\mathcal{H}}(u) - \Delta_1 u + \partial p = 0$, div $u = \partial^* u = 0$. Therefore any weak solution $u$ of (1) is unique, harmonic and stationary (i.e. $u_t = u_0$ is independent of $t \in [0, \infty)$) for any divergence free initial condition $u_0$. 

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**Dirichlet forms and vector analysis on fractal spaces**
Goals

- General goal: **Linear and non-linear equations on rough domains and fractals** involving
  - Gradient terms
  - Magnetic potentials
- Here in this talk:
  **Domain with fractal snowflake boundary**
  - Tangential derivative along boundary and *pointwise interpretation*
  - Model problem: Parabolic Ventsell problem with drifts in interior domain and on fractal boundary
  - Lipschitz and $C^1$-extensions of functions on the boundary *(preparation for Lipschitz coordinates and studies of magnetic fields)*
Ventsell problem

Consider a 'mixed'/interacting diffusion in the interior and on the boundary

Figure: Fractal (closed) snowflake domain $\Omega$. 
Central tool: bilinear form

\[ \mathcal{E}_A(f, f) = \int_{\Omega} \left( A(x) \cdot \nabla f(x) \right) \cdot \nabla f(x) \, d\mathcal{L}^2(\mathcal{L}^2) + c_0 \mathcal{E}_{\partial \Omega}(f, f) \]

- A bounded measurable uniformly elliptic coefficient matrix,
- \( \mathcal{L}^2 \) two-dim. Lebesgue,
- \( c_0 > 0 \) fixed constant,
- \( \mathcal{E}_{\partial \Omega} \) Kusuoka-Kigami Dirichlet form on snowflake boundary \( \partial \Omega \)
- \( \mathcal{E}_A \), equipped with suitable domain, becomes Dirichlet form
Consider the parabolic Ventsell problem

\[ u_t(t, x) - L_A u(t, x) - \vec{b}(x) \cdot \nabla u(t, x) = f(t, x) \]

in \((0, T] \times \Omega\)

\[ u_t(t, x) - c_0 \Delta_{\partial \Omega} u(t, x) - b_{\partial \Omega}(x) D_{\partial \Omega} u(t, x) + c(x) u(t, x) \]

\[ = - \frac{\partial u(t, x)}{\partial n_A} + f(t, x) \]

in \((0, T] \times \partial \Omega\)

\[ u(0, x) = u_0(x) \]

in \(\Omega\).

(generalizes a problem of Lancia/Vernole ’14)
Here

- \( \mathcal{L}_A u(t, \cdot) = \text{div}(A \cdot \nabla u(t, \cdot)) \)
- \( \Delta_{\partial \Omega} \) Kusuoka-Kigami Laplacian on \( \partial \Omega \)
- \( c \) stationary scalar potential on \( \partial \Omega \), \( u_0 \) given initial condition
- \( \frac{\partial u(t, \cdot)}{\partial n_A} \) is the co-normal derivative of \( u(t, \cdot) \) across \( \partial \Omega \)
- \( \vec{b} \) stationary drift vector field in \( \Omega \), \( b_{\partial \Omega} \) is a drift vector field on \( \partial \Omega \) (not necessarily related)
- \( D_{\partial \Omega} u(t, \cdot) \) 'tangential derivative' of \( u(t, \cdot) \) along \( \partial \Omega \)
Energy and tangential gradients

Discrete energy forms on Koch curve:

\[ \mathcal{E}_K^{(n)}(u) := \frac{1}{2} 4^n \sum_{p \in V_n(K)} \sum_{q \sim n p} (u(p) - u(q))^2 \]

- \( V_n \) ‘dyadic points of level \( n \)’ (images of boundary points \( V_0 = \{0, 1\} \) under similarity maps)
- non-decreasing in \( n \) with non-trivial limit \( \mathcal{E}_K(u) := \lim_{n \to \infty} \mathcal{E}_K^{(n)}(u) \)
- obtain regular resistance form \((\mathcal{E}_K, \mathcal{D}(\mathcal{E}_K))\) in the sense of Kigami
- \( \mathcal{D}(\mathcal{E}_K) \subset C(K) \)
- resistance metric

\[ d_R(p, q) := \sup \left\{ \frac{|u(p) - u(q)|^2}{\mathcal{E}_K(u)} : u \in \mathcal{D}(\mathcal{E}_K), \mathcal{E}_K(u) > 0 \right\} \]
Figure: Koch curve with boundary $V_0 = \{0, 1\}$
• $h$ unique harmonic function on $K$ with boundary values $h(0) = 0$ and $h(1) = 1$ (coordinate function)
• $h : K \to [0, 1]$ homeomorphism onto $[0, 1]$
• with $\delta = \dim_H K = \frac{\ln 4}{\ln 3}$ and quasi-distance

\[ d_\delta(x, y) := |x - y|^\delta, \]

$K$ is variational fractal in the sense of Mosco '97, and for any $p, q \in V_n(K)$ with $p \sim_q q$, have

\[ |h(p) - h(q)| = d_\delta(p, q) = |p - q|^\delta \]
energy measure $\mu_K$ of $h$,

$$\mathcal{E}_K(h, uh) - \frac{1}{2} \mathcal{E}_K(u, h^2) = \int_K u \, d\mu_K,$$

equals $\delta$-dim. Hausdorff measure (up to constant)

- subspace

$$S_K := \left\{ F \circ h : F \in C^1(\mathbb{R}) \right\}$$

is dense in $\mathcal{D}(\mathcal{E}_K)$

- $\mathcal{E}_K(u) = \int_K F'(h(x))^2 \mu_K(dx)$ for any $u = F \circ h$ from $S_K$

- $(\mathcal{E}_K, \mathcal{D}(\mathcal{E}_K))$ is a strongly local regular Dirichlet form on $L^2(K, \mu_K)$ (Kigami-Kusuoka energy)
Koch snowflake $\partial \Omega = \text{union of three congruent copies } K_1, K_2 \text{ and } K_3 \text{ resp. } K_4, K_5 \text{ and } K_6 \text{ of } K$

**Figure:** Using the copies $K_1$, $K_2$ and $K_3$.  

**Figure:** Using the copies $K_4$, $K_5$ and $K_6$.  

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\( \varphi_i \) Euclidean motion that maps \( K_i \) into \( K \)

\[ \mathcal{D}(\mathcal{E}_{\partial \Omega}) := \left\{ u : \partial \Omega \rightarrow \mathbb{R} : u|_{K_i} \circ \varphi_i^{-1} \in \mathcal{D}(\mathcal{E}_K), \ i = 1, \ldots, 6 \right\} \]

\[ \mathcal{E}_{\partial \Omega}(u) := \mathcal{E}_{K_1}(u|_{K_1}) + \mathcal{E}_{K_2}(u|_{K_2}) + \mathcal{E}_{K_3}(u|_{K_3}), \ u \in \mathcal{D}(\mathcal{E}_{\partial \Omega}). \]

identify \( u|_{K_i} \) with \( u|_{K_i} \circ \varphi_i^{-1} \), identify \( \mu_{K_i} \) with image measures of \( \mu_K \) and so on

equip \( \partial \Omega \) with measure \( \mu := \mu_{K_1} + \mu_{K_2} + \mu_{K_3} \)

\( (\mathcal{E}_{\partial \Omega}, \mathcal{D}(\mathcal{E}_{\partial \Omega})) \) strongly local Dirichlet form on \( L^2(\partial \Omega, \mu) \)...

Kusuoka-Kigami energy form on \( \partial \Omega \) (see Freiberg/Lancia ’04)

assoc. Laplacian in variational sense is

\[ \langle \Delta_{\partial \Omega} f, g \rangle_{((\mathcal{D}(\mathcal{E}_{\partial \Omega}))', \mathcal{D}(\mathcal{E}_{\partial \Omega}))} = -\mathcal{E}_{\partial \Omega}(f, g), \ f, g \in \mathcal{D}(\mathcal{E}_{\partial \Omega}) \]
Set

\[ S_{\partial\Omega} := \{ u : \partial\Omega \to \mathbb{R} : u|_{K_i} = F_i \circ h_i \in S_{K_i}, i = 1, \ldots, 6 \} \]

with \( S_{K_i} \) similarly as before.

**Lemma**

*The space* \( S_{\partial\Omega} \) *is dense in* \( \mathcal{D}(\mathcal{E}_{\partial\Omega}) \).
Write \( V_\ast(\partial \Omega) := \bigcup_{i=1}^{6} V_\ast(K_i) \), where for any \( K_i \) the set \( V_\ast(K_i) \) is 'set of all dyadic points' (images of bd. points).

**Lemma**

Let \( u \in S_{\partial \Omega} \). Then for any \( p \in V_\ast(\partial \Omega) \) the limit

\[
D_{\partial \Omega} u(p) := \lim_{V_\ast(\partial \Omega) \ni q \to p} \frac{u(p) - u(q)}{|p - q|^\delta},
\]

exists uniformly in \( p \in V_\ast(\partial \Omega) \). If \( p \in K_i \) and \( u|_{K_i} = F_i \circ h_i \), then \( D_{\partial \Omega} u(p) = F'_i(h_i(p)) \).

Moreover, if \( p \in K_i \cap K_j \cap V_\ast(\partial \Omega) \) and \( u|_{K_j} = F_j \circ h_j \) then \( F'_i(h_i(p)) = F'_j(h_j(p)) \).

(concept of limit goes back to Mosco)
**Pointwise definition of tangential gradient of composed functions:**

**Definition**

Given \( x \in \mathcal{K}_i \) and \( u \in S_{\partial\Omega} \) such that \( u|_{\mathcal{K}_i} = F_i \circ h_i \), we set

\[
D_{\partial\Omega} u(x) := F'_i(h_i(x)).
\] (2)

We refer to \( D_{\partial\Omega} u(x) \) as the **tangential gradient** of \( u \) along \( \partial\Omega \) in \( x \in \partial\Omega \).

Pointwise definition (’new’) useful for numerical schemes. Connections/applications to research of Rozanova-Pierrat (acoustics, heat content) and Sapoval.

**Corollary**

*The definition (2) is independent of the particular choice of \( h_i \) and \( F_i \).*

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Dirichlet forms and vector analysis on fractal spaces
Interpreted as a linear operator:

**Corollary**

The tangential gradient $D_{\partial \Omega}$ defines a bounded linear operator $D_{\partial \Omega} : S_{\partial \Omega} \rightarrow L^2(\partial \Omega, \mu)$, more precisely, we have

$$\|D_{\partial \Omega} u\|_{L^2(\partial \Omega, \mu)}^2 = \mathcal{E}_{\partial \Omega}(u)$$

for all $u \in S_{\partial \Omega}$.

The closedness of $(\mathcal{E}_{\partial \Omega}, D(\mathcal{E}_{\partial \Omega}))$ implies:

**Proposition**

$D_{\partial \Omega}$ extends to a closed unbounded operator $D_{\partial \Omega} : L^2(\partial \Omega, \mu) \rightarrow L^2(\partial \Omega, \mu)$ with domain $D(\mathcal{E}_{\partial \Omega})$. Moreover, for any $u \in D(\mathcal{E}_{\partial \Omega})$ we have $\mathcal{E}_{\partial \Omega}(u) = \|D_{\partial \Omega} u\|_{L^2(\partial \Omega, \mu)}^2$ and the energy measure $\nu_u$ of $u$ is absolutely continuous with respect to $\mu$ with density $\Gamma_{\partial \Omega}(u) = (D_{\partial \Omega} u)^2$ $\mu$-a.e.
\( D_{\partial \Omega} \) version of first order derivation \( \partial \) (abstract gradient) in the sense of Cipriani/Sauvageot ’03

special situation here (on curve): gradients themselves are scalar valued \( L^2 \)-functions
Cauchy problem and interpretation

Consider the function space

\[ V(Ω, \partialΩ) := \{ u \in H^1(Ω) : u|_{\partialΩ} \in D(E_{\partialΩ}) \} \]

equipped with

\[ \langle u, v \rangle_{V(Ω, \partialΩ)} = \langle u, v \rangle_{H^1(Ω)} + E_{\partialΩ}(u|_{\partialΩ}, v|_{\partialΩ}) + \langle u|_{\partialΩ}, v|_{\partialΩ} \rangle_{L^2(\partialΩ, \mu)} \]

Hilbert.

Consider \( m := \mathcal{L}^2|_{Ω} + \mu \) measure on \( \overline{Ω} \)

Then

\[ E_0(u, v) := \int_{Ω} \nabla u \cdot \nabla v \ d\mathcal{L}^2 + c_0 E_{\partialΩ}(u|_{\partialΩ}, v|_{\partialΩ}) \]

defines Dirichlet form \((E_0, V(Ω, \partialΩ))\) on \( L^2(\overline{Ω}, m) \) (Lancia ’02/’03, Lancia/Vernole ’14)
Assume we are given the following data:

(i) \( A = (a_{ij})^2_{i,j=1}, a_{ji} = a_{ij} \) bounded,
\[
\sum_{i,j=1}^{2} a_{ij}\xi_i\xi_j \geq \lambda \sum_{i=1}^{2} |\xi_i|^2, \quad \xi = (\xi_1, \xi_2) \in \Omega
\]

(ii) a vector field \( \vec{b} = (b_1, b_2) \in L^2(\Omega, \mathbb{R}^2) \) such that
\[
\int_{\Omega} u^2 (\vec{b} \cdot \vec{b}) \, dL^2 \leq \gamma_1 \int_{\Omega} (\nabla u)^2 \, dL^2 + \gamma_2 \|u\|^2_{L^2(\Omega)}, \quad u \in V(\Omega, \partial\Omega),
\]
with positive constants \( \gamma_1 \) and \( \gamma_2 \) such that \( \sqrt{2\gamma_1} < \lambda \),

(iii) a ’boundary vector field’ \( b_{\partial\Omega} \in L^2(\partial\Omega, \mu) \) such that
\[
\int_{\partial\Omega} u^2 b_{\partial\Omega}^2 \, d\mu \leq \delta_1 \mathcal{E}_{\partial\Omega}(u|\partial\Omega) + \delta_2 \|u\|^2_{L^2(\partial\Omega, \mu)}, \quad u \in V(\Omega, \partial\Omega),
\]
with positive constants \( \delta_1 \) and \( \delta_2 \) such that \( \sqrt{2\delta_1} < c_0 \),

(iv) a continuous function \( c \) on \( \partial\Omega \).
Consider

\[ \mathcal{E}(u, v) = \int_{\Omega} (A(x) \cdot \nabla u(x)) \cdot \nabla v(x) L^2(dx) - \int_{\Omega} (\vec{b}(x) \cdot \nabla u(x)) v(x) L^2(dx) \]

\[ + c_0 \mathcal{E}_{\partial \Omega}(u, v) - \int_{\partial \Omega} b_{\partial \Omega}(x) D_{\partial \Omega} u(x) v(x) \mu(dx) + \int_{\partial \Omega} c(x) u(x) v(x) m(dx), \]

\[ u, v \in V(\Omega, \partial \Omega). \]

Given \( \alpha > 0 \) write

\[ \mathcal{E}_\alpha(u, v) := \mathcal{E}(u, v) + \alpha \langle u, v \rangle_{L^2(\Omega, m)}. \]
Proposition

Let $A$, $\vec{b}$, $b_{\partial \Omega}$ and $c$ be as above. Then $(\mathcal{E}, V(\Omega, \partial \Omega))$ is a closed coercive form in the wide sense, i.e. there is some $\alpha > 0$ such that $u \mapsto \mathcal{E}_\alpha(u, u)$ defines a positive definite closed quadratic form on $L^2(\overline{\Omega}, m)$ with domain $V(\Omega, \partial \Omega)$ and

$$|\mathcal{E}_{\alpha+1}(u, v)| \leq K \mathcal{E}_{\alpha+1}(u)^{1/2} \mathcal{E}_{\alpha+1}(v)^{1/2}, \quad u, v \in V(\Omega, \partial \Omega),$$

with a universal constant $K > 0$. 

\((\mathcal{E}, V(\Omega, \partial\Omega))\) generates an analytic semigroup \((T_t)_{t \geq 0}\) on \(L^2(\Omega, m)\) with generator \((A, \mathcal{D}(A))\).

Given \(f : [0, T] \to L^2(\Omega, m)\) and \(u_0 \in L^2(\Omega, m)\), consider abstract Cauchy problem
\[
\begin{cases}
\frac{du(t)}{dt} = Au(t) + f(t), & 0 < t \leq T, \\
u(0) = u_0
\end{cases}
\]

in \(L^2(\Omega, m)\).
$u \in C([0, T], L^2(\Omega, m))$ classical solution if
$u \in C^1((0, T], L^2(\Omega, m)) \cap C((0, T], D(A))$ and $u$ satisfies (3).

**Theorem**

Suppose $0 < \theta < 1$ and $f \in C^\theta((0, T], L^2(\Omega, m))$ and that assumptions above are satisfied. Then (3) has a unique classical solution $u$, given by

$$u(t) = T_t u_0 + \int_0^t T_{t-s} f(s) ds, \quad 0 \leq t \leq T.$$
Interpretation of abstract Cauchy solution:

- consider

\[ V(\Omega) = \left\{ g \in H^1(\Omega) : L_A g \in L^2(\Omega) \right\}, \]

where \( L_A g = \text{div}(A \nabla g) \), with norm

\[ \|g\|_{V(\Omega)} := \|L_A g\|_{L^2(\Omega)} + \|\nabla g\|_{L^2(\Omega,\mathbb{R}^2)} + \|g\|_{L^2(\Omega)}. \]

- similarly as in Lancia ’02 and Lancia/Vernole ’06, for any \( g \in V(\Omega) \) we can define a distribution \( I_g \in (H^1(\Omega))' \) by

\[ I_g(v) := \int_{\Omega} (A \cdot \nabla g) \cdot \nabla v \, d\mathcal{L}^2 + \int_{\Omega} v L_A g \, d\mathcal{L}^2, \quad v \in H^1(\Omega). \]
can view each $l_g$ as distribution $\frac{\partial g}{\partial n_A}$ in $(B^{2,2}_{\delta/2}(\partial \Omega))'$ and write

$$\langle \frac{\partial g}{\partial n_A}, v|_{\partial \Omega} \rangle_{(B^{2,2}_{\delta/2}(\partial \Omega))', B^{2,2}_{\delta/2}(\partial \Omega))} = l_g(v), \quad v \in H^1(\Omega)$$

(dual pairing)

- can show classical solution $u$ is in $C((0, T], V(\Omega))$

- by embedding of $\mathcal{D}(\mathcal{E}_{\partial \Omega})$ into $B^{2,2}_{\delta/2}(\partial \Omega)$, may view $\frac{\partial u}{\partial n_A}(t)$ as an element of $(\mathcal{D}(\mathcal{E}_{\partial \Omega}))'$
Arrive at rigorous version of parabolic Ventsell problem:

\[
    u_t(u) - L_A u(t) - \vec{b} \cdot \nabla u(t) = f(t)
\]

in \( L^2(\Omega), \ t \in (0, T] \),

\[
    u_t(t) - c_0 \Delta_{\partial \Omega} u(t)|_{\partial \Omega} - b_{\partial \Omega} D_{\partial \Omega} u(t)|_{\partial \Omega} + cu(t) = -\frac{\partial u}{\partial n_A}(t) + f(t)
\]

in \( (\mathcal{D}(\mathcal{E}_{\partial \Omega}))', \ t \in (0, T] \),

\[
    u(0) = u_0
\]

in \( L^2(\overline{\Omega}, m) \).
A core of Lipschitz functions

- Difficulty for vector analysis on $\overline{\Omega}$: $\Omega$ no Lipschitz domain
- Idea here: consider intrinsic Lipschitz functions on $\partial \Omega$:

$$d_{\mathcal{E}_{\partial \Omega}}(x, y) := \sup \{ u(x) - u(y) : u \in \mathcal{D}(\mathcal{E}_{\partial \Omega}) \text{ with } \Gamma_{\partial \Omega}(u) \leq 1 \mu\text{-a.e.} \},$$

$x, y \in \partial \Omega$.
Here $\Gamma_{\partial \Omega}(u) = (D_{\partial \Omega} u)^2$ energy density of $u \in \mathcal{D}(\mathcal{E}_{\partial \Omega})$. 

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Locally, the intrinsic metric on $\partial \Omega$ can be expressed via coordinates.

**Corollary**

For any $i$ and any $x, y \in \mathcal{K}_i$ we have $d_{E_{\partial \Omega}}(x, y) = |h_i(x) - h_i(y)|$.

Note that also $d_R(x, y) = d_{E_{\partial \Omega}}(x, y)$, $x, y \in \mathcal{K}_i$.

The corollary allows for the following extension principle.

**Theorem**

Any $\mathcal{E}_{\partial \Omega}$-intrinsically Lipschitz function on $\partial \Omega$ has an Euclidean-Lipschitz extension to $\overline{\Omega}$. 
Recall the notation

\[ E_0(u, v) := \int_{\Omega} \nabla u \cdot \nabla v \, dL^2 + c_0 E_{\partial \Omega}(u|_{\partial \Omega}, v|_{\partial \Omega}), \quad u, v \in V(\Omega, \partial \Omega), \]

\[ u, v \in V(\Omega, \partial \Omega). \]

**Corollary**

(i) The Dirichlet form \( E_0 \) has a core consisting of functions in \( C(\overline{\Omega}) \) whose restriction to \( \Omega \) is in \( C^1(\Omega) \), and whose restriction to \( \partial \Omega \) is a \( E_{\partial \Omega} \)-intrinsic \( C^1 \)-function. For a core function \( u \) the gradients \( \nabla u \) and \( D_{\partial \Omega} u \) are well defined pointwise in \( \Omega \) and on \( \partial \Omega \), respectively.

(ii) There are two coordinate functions \( y_1, y_2 \) which are contained in the core and separate the points of \( \overline{\Omega} \).
For a core function $u$ have $D_{\partial \Omega} u$ continuous on $\partial \Omega$ and $\nabla u$ continuous on $\Omega$, and usual calculus rules.

$\nabla u$ not necessarily continuous on $\overline{\Omega}$, a typical function in Euclidean $C^1(\overline{\Omega})$ is not in the domain of $\mathcal{E}$.

Can be used to obtain a coordinate representation for the gradients.
Have two 'constructios / proofs' of the Theorem, suitable to create numerical schemes. Assume that $\Omega$ is embedded in $\mathbb{C}$ and inscribed in the circle of radius $\sqrt{3}$

- the six most outward points of $\overline{\Omega}$ are $\sqrt{3}e^{ik\pi/3+i\pi/6}$, $k = 0, 1, \ldots, 5$;
- the inscribed circle is the unit circle
- the six most inward points of $\partial\Omega$ are the 6th roots of unity $e^{ik\pi/3}$, $k = 0, 1, \ldots, 5$
- most left and right most inward points of $\partial\Omega$ are $-1, 1 \in \mathbb{C}$ respectively

Slight change of notation: now points $p$ and $q$ from $V_{n+1}(\partial\Omega)$ that are neighbors, $p \sim_{n+1} q$, have Euclidean distance $3^{-n}$
Assume $f$ be an $\mathcal{E}_{\partial \Omega}$-intrinsically Lipschitz on $\partial \Omega$, like to construct a Euclidean-Lipschitz extension $g$ of $f$ to $\overline{\Omega}$.

Use natural “approximate” triangulations of $\Omega$.

First method is reminiscent of Lapidus/Pearse ’06 and Evans ’12, ’12 and uses weakly self-similar triangulation, which is nicely separated from the boundary.
Consider lattice $L_n = 3^{-n}\mathbb{Z}\{e^{ik\pi/3}| k = 0, 1, \ldots, 5\}$. For each $x \in L_n \cap \Omega$ there are finitely many points of $L_n \cap \partial \Omega = V_{n+1}(\partial \Omega)$ that closest to $x$. On $L_n \cap \Omega$ we obtain a function $g_n$ by defining $g_n(x)$ to be the average of $f$ at these points.

Assume that points of each lattice $L_n$ are the vertices of closed equilateral triangles $T_{n,m}$ of sides $3^{-n}$, and so each lattice $L_n$ defines a triangulation of $\mathbb{R}^2 = \bigcup_m T_{n,m}$ into such triangles $T_{n,m}$.

Define $\mathcal{T}_n := \bigcup_{m: T_{n,m} \subseteq \Omega} T_{n,m}$ as the union of triangles $T_{n,m}$ that lie inside $\Omega$.

$\mathcal{T}_n$ compact set contained in $\Omega$, and $\mathcal{T}_n$ is separated from $\partial \Omega$. Denote boundary of $\mathcal{T}_n$ by $\partial \mathcal{T}_n$. 

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Dirichlet forms and vector analysis on fractal spaces
Figure: The set $\mathcal{T}_1 \subset \Omega$, triangulated by triangles of side length $\frac{1}{3}$. 
Figure: The set $\mathcal{T}_2 \subset \Omega$, with $\mathcal{T}_1 \subset \mathcal{T}_2$ triangulated by triangles of side length $\frac{1}{3}$ and $\mathcal{T}_2 \setminus \mathcal{T}_1$ triangulated by triangles of side length $\frac{1}{9}$.
From the functions $g_n$ on $L_n \cap \Omega$ now inductively define $g$:

- on $\mathcal{T}_1$ define $g$ as piece-wise linear interpolation of $g_1$ on the triangulation by triangles $T_{1,m} \subset \mathcal{T}_1$.
- extend $g$ from $\mathcal{T}_1$ to $\mathcal{T}_2$ using piece-wise linear interpolation of $g_2$ on the triangles $T_{2,m}$ contained in $\mathcal{T}_2$ but not in $\mathcal{T}_1$.
- if $g$ is already defined on $\mathcal{T}_{n-1}$, we extend it to $\mathcal{T}_n$ using piece-wise linear interpolation of $g_n$ on the triangles $T_{n,m}$ contained in $\mathcal{T}_n$ but not in $\mathcal{T}_{n-1}$.
To show $g$ is Lipschitz use:

For any $p \in V_n(\partial \Omega)$ we have

$$L_{\mathcal{E}_\partial \Omega} (f) \geq \sup_{q \in V_n(\partial \Omega), q \sim np} \frac{|f(p) - f(q)|}{d_{\mathcal{E}_\partial \Omega}(p, q)} = \left(\frac{4}{3}\right)^n \sup_{q \in V_n(\partial \Omega), q \sim np} \frac{|f(p) - f(q)|}{|p - q|},$$

where $L_{\mathcal{E}_\partial \Omega} (f)$ denotes the $\mathcal{E}_\partial \Omega$-intrinsic Lipschitz constant of $f$. 
Figure: Chain of 8 consecutive points from $V_{n+1}(\partial \Omega)$ that contains all points considered when computing the values $g_n(x)$ and $g_n(x')$ for neighbours $x$ and $x'$ on the shell $\partial T_n$. 
Second method of construction / proof: weakly self-similar triangulation \textit{which is not nicely separated from the boundary} \( \partial \Omega \).

- define \( g(0) \) to be the average of \( f \) at the 6th roots of unity \( e^{ik\pi/3} \), \( k = 0, 1, \ldots, 5 \)
- at each 6th root of unity \( e^{ik\pi/3} \), \( k = 0, 1, \ldots, 5 \), define \( g(e^{ik\pi/3}) = f(e^{ik\pi/3}) \)
- interpolate \( g \) linearly in each unit equilateral triangle with vertices \( 0, e^{ik\pi/3}, e^{i(k+1)\pi/3} \)

This defines \( g \) in the regular closed convex unit hexagon \( \overline{\Omega}_1 \) which is the convex span of the 6th roots of unity.
Figure: The triangulated hexagon $\Omega_1$. 
\(\Omega \setminus \overline{\Omega}_1\) consists of six disjoint isometric open sets
closure of each of these contains a regular closed convex hexagon with sides \(\frac{1}{3}\). On the joint facet with \(\overline{\Omega}_1\) the function \(g\) is already defined (and on incident vertices)
remaining four vertices of this hexagon lie on \(\partial \Omega\), and so we define \(g(x) = f(x)\) at these vertices
define \(g\) at center of hexagon as the average at the vertices
interpolate \(g\) linearly in each of these triangle

Have defined \(g\) in the closed set which is the union of the unit regular hexagon \(\overline{\Omega}_1\) and the six adjacent closed hexagons with sides \(\frac{1}{3}\), denote this union by \(\overline{\Omega}_2\).
Figure: The triangulated set $\Omega_2$. 
\( \Omega \setminus \overline{\Omega_2} \) consists of 30 open components of two shapes:

- 18 components are \( \frac{1}{3} \) in size in comparison to the components considered before
- 12 component are of the same scale, but have only one fractal side
- closure of each of these 30 components has an inscribed regular closed convex hexagon with sides \( \frac{1}{9} \).
Figure: The triangulated set $\Omega_3$. 

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Dirichlet forms and vector analysis on fractal spaces
Generalization

Assume $\Omega \subset \mathbb{R}^2$ domain with boundary $\partial \Omega$ homeo to circle. Assume that locally $R = R(x, y)$ is a (geodesic) distance along the boundary, assume that locally

$$R(x, y) \leq c|x - y|.$$  

(prevents near-intersections)

Assume that Dirichlet form $\mathcal{E}$ on interior $\Omega$ satisfies $\mathcal{E}(f) \leq c \| f \|_{\text{Lip}}.$
Consider resistance form $\mathcal{E}_{\partial \Omega}$ on $\partial \Omega$ associated with $R$.

**Corollary**

The form $\mathcal{E} + \mathcal{E}_{\partial \Omega}$ is closable on a Lipschitz core, closure is Dirichlet form on $\overline{\Omega}$ (with sum of measures).

**Theorem**

Any $R$-Lipschitz function on $\partial \Omega$ has an Euclidean-Lipschitz extension to $\overline{\Omega}$.

Do no longer need boundary to be $d$-set / self-similar (usual trace results omitted).
Can again define tangential derivative, now by

\[
D_{\partial \Omega} f(x) := \lim_{y \to x} \frac{f(x) - f(y)}{R(x, y)}
\]

(or with other sign, depending on orientation)

Example:

Assume \( \partial \Omega \) has Hausdorff-dim. \( \delta \), then set

\[
R(x, y) = H^\delta([x, y]),
\]

where \([x, y]\) is the piece of \( \partial \Omega \) joining two points \( x \) and \( y \) and \( H^\delta \) \( \delta \)-dim. Hausdorff measure on \( \partial \Omega \) (parametrization by Hausdorff measure).
Mixed boundary valued problem for linear and nonlinear wave equations in domains with fractal boundaries
Adrien Dekkers, Anna Rozanov-Plerrat, A.T.

Fractal shape optimization in linear acoustics,
Michael Hinz, Anna Rozanova-Pierrat, and A.T.
DISCRETIZATION OF THE KOCH SNOWFLAKE DOMAIN WITH BOUNDARY AND INTERIOR ENERGIES

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Abstract. We show how to discretize the energy form on the Koch snowflake domain so that both the fractal boundary and the Euclidean interior can support positive energy. We compute eigenvalues and eigenfunctions, and demonstrate the high energy localization of the eigenfunctions on the boundary, following a modified Filoche-Mayboroda arguments.

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1. Introduction

The main objective of the paper is to investigate a discrete version of the eigenvalue problem $\Delta u = \lambda u$ on the Koch snowflake domain which we denote by $\Omega$, in which both the Euclidean interior and the fractal boundary carry energy forms. We consider the discrete Laplacian that is the generator of the discrete Dirichlet form that takes into account the interior of $\Omega$ as well as the fractal boundary.
Our work is part of the long term study that aims to provide robust computational tools to address, in a fractal setting, a number of linear and nonlinear problems arising from physics [GDG+00, BCP04, AM99, Akk13, Dun12, ADT10]. The physics problems involving magnetic fields and vector equations are particularly challenging on fractal spaces. Also, the discretization is expected to be essential to study quantum walks [ABM10, Ord83, APSS12, ADZ93, Kem03]. On the mathematical side, our work is related to [AST06, AST07, HKM18, CAV14, CLV+18, BCH+17, RT10, HR16, HT13, SST13, ADS13, FS12]. Our paper follows results on diffusion problems involving fractal membranes [Lan02, CL14, LV14]. Further investigations have been addressed in a series of papers, see for instance [FLV95, Lap91, LP95, SW19]. We also study a discretized version of the eigenvalue problem with a zero-Dirichlet boundary condition. We compare our computations for the Dirichlet Laplacian mainly to the results in Lapidus et. al. Snowflake harmonics [LNRC96]. Physical experiments on mechanical vibrations of fractal drums motivated Lapidus work.

In particular, we follow [HLTV18] where Dirichlet forms on the Koch snowflake domain were investigated. They represent a central tool in this paper and we consider the following concrete version for the computations,

\[ E(u) := \int_{\Omega} (\nabla u)^2 d\mathcal{L}^2 + E_{\partial\Omega}(u|_{\partial\Omega}), \]

together with a suitable domain of definition, where \(\mathcal{L}^2\) is the usual Lebesgue measure on \(\mathbb{R}^2\) and \(E_{\partial\Omega}\) denotes the Kusuoka-Kigami Dirichlet form on the Koch snowflake boundary \(\partial\Omega\).

We use a graph (triangular grid) to approximate the Koch snowflake domain and define a corresponding sequence of graph energies \(\{E_n\}_{n\in\mathbb{N}}\) to approximate \(E\) in 1.1 in a sense explained in section 2. Then we identify a discrete Laplacian as the generator of \(E_n\) and denote it by \(L_n\). The Dirichlet Laplacian is obtained by deleting the rows and columns in \(L_n\) corresponding to boundary vertices.
where $c_n(p_1, p_2)$ is the conductance between points $p_1, p_2$. Unless stated otherwise we take

$$c_n(p_1, p_2) = \begin{cases} 
1 & \text{if } p_1 \text{ and } p_2 \text{ are connected by an interior edge} \\
4^n & \text{if } p_1 \text{ and } p_2 \text{ are connected by an outer boundary edge} \\
0 & \text{if } p_1 \text{ and } p_2 \text{ are not connected by an edge}.
\end{cases}$$

We introduce a measure on the vertices $V_n$:

$$m_n(p) = \begin{cases} 
\frac{1}{9^n} & \text{if } p \text{ is an interior vertex} \\
\frac{1}{4^n} & \text{if } p \text{ is an outer boundary vertex}.
\end{cases}$$

The sequence of graph energies $\{\mathcal{E}_n(u)\}_{n \geq 1}$ serves as an approximation of $\mathcal{E}(u)$ as justified in the following proposition.

**Theorem 3.1.** Let $u$ be a core function as described in Corollary 2.4. Then $\mathcal{E}(u) = \lim_{n \to \infty} \mathcal{E}_n(u)$.

**Remark 3.2.** $\mathcal{E}_n(u)$ is understood as the graph energy of the restriction of $u$ on $V_n$. 

**Figure 1.** Mesh construction through scaled equilateral triangles.
Figure 2. Algorithm to generate the vertices of the graph $\Gamma_n$.

After the construction of $\Gamma_n$ we generate the corresponding discrete Laplacian $L_n$ as well as the Dirichlet Laplacian. To generate the discrete Laplacian $L_n$ we use KD Trees to determine efficiently which vertices’s are boundary points and which are interior. We then create a weighted adjacency matrix with this information where the weights are defined as in (3.2). From the constructed discrete Laplacian $L_n$ we are able to generate
Figure 3. Eigenvectors of $L_n$ (left) compared with Dirichlet eigenvectors (right). (a) 1st eigenvector of $L_n$, eigenvalue 0. (b) 1st Dirichlet eigenvector, eigenvalue 118.8. (c) 2nd eigenvector of $L_n$, eigenvalue 15.1. (d) 2nd Dirichlet eigenvector, eigenvalue 294.5. (e) 4th eigenvector of $L_n$, eigenvalue 48.1. (f) 4th Dirichlet eigenvector, eigenvalue 499.8.
Figure 4. Eigenvector of $L_n$ (left) compared with Dirichlet eigenvectors (right). (a) 34th eigenvector of $L_n$, eigenvalue 1098.6. (b) 13th Dirichlet eigenvector, eigenvalue 1084.6. (Level 4 graph approximation)

to the 13th Dirichlet eigenvector but also correspond to the eigenvalue 1098.6, which is very close to the 13th Dirichlet eigenvalue (1084.6).

In general we observe that the Dirichlet eigenvectors exhibit more complex pattern than the their corresponding eigenvectors of $L_n$. Note that from a physics point of view, this is expected as the Dirichlet eigenvectors correspond in our computations to a higher eigenvalues (energy) than their corresponding eigenvectors of $L_n$. Moreover, eigenvectors correspond to higher eigenvalues show increasing oscillatory behavior which limits the graphical representation. For a better view of such eigenvectors, in particular regarding their symmetries, we display in Figure 5 contour plots for a selection of eigenvectors of $L_n$. The blue regions
Figure 5. Contour Plots of the Eigenvectors of $L_n$ corresponding to eigenvalues $\lambda$: (a) 4th eigenvector, $\lambda = 48.1$. (b) 5th eigenvector, $\lambda = 48.1$. (c) 6th eigenvector, $\lambda = 85.1$. (d) 8th eigenvector $\lambda = 125.4$. (e) 1153rd eigenvector $\lambda = 49965.7$. (f) 1157th eigenvector $\lambda = 50156.6$. (g) 1161st eigenvector, $\lambda = 50188.8$ and (h) 1162nd eigenvector, $\lambda = 50188.83$. Blue regions indicate the values of an eigenvector in $(-\epsilon, \epsilon)$, red regions in $(\epsilon, \infty)$ and green regions in $(-\infty, -\epsilon)$, where $\epsilon = 0.01$. (Level 4 graph approximation)
Figure 6. (Upper) Eigenvalue counting functions of Dirichlet Laplacian (orange) and $L_n$ (blue). (Lower) Log-Log plot of the eigenvalue counting functions of Dirichlet Laplacian (orange) and $L_n$ (blue) (Level 4 graph approximation).
Our numerical observations show that taking the boundary energy into account while defining the discrete Laplacian $L_n$ induces eigenvectors corresponding to eigenvalues in the higher part of the spectrum that demonstrate features of whispering gallery type modes. The geometrical information of the boundary is encoded in $L_n$ via the edge weights (conductance) and the vertices measure. Hence we are confronted with the question whether it is possible to predict such modes by directly studying the Laplacian $L_n$. We will give a partial answer in the next section by using ideas of Filoche and Mayboroda introduced in [FM].
5. Landscape Mapping: A Filoche-Mayboroda Argument

Wave localization in general is a puzzling phenomena as we mentioned last section, in particular the question about the mechanisms that enable systems of complicated geometries to confine vibration modes in some subregions of their domain even in the absence of clearly identifiable physical potentials. For the general study of the localizations in fractal domain, see [SGM91] and related works.
Figure 9. $L_n$ eigenvectors localization with eigenvalues $\lambda$: (a) 5030th eigenvector, $\lambda = 118048.66$. (b) 5031th eigenvector, $\lambda = 119678.65$. (c) 5032th eigenvector, $\lambda = 119678.65$. (d) 5033th eigenvector, $\lambda = 121460.72$. (e) 5100th eigenvector, $\lambda = 185367.41$. (f) 5200th eigenvector, $\lambda = 291364.38$. (g) 5300th eigenvector, $\lambda = 392584.97$. (h) 5557th eigenvector, $\lambda = 524999.69$. Blue regions indicate the values of an eigenvector in $(-\epsilon, \epsilon)$, red regions in $(\epsilon, \infty)$ and green regions in $(-\infty, -\epsilon)$, where $\epsilon = 0.01$ (Level 4 graph approximation).
Filoche and Mayboroda [FM, FM12] introduced a general theory which demonstrate that both Anderson (disordered systems) and weak localizations (irregular geometries) originate from the same universal mechanism. The theory gives rise to what they call a landscape Mapping, which we explain briefly with an example following [FM]. They consider a vibrating system described by the wave equation associated to a suitable elliptic differential operator $L$ on an open bounded set $\Omega \subset \mathbb{R}^n$ and for the sake of brevity they impose Dirichlet boundary conditions. By investigating the eigenvalue problem for $L$,

$$L\phi(x) = \lambda \phi(x), \quad x \in \Omega, \quad u|_{\partial \Omega} = 0,$$

they demonstrate that eigenfunctions are controlled by the landscape mapping defined for $x \in \Omega$,

$$u(x) := \int_{\Omega} |G(x, y)| dy,$$

(5.1)
where $G(x, y)$ is the Green’s function associated to $L$ with Dirichlet boundary conditions. For a normalized eigenfunction $\phi$ with respect to the uniform norm, i.e. $\sup_{x \in \Omega} |\phi(x)| = 1$, corresponding to the eigenvalue $\lambda$, they prove the inequality,

$$|\phi(x)| \leq \lambda \int_\Omega |G(x, y)| dy = \lambda \ u(x).$$

(5.2)

Subsequently they show that a graphical representation of the landscape mapping $u(x)$ reveals usually a complex partition of the domain into subregions and thereby helping to predict possible locations of localized eigenfunctions. The idea is to track the low values of the landscape mapping $u(x)$ and by doing so they