Dirichlet forms on Fractals (a preliminary report)

Alexander Teplyaev

University of Connecticut

Special Session on Analysis and Probability on Metric Spaces and Fractals
University of Wisconsin-Madison, Madison, WI

September 14-15, 2019

7th Cornell Conference on Analysis, Probability, and Mathematical Physics on Fractals: June 9–13, 2020

Department of Mathematics

7th Cornell Conference on Analysis, Probability, and Mathematical Physics on Fractals

Home » 7th Cornell Conference on Analysis, Probability, and Mathematical Physics on Fractals

Welcome!

Planning has begun for Fractals 7 (June 9-13, 2020). The purpose of this conference, held every three years, is to bring together mathematicians who are already working in the area of analysis and probability on fractals with students and researchers from related areas. Information will be posted here as it becomes available.

Financial support will be provided to a limited number of participants to cover the cost of housing in Cornell single dormitory rooms and partially support other travel expenses. Students and junior researchers from underrepresented groups in STEM are particularly encouraged to apply for travel funding. Well-established researchers are encouraged to use their own travel funding; the NSF expects that most funds will be expended on otherwise unfunded mathematicians.

Registration details will be publicized once available.

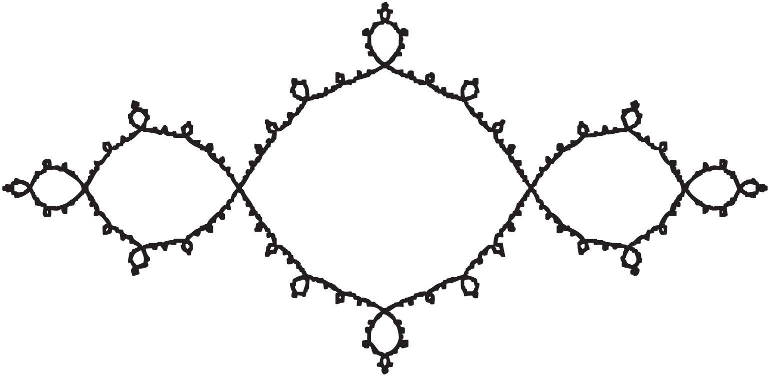
All general inquiries can be sent to: fractals_math@cornell.edu

Conference Organizers:

- Robert Strichartz (chair), Cornell University
- Patricia Alonso Ruiz, Texas A&M University
- Michael Hinz, Bielefeld University
- Luke Rogers, University of Connecticut
- Alexander Teplyaev, University of Connecticut

Plan of the talk:

- Introduction and examples of fractals
- Existence, uniqueness, heat kernel estimates
  - F-invariant Dirichlet forms
- Selected results: spectral analysis
- Open problems and further directions
  - Introduction
  - Classical Curl
  - Sierpinski carpets
  - Non-closable curl
  - Generalization
- Canonical diffusions on the pattern spaces of aperiodic Delone sets
  - (Patricia Alonso-Ruiz, Michael Hinz, Rodrigo Trevino, T.)
- BV and Besov spaces on fractals with Dirichlet forms
  - (Patricia Alonso-Ruiz, Fabrice Baudoin, Li Chen, Luke Rogers, Nages Shanmugalingam, T.)



The basilica Julia set, the Julia set of $z^2 - 1$ and the limit set of the basilica group of exponential growth (Grigorchuk, Żuk, Bartholdi, Virág, Nekrashevych, Kaimanovich, Nagnibeda et al.).

Asymptotic aspects of Schreier graphs and Hanoi Towers groups

Rostislav Grigorchuk[1], Zoran Šunić

Department of Mathematics, Texas A&M University, MS-3368, College Station, TX, 77843-3368, USA

Received 23 January, 2006; accepted after revision +++++

Presented by Étienne Ghys

Abstract

We present relations between growth, growth of diameters and the rate of vanishing of the spectral gap in Schreier graphs of automaton groups. In particular, we introduce a series of examples, called Hanoi Towers groups since they model the well known Hanoi Towers Problem, that illustrate some of the possible types of behavior. *To cite this article: R. Grigorchuk, Z. Šunić, C. R. Acad. Sci. Paris, Ser. I 344 (2006).*

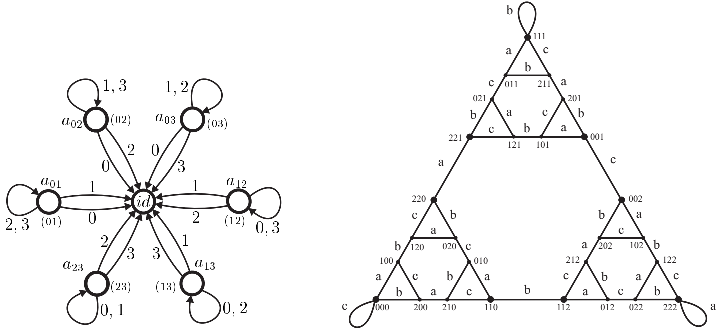
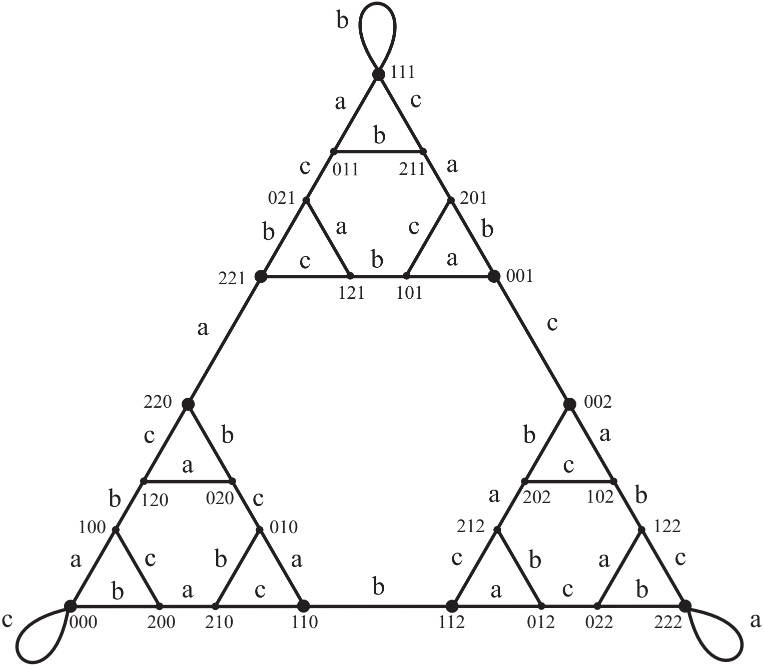


Figure 1. The automaton generating $H^{(4)}$ and the Schreier graph of $H^{(3)}$ at level 3 / L'automate engendrant $H^{(4)}$ et le graphe de Schreier de $H^{(3)}$ au niveau 3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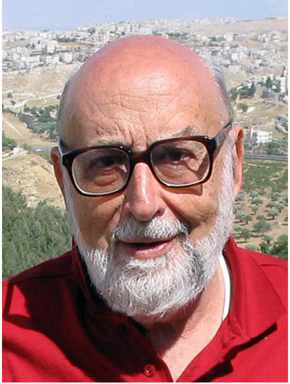
François Englert

From Wikipedia, the free encyclopedia

François Baron Englert (French: [ɑ̃glɛʁ]; born 6 November 1932) is a Belgian theoretical physicist and 2013 Nobel prize laureate (shared with Peter Higgs). He is Professor emeritus at the Université libre de Bruxelles (ULB) where he is member of the Service de Physique Théorique. He is also a Sackler Professor by Special Appointment in the School of Physics and Astronomy at Tel Aviv University and a member of the Institute for Quantum Studies at Chapman University in California. He was awarded the 2010 J. J. Sakurai Prize for Theoretical Particle Physics (with Gerry Guralnik, C. R. Hagen, Tom Kibble, Peter Higgs, and Robert Brout), the Wolf Prize in Physics in 2004 (with Brout and Higgs) and the High Energy and Particle Prize of the European Physical Society (with Brout and Higgs) in 1997 for the mechanism which unifies short and long range interactions by generating massive gauge vector bosons. He has made contributions in statistical physics, quantum field theory, cosmology, string theory and supergravity.[4] He is the recipient of the 2013 Prince of Asturias Award in technical and scientific research,

François Englert



François Englert in Israel, 2007

Nuclear Physics B280 [FS 18] (1987) 147–180
North-Holland, Amsterdam

METRIC SPACE-TIME AS FIXED POINT OF THE RENORMALIZATION GROUP EQUATIONS ON FRACTAL STRUCTURES

F. ENGLERT, J.-M. FRÈRE[1] and M. ROOMAN[2]

Physique Théorique, C.P. 225, Université Libre de Bruxelles, 1050 Brussels, Belgium

Ph. SPINDEL

Faculté des Sciences, Université de l'Etat à Mons, 7000 Mons, Belgium

Received 19 February 1986

We take a model of foamy space-time structure described by self-similar fractals. We study the propagation of a scalar field on such a background and we show that for almost any initial conditions the renormalization group equations lead to an effective highly symmetric metric at large scale.

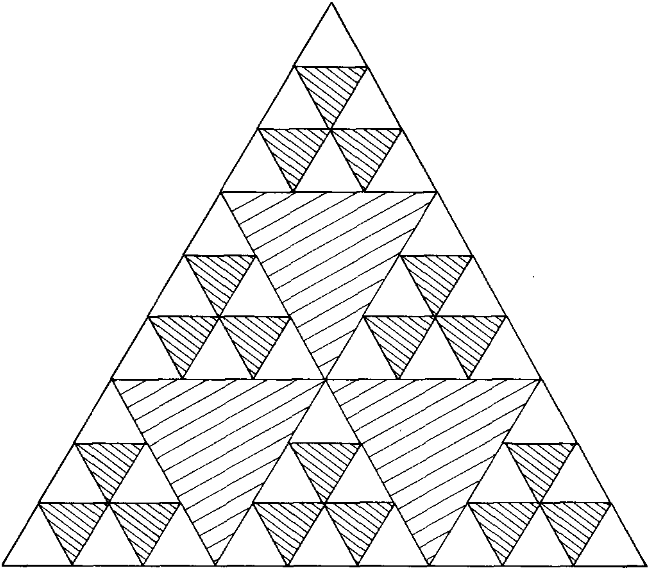


Fig. 1. The first two iterations of a 2-dimensional 3-fractal.

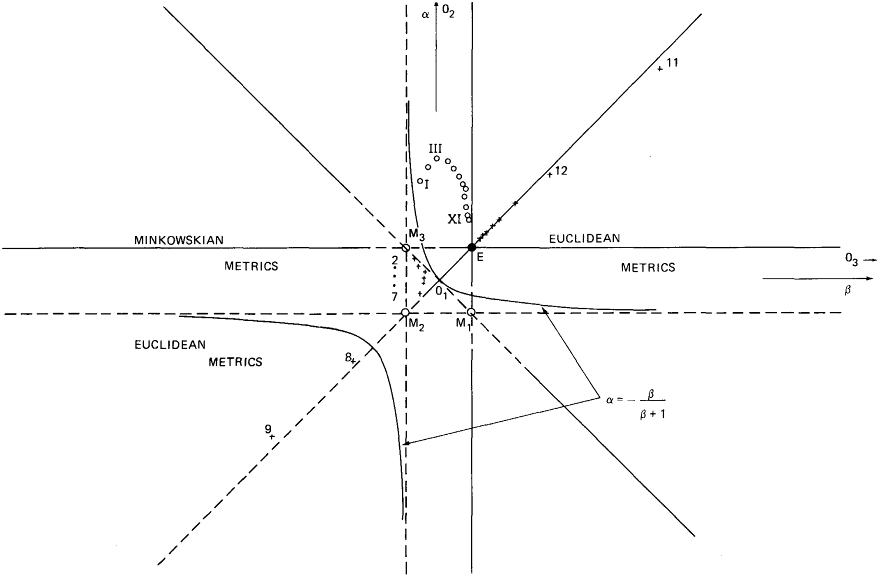


Fig. 5. The plane of 2-parameter homogeneous metrics on the Sierpinski gasket. The hyperbole $\alpha = -\beta/(\beta+1)$ separates the domain of euclidean metrics from minkowskian metrics and corresponds – except at the origin – to 1-dimensional metrics. M_1, M_2, M_3 denote unstable minkowskian fixed geometries while E corresponds to the stable euclidean fixed point. The unstable fixed points 0_1, 0_2 and 0_3 associated to 0-dimensional geometries are located at the origin and at infinity on the (α, β) coordinates axis. The six straight lines are subsets invariant with respect to the recursion relation but repulsive in the region where they are dashed. The first points of two sequences of iterations are drawn. Note that for one of them the 10th point ($\alpha = -56.4$, $\beta = -52.5$) is outside the frame of the figure.

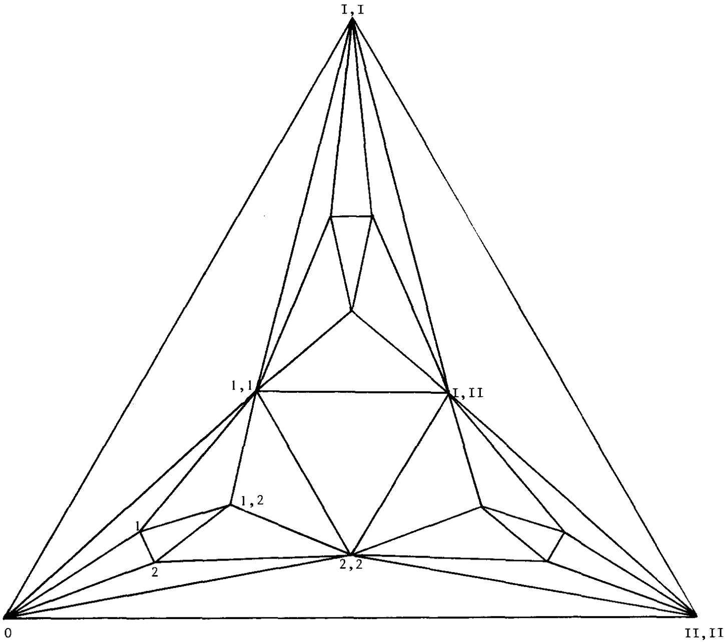


Fig. 10. A metrical representation of the two first iterations of a 2-dimensional 2-fractal corresponding to the euclidean fixed point. Vertices are labelled according to fig. 4.

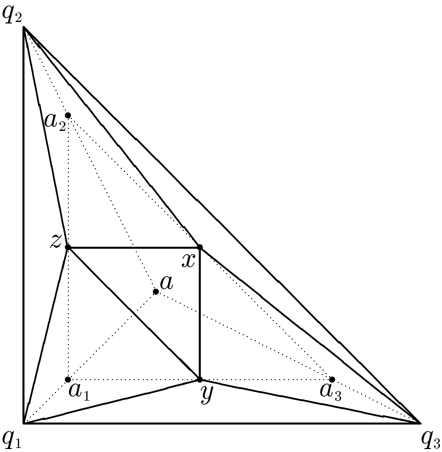


Figure 6.4. Geometric interpretation of Proposition 6.1.

PRL **95,** 171301 (2005) PHYSICAL REVIEW LETTERS week ending 21 OCTOBER 2005

The Spectral Dimension of the Universe is Scale Dependent

J. Ambjørn,[1,3,*] J. Jurkiewicz,[2,†] and R. Loll[3,‡]

[1]*The Niels Bohr Institute, Copenhagen University, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark*

[2]*Mark Kac Complex Systems Research Centre, Marian Smoluchowski Institute of Physics, Jagellonian University, Reymonta 4, PL 30-059 Krakow, Poland*

[3]*Institute for Theoretical Physics, Utrecht University, Leuvenlaan 4, NL-3584 CE Utrecht, The Netherlands*

(Received 13 May 2005; published 20 October 2005)

We measure the spectral dimension of universes emerging from nonperturbative quantum gravity, defined through state sums of causal triangulated geometries. While four dimensional on large scales, the quantum universe appears two dimensional at short distances. We conclude that quantum gravity may be "self-renormalizing" at the Planck scale, by virtue of a mechanism of dynamical dimensional reduction.

DOI: 10.1103/PhysRevLett.95.171301 PACS numbers: 04.60.Gw, 04.60.Nc, 98.80.Qc

Quantum gravity as an ultraviolet regulator?—A shared hope of researchers in otherwise disparate approaches to quantum gravity is that the microstructure of space and time may provide a physical regulator for the ultraviolet infinities encountered in perturbative quantum field theory. *tral dimension*, a diffeomorphism-invariant quantity obtained from studying diffusion on the quantum ensemble of geometries. On large scales and within measuring accuracy, it is equal to four, in agreement with earlier measurements of the large-scale dimensionality based on the other hand, the "short-distance spectral dimension," obtained by extrapolating Eq. (12) to $\sigma \rightarrow 0$ is given by

$$D_S(\sigma = 0) = 1.80 \pm 0.25, \tag{15}$$

and thus is compatible with the integer value two.

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Fractal space-times under the microscope: a renormalization group view on Monte Carlo data

Martin Reuter and Frank Saueressig

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E-mail: reuter@thep.physik.uni-mainz.de,
saueressig@thep.physik.uni-mainz.de

Abstract: The emergence of fractal features in the microscopic structure of space-time is a common theme in many approaches to quantum gravity. In this work we carry out a detailed renormalization group study of the spectral dimension d_s and walk dimension d_w associated with the effective space-times of asymptotically safe Quantum Einstein Gravity (QEG). We discover three scaling regimes where these generalized dimensions are approximately constant for an extended range of length scales: a classical regime where $d_s = d, d_w = 2$, a semi-classical regime where $d_s = 2d/(2+d), d_w = 2+d$, and the UV-fixed point regime where $d_s = d/2, d_w = 4$. On the length scales covered by three-dimensional Monte Carlo simulations, the resulting spectral dimension is shown to be in very good agreement with the data. This comparison also provides a natural explanation for the apparent puzzle between the short distance behavior of the spectral dimension reported from Causal Dynamical Triangulations (CDT), Euclidean Dynamical Triangulations (EDT), and Asymptotic Safety.

Keywords: Models of Quantum Gravity, Renormalization Group, Lattice Models of Gravity, Nonperturbative Effects

JHEP12(2011)012

Fractal space-times under the microscope: A Renormalization Group view on Monte Carlo data

Martin Reuter and Frank Saueressig

a classical regime where $d_s = d, d_w = 2$, a semi-classical regime where $d_s = 2d/(2+d), d_w = 2+d$, and the UV-fixed point regime where $d_s = d/2, d_w = 4$. On the length scales covered

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Norbert Wiener

From Wikipedia, the free encyclopedia

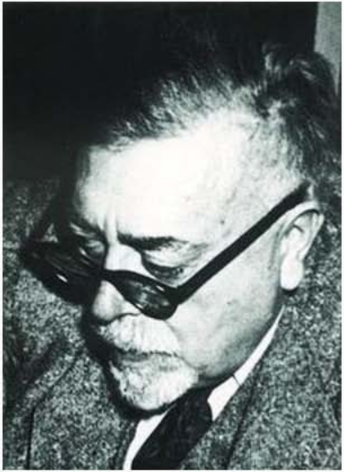
Norbert Wiener (November 26, 1894, Columbia, Missouri – March 18, 1964, Stockholm, Sweden) was an American mathematician.

A famous child prodigy, Wiener (*pronounced WEE-nur*) later became an early studier of stochastic and noise processes, contributing work relevant to electronic engineering, electronic communication, and control systems.

Wiener is wrongly regarded as the originator of cybernetics(see Ştefan Odobleja), a formalization of the notion of feedback, with many implications for engineering, systems control, computer science, biology, philosophy, and the organization of society.

Contents [hide]

Norbert Wiener



Born	November 26, 1894 Columbia, Missouri, U.S.
Died	March 18, 1964 (aged 69)

Andrey Kolmogorov

From Wikipedia, the free encyclopedia

Andrey Nikolaevich Kolmogorov (Russian: Андре́й Никола́евич Колмого́ров) (25 April 1903 – 20 October 1987) was a Soviet Russian mathematician, preeminent in the 20th century, who advanced various scientific fields, among them probability theory, topology, intuitionistic logic, turbulence, classical mechanics and computational complexity.

Andrey Kolmogorov

Born	25 April 1903 Tambov, Imperial Russia
Died	20 October 1987 (aged 84)

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Wacław Sierpiński

From Wikipedia, the free encyclopedia

Wacław Franciszek Sierpiński (Polish pronunciation: [ˈvatswaf franˈtɕiʂɛk ɕɛrˈpʲiɲskʲi]) (March 14, 1882, Warsaw — October 21, 1969, Warsaw) was a Polish mathematician. He was known for outstanding contributions to set theory (research on the axiom of choice and the continuum hypothesis), number theory, theory of functions and topology. He published over 700 papers and 50 books.

Three well-known fractals are named after him (the Sierpinski triangle, the Sierpinski carpet and the Sierpinski curve), as are Sierpinski numbers and the associated Sierpiński problem.

Contents [hide]

1 Education

Wacław Sierpiński

Born	March 14, 1882 Warsaw, Poland
Died	October 21, 1969 (aged 87)

ANALYSE MATHÉMATIQUE. — *Sur une courbe dont tout point est un point de ramification.* Note [1] de M. **W. Sierpinski**, présentée par M. Émile Picard.

Le but de cette Note est de donner un exemple d'une courbe cantorienne et jordanienne en même temps, dont tout point est un point de ramification. (Nous appelons *point de ramification* d'une courbe $\mathcal{C}$ un point p de cette courbe, s'il existe trois continus, sous-ensembles de $\mathcal{C}$, ayant deux à deux le point p et seulement ce point commun.)

Soient T un triangle régulier donné; A, B, C respectivement ses sommets : gauche, supérieur et droit. En joignant les milieux des côtés du triangle T, nous obtenons quatre nouveaux triangles réguliers (*fig.* 1), dont trois, T_0, T_1, T_2, contenant respectivement les sommets A, B, C, sont situés parallèlement à T et le quatrième triangle U contient le centre du triangle T; nous exclurons tout l'intérieur du triangle U.

Les sommets des triangles T_0, T_1, T_2 nous les désignerons respectivement :

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[1] Séance du 1er février 1915.

triangles U_0, U_1, U_2, situés parallèlement à U, dont les intérieurs seront

Fig. 1.

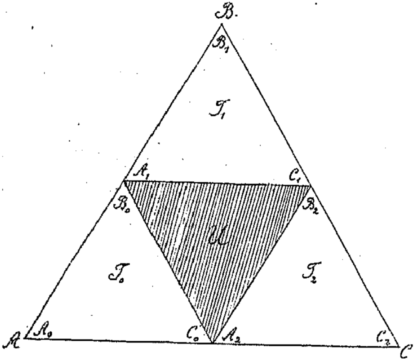
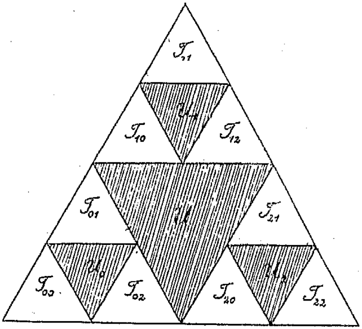


Fig. 2.



exclus (*fig.* 2). Avec chacun des triangles $T_{\lambda_1\lambda_2}$ procédons de même et ainsi

Fig. 3.

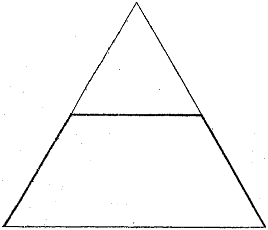
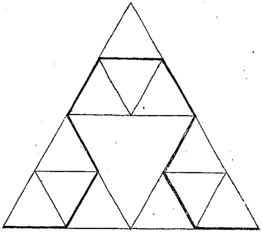


Fig. 4.



d'eux se rencontrent quatre segments différents, situés entièrement sur l'ensemble $\mathcal{S}$.

Donc, tous les points de la courbe $\mathcal{S}$, sauf peut-être les points A, B, C, sont ses points de ramification.

Pour obtenir une courbe dont tous les points sans exception sont ses

Fig. 5.

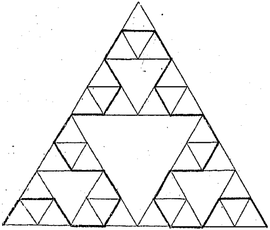
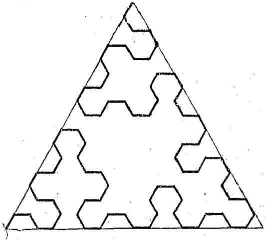


Fig. 6.



points de ramification, il suffit de diviser un hexagone régulier en six triangles réguliers et dans chacun d'eux inscrire une courbe $\mathcal{S}$.

Fig. 5.

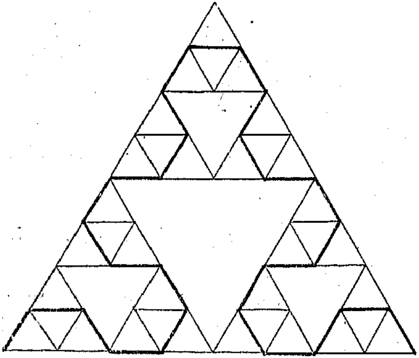
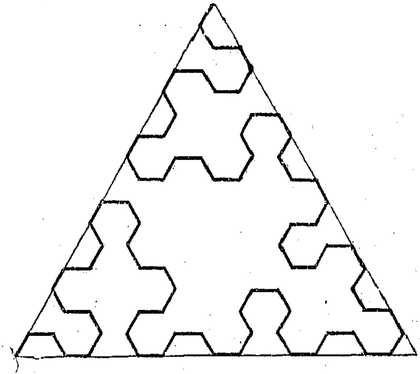
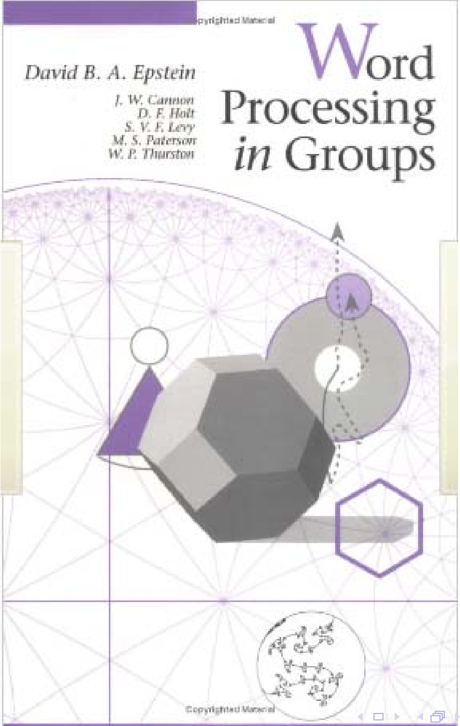


Fig. 6.

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Initial motivation

- R. Rammal and G. Toulouse, *Random walks on fractal structures and percolation clusters.* J. Physique Letters **44** (1983)
- R. Rammal, *Spectrum of harmonic excitations on fractals.* J. Physique **45** (1984)
- E. Domany, S. Alexander, D. Bensimon and L. Kadanoff, *Solutions to the Schrödinger equation on some fractal lattices.* Phys. Rev. B (3) **28** (1984)
- Y. Gefen, A. Aharony and B. B. Mandelbrot, *Phase transitions on fractals. I. Quasilinear lattices. II. Sierpiński gaskets. III. Infinitely ramified lattices.* J. Phys. A **16** (1983)**17** (1984)

Main early results

Sheldon Goldstein, *Random walks and diffusions on fractals.* Percolation theory and ergodic theory of infinite particle systems (Minneapolis, Minn., 1984–1985), IMA Vol. Math. Appl., 8, Springer

Summary: we investigate the asymptotic motion of a random walker, which at time n is at $X(n)$, on certain 'fractal lattices'. For the 'Sierpiński lattice' in dimension d we show that, as $L \to \infty$, the process $Y_L(t) \equiv X([(d+3)^L t])/2^L$ converges in distribution to a diffusion on the Sierpin'ski gasket, a Cantor set of Lebesgue measure zero. The analysis is based on a simple 'renormalization group' type argument, involving self-similarity and 'decimation invariance'. In particular,

$$|X(n)| \sim n^{\gamma},$$

where $\gamma = (\ln 2)/\ln(d+3)) \leqslant 2$.

Shigeo Kusuoka, *A diffusion process on a fractal.* Probabilistic methods in mathematical physics (Katata/Kyoto, 1985), 1987.

- M.T. Barlow, E.A. Perkins, *Brownian motion on the Sierpinski gasket.* (1988)
- M. T. Barlow, R. F. Bass, *The construction of Brownian motion on the Sierpiński carpet.* Ann. Inst. Poincaré Probab. Statist. (1989)
- S. Kusuoka, *Dirichlet forms on fractals and products of random matrices.* (1989)
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- J. Béllissard, *Renormalization group analysis and quasicrystals,* Ideas and methods in quantum and statistical physics (Oslo, 1988) Cambridge Univ. Press, 1992.
- M. Fukushima and T. Shima, *On a spectral analysis for the Sierpiński gasket.* (1992)
- J. Kigami, *Harmonic calculus on p.c.f. self–similar sets.* Trans. Amer. Math. Soc. **335** (1993)
- J. Kigami and M. L. Lapidus, *Weyl's problem for the spectral distribution of Laplacians on p.c.f. self-similar fractals.* Comm. Math. Phys. **158** (1993)

Main classes of fractals considered

- $[0, 1]$

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- **[0, 1]**
- Sierpiński gasket

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- **[0, 1]**
- Sierpiński gasket
- nested fractals

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- finitely ramified self-similar sets, possibly with various symmetries

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- finitely ramified self-similar sets, possibly with various symmetries
- infinitely ramified self-similar sets, with local symmetries, and with heat kernel estimates (such as the Generalized Sierpiński carpets)

Main classes of fractals considered

- $[0, 1]$
- Sierpiński gasket
- nested fractals
- p.c.f. self-similar sets, possibly with various symmetries
- finitely ramified self-similar sets, possibly with various symmetries
- infinitely ramified self-similar sets, with local symmetries, and with heat kernel estimates (such as the Generalized Sierpiński carpets)
- metric measure Dirichlet spaces, possibly with heat kernel estimates (MMD+HKE)

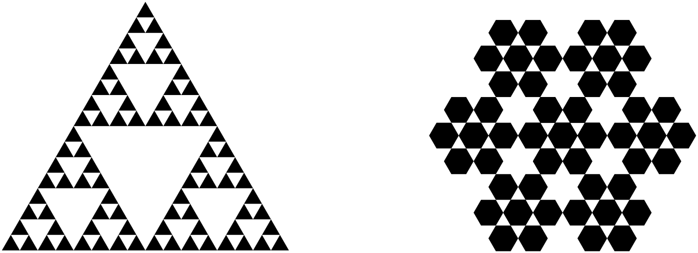

Figure: Sierpiński gasket and Lindstrøm snowflake (nested fractals), p.c.f., finitely ramified)

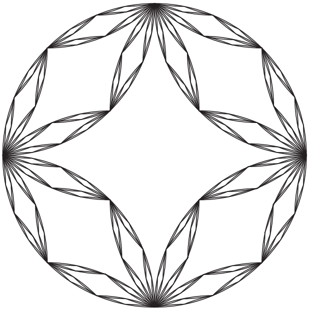

Figure: Diamond fractals, non-p.c.f., but finitely ramified

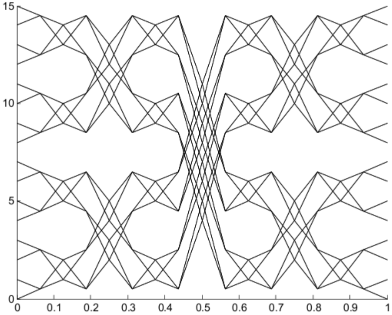

Figure: Laakso Spaces (Ben Steinhurst), infinitely ramified

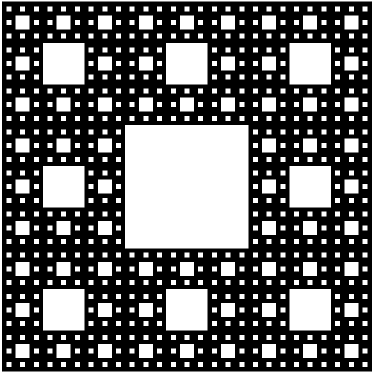

Figure: Sierpiński carpet, infinitely ramified

Existence, uniqueness, heat kernel estimates

Brownian motion:
Thiele (1880), Bachelier (1900)
Einstein (1905), Smoluchowski (1906)
Wiener (1920'), Doob, Feller, Levy, Kolmogorov (1930'),
Doeblin, Dynkin, Hunt, Ito ...

Wiener process in $\mathbb{R}^n$ satisfies $\frac{1}{n}\mathbb{E}|W_t|^2 = t$ and has a Gaussian transition density:

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

$$\textit{distance} \sim \sqrt{\textit{time}}$$

"Einstein space–time relation for Brownian motion"

De Giorgi-Nash-Moser estimates for elliptic and parabolic PDEs;

Li-Yau (1986) type estimates on a geodesically complete Riemannian manifold with $\boldsymbol{Ricci} \geqslant \mathbf{0}$:

$$p_t(x,y) \sim \frac{1}{V(x,\sqrt{t})} \exp\left(-c\frac{d(x,y)^2}{t}\right)$$

$$\textit{distance} \sim \sqrt{\textit{time}}$$

Gaussian:

$$p_t(x, y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

Li-Yau Gaussian-type:

$$p_t(x, y) \sim \frac{1}{V(x, \sqrt{t})} \exp\left(-c\frac{d(x, y)^2}{t}\right)$$

Sub-Gaussian:

$$p_t(x, y) \sim \frac{1}{t^{d_H/d_w}} \exp\left(-c\left(\frac{d(x, y)^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right)$$

$$distance \sim (time)^{\frac{1}{d_w}}$$

Brownian motion on $\mathbb{R}^d$: $\mathbb{E}|X_t - X_0| = ct^{1/2}$.

Anomalous diffusion: $\mathbb{E}|X_t - X_0| = o(t^{1/2})$, or (in regular enough situations),

$$\mathbb{E}|X_t - X_0| \approx t^{1/d_w}$$

with $d_w > 2$.

Here d_w is the so-called **walk dimension** (should be called "**walk index**" perhaps).

This phenomena was first observed by mathematical physicists working in the transport properties of disordered media, such as (critical) percolation clusters.

$$p_t(x,y) \sim \frac{1}{t^{d_H/d_w}} \exp\left(-c\frac{d(x,y)^{\frac{d_w}{d_w-1}}}{t^{\frac{1}{d_w-1}}}\right)$$

$$distance \sim (time)^{\frac{1}{d_w}}$$

$d_H =$ Hausdorff dimension

$\frac{1}{\gamma} = d_w =$ "walk dimension" (γ=diffusion index)

$\frac{2d_H}{d_w} = d_S =$ "spectral dimension" (diffusion dimension)

First example: Sierpiński gasket; Kusuoka, Fukushima, Kigami, Barlow, Bass, Perkins (mid 1980'—)

Theorem (Barlow, Bass, Kumagai (2006)).

Under natural assumptions on the MMD (geodesic Metric Measure space with a regular symmetric conservative Dirichlet form), the sub-Gaussian ***heat kernel estimates are stable under rough isometries***, *i.e. under maps that preserve distance and energy up to scalar factors.*

Gromov-Hausdorff $+$ ***energy***

Theorem. (Barlow, Bass, Kumagai, T. (1989–2010).) On any fractal in the class of generalized Sierpiński carpets there exists a unique, up to a scalar multiple, local regular Dirichlet form that is invariant under the local isometries.

Therefore there there is a unique corresponding symmetric Markov process and a unique Laplacian. Moreover, the Markov process is Feller and its transition density satisfies sub-Gaussian heat kernel estimates.

Main difficulties:

If it is not a cube in $\mathbb{R}^n$, then

- $d_S < d_H$, $d_w > 2$
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- in fact, we can not compute any functions of finite energy;
- Fourier and complex analysis methods seem to be not applicable.

The key result in the center of the proof: the classical elliptic Harnack inequality. Any harmonic function (a local energy minimizer) $u \geqslant 0$ satisfies

$$\sup_{B(x,R/2)} u \leq c_1 \inf_{B(x,R/2)} u$$

where **the constant c_1 is determined only by the geometry of the generalized Sierpiński carpet**.

Remark. This lemma is a hard mix of analysis (commutativity of certain geometric projections and the Laplacian) and probability (coupling).

Corollary. Harmonic functions are quasi-everywhere Hölder continuous.

Theorem. (Grigor'yan and Telcs, also [BBK])

On a MMD space the following are equivalent

- **(*VD*)**, **(*EHI*)** and **(*RES*)**
- **(*VD*)**, **(*EHI*)** and **(*ETE*)**
- **(*PHI*)**
- **(*HKE*)**

and the constants in each implication are effective.

Abbreviations: Metric Measure Dirichlet spaces, Volume Doubling, Elliptic Harnack Inequality, Exit Time Estimates, Parabolic Harnack Inequality, Heat Kernel Estimates.

Theorem 1. Let $(\mathcal{A}, \mathcal{F})$, $(\mathcal{B}, \mathcal{F})$ be **regular local conservative** irreducible Dirichlet forms on $L^2(F, m)$ and

$$(1+\delta)\mathcal{A}(u, u) \leq \mathcal{B}(u, u) \quad \text{for all } u \in \mathcal{F}$$

where $\delta > 0$. Then $(\mathcal{B}-\mathcal{A}, \mathcal{F})$ is a regular local conservative irreducible Dirichlet form on $L^2(F, m)$.

Technical lemma. If $\mathcal{E}$ is a local regular Dirichlet form with domain $\mathcal{F}$, then for any $f \in \mathcal{F} \cap L^\infty(F)$ we have $\Gamma(f, f)(A) = 0$, if $A = \{x \in F : f(x) = 0\}$ where $\Gamma(f, f)$ is the energy measure or the "square field operator"

$$\int_F g d\Gamma(f, f) = 2\mathcal{E}(f, fg) - \mathcal{E}(f^2, g), \ g \in \mathcal{F}_b.$$

Definition

Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(F, \mu)$. We say that $\mathcal{E}$ *is invariant with respect to all the local symmetries of* F (F-invariant or $\mathcal{E} \in \mathfrak{E}$) if

- (1) If $S \in \mathcal{S}_n(F)$, then $U_S R_S f \in \mathcal{F}$ for any $f \in \mathcal{F}$.
- (2) Let $n \geq 0$ and S_1, S_2 be any two elements of $\mathcal{S}_n$, and let Φ be any isometry of $\mathbb{R}^d$ which maps S_1 onto S_2. If $f \in \mathcal{F}^{S_2}$, then $f \circ \Phi \in \mathcal{F}^{S_1}$ and $\mathcal{E}^{S_1}(f \circ \Phi, f \circ \Phi) = \mathcal{E}^{S_2}(f, f)$ where

$$\mathcal{E}^S(g, g) = \frac{1}{m_F^n} \mathcal{E}(U_S g, U_S g)$$

  and $Dom(\mathcal{E}^S) = \{g : g \text{ maps } S \text{ to } \mathbb{R}, U_S g \in \mathcal{F}\}$.
- (3) $\mathcal{E}(f, f) = \sum_{S \in \mathcal{S}_n(F)} \mathcal{E}^S(R_S f, R_S f)$ for all $f \in \mathcal{F}$

Lemma

Let $(\mathcal{A}, \mathcal{F}_1), (\mathcal{B}, \mathcal{F}_2) \in \mathfrak{E}$ *with* $\mathcal{F}_1 = \mathcal{F}_2$ *and* $\mathcal{A} \geq \mathcal{B}$*. Then* $\mathcal{C} = (1 + \delta)\mathcal{A} - \mathcal{B} \in \mathfrak{E}$ *for any* $\delta > 0$.

$$\Theta f = \frac{1}{m_F^n} \sum_{S \in \mathcal{S}_n(F)} U_S R_S f.$$

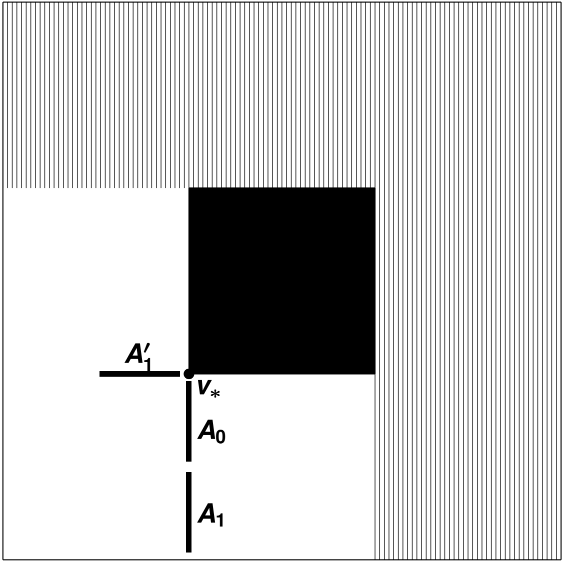
Note that Θ is a projection operator because $\Theta^2 = \Theta$. It is bounded on $C(F)$ and is an orthogonal projection on $L^2(F, \mu)$.

Lemma

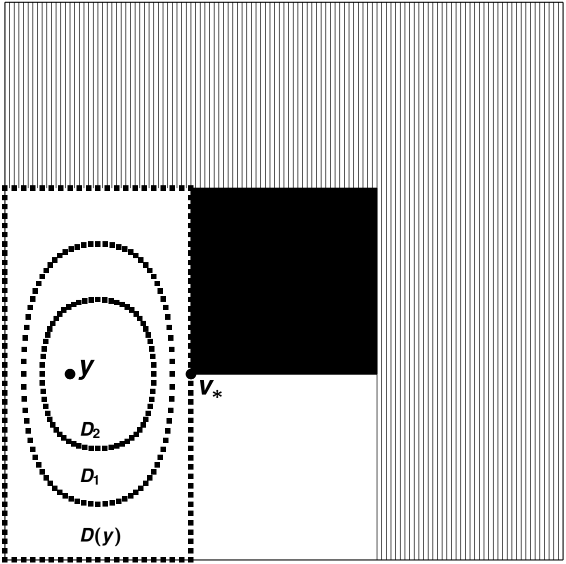
Assume that $\mathcal{E}$ is a local regular Dirichlet form on F, T_t is its semigroup, and $U_S R_S f \in \mathcal{F}$ whenever $S \in \mathcal{S}_n(F)$ and $f \in \mathcal{F}$. Then the following, for all $f, g \in \mathcal{F}$, are equivalent:

$$\text{(a):} \quad \mathcal{E}(f, f) = \sum_{S \in \mathcal{S}_n(F)} \mathcal{E}^S(R_S f, R_S f)$$

(b): $\mathcal{E}(\Theta f, g) = \mathcal{E}(f, \Theta g)$ (c): $T_t \Theta f = \Theta T_t f$



The half-face A_1 corresponds to a “slide move”,
and the half-face A_1' corresponds to a “corner move”,
analogues of the “corner” and “knight’s” moves in [BB89].



y
v_*
D_2
D_1
$D(y)$

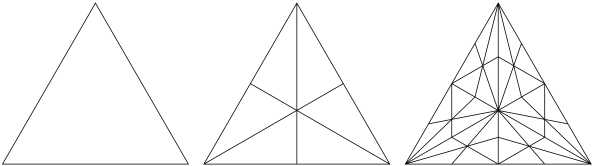


FIGURE 1. Barycentric subdivision of a 2-simplex, the graphs G_0^T, G_1^T and G_2^T.

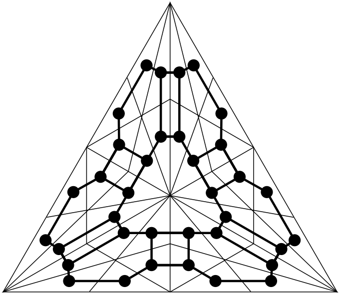


FIGURE 2. Adjacency (dual) graph G_2, in bold, and the barycentric subdivision graph pictured together with the thin image of G_2^T.

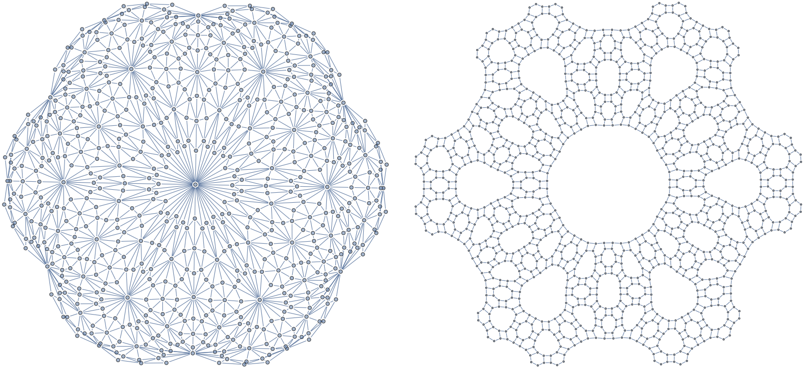


FIGURE 3. On the left: the graph G_4^T for barycentric subdivision of a 2-simplex. On the right: the adjacency (dual) graph G_4.

Theorem 1.1. *The resistances across graphs G_n^T and G_n^H (defined in Subsection 2.2) are reciprocals, that is $R_n^T = 1/R_n$, and the asymptotic limits*

$$\log \rho^T = \lim_{n\to\infty} \frac{1}{n} \log R_n^T \qquad \text{and} \qquad \log \rho = \lim_{n\to\infty} \frac{1}{n} \log R_n$$

exist (and $\rho^T = 1/\rho$). Furthermore, $2/3 \le \rho^T \le 4/5$ and $5/4 \le \rho \le 3/2$.

These estimates agree with the numerical experiments from [12], which suggest that there exists a limiting Dirichlet form on these fractals and estimates $\rho \approx 1.306$, and hence $\rho^T \approx 0.7655$.

Conjecture 1. *In the case $5/4 \le \rho \le 3/2$ ($\rho \approx 1.306$), we conjecture that the recent results of A. Grigor'yan, J. Hu, K.-S. Lau and M. Yang in [24–26, 28] can imply existence of the Dirichlet form.*

Conjecture 2. *Since $2/3 \le \rho^T \le 4/5 < 5/4 \le \rho \le 3/2$, we conjecture that there is essentially no uniqueness of the Dirichlet forms, spectral dimensions, resistance scaling factors etc for repeated barycentric subdivisions.*

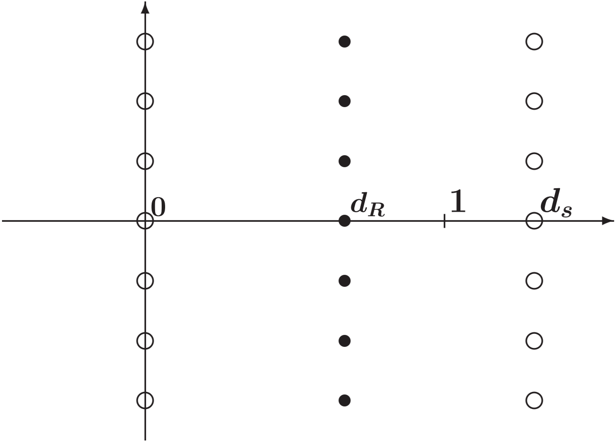
Selected results: spectral analysis

Theorem. (Derfel, Grabner, Vogl; T.; Kajino (2007–2011)) For a large class of **finitely ramified symmetric fractals**, which includes the Sierpiński gaskets, and may include the Sierpiński carpets, the spectral zeta function

$$\zeta(s) = \sum \lambda_j^{s/2}$$

has a meromorphic continuation from the half-pain $Re(s) > d_S$ to $\mathbb{C}$. Moreover, all the poles and residues are computable from the geometric data of the fractal. Here λ_j are the eigenvalues if the unique symmetric Laplacian.

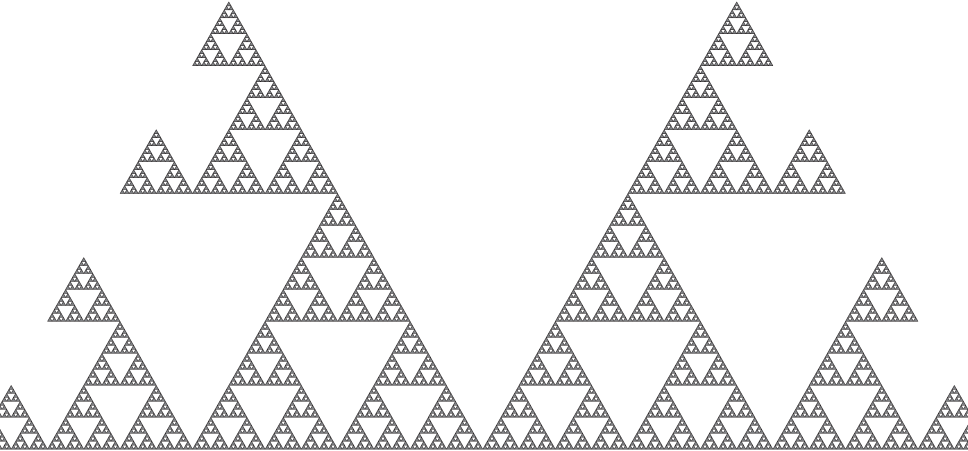
- Example: $\zeta(s)$ is the Riemann zeta function up to a trivial factor in the case when our fractal is $[0, 1]$.
- In more complicated situations, such as the Sierpiński gasket, there are infinitely many non-real poles, which can be called complex spectral dimensions, and are related to oscillations in the spectrum.



$$d_s = \frac{\log 9}{\log 5}$$

$$d_R = \frac{\log 4}{\log 5}$$

Poles (white circles) of the spectral zeta function of the Sierpiński gasket.



A part of an infinite Sierpiński gasket.

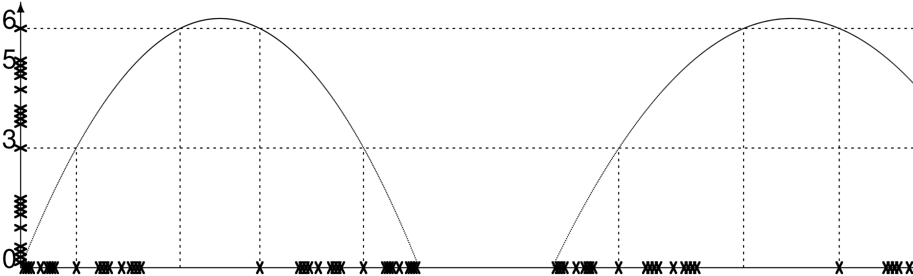
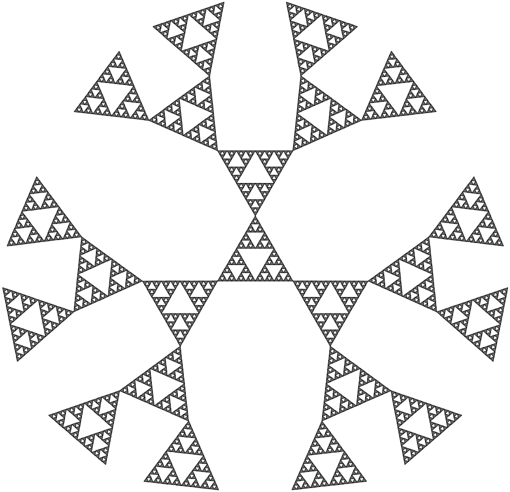
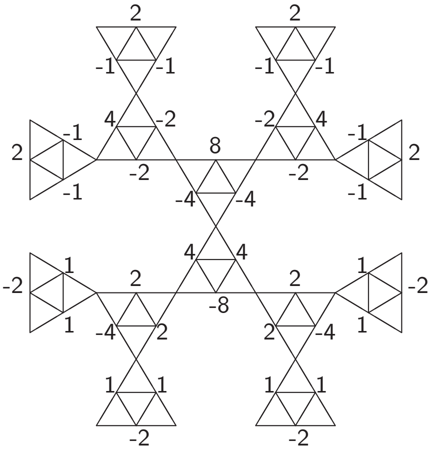


Figure: An illustration to the computation of the spectrum on the infinite Sierpiński gasket. The curved lines show the graph of the function $\mathfrak{R}(\cdot)$.

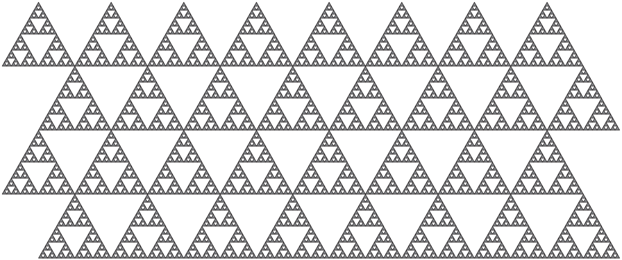
Theorem. (T. 1998, Quint 2009) On the Barlow-Perkins infinite Sierpiński fractafold the spectrum of the Laplacian consists of a **dense set of eigenvalues** $\mathfrak{R}^{-1}(\Sigma_0)$ **of infinite multiplicity** and a **singularly continuous component of spectral multiplicity one supported on** $\mathfrak{R}^{-1}(\mathcal{J}_R)$.

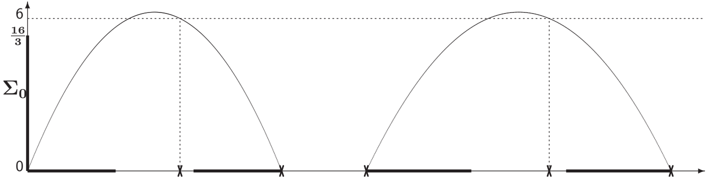


The Tree Fractafold.



An eigenfunction on the Tree Fractafold.





Theorem. (Strichartz, T. 2010) The Laplacian on the periodic triangular lattice finitely ramified Sierpiński fractal field consists of absolutely continuous spectrum and pure point spectrum. The **absolutely continuous spectrum is $\mathfrak{R}^{-1}[0, \frac{16}{3}]$.** The **pure point spectrum** consists of two infinite series of eigenvalues of infinite multiplicity. The spectral resolution is given in the main theorem.

Open problems

- Existence of self-similar diffusions on finitely ramified fractals? on any self-similar fractals? on limit sets of self-similar groups? is there a natural diffusion on any connected set with a finite Hausdorff measure?

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- PDEs involving derivatives, such as the Navier-Stokes equation.

More on motivations and connections to other areas: Cheeger, Heinonen, Koskela, Shanmugalingam, Tyson

J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces,* Geom. Funct. Anal. **9** (1999) J. Heinonen, *Lectures on analysis on metric spaces.* Universitext. Springer-Verlag, New York, 2001. J. Heinonen, *Nonsmooth calculus,* Bull. Amer. Math. Soc. (N.S.) **44** (2007)

J. Heinonen, P. Koskela, N. Shanmugalingam, J. Tyson, *Sobolev classes of Banach space-valued functions and quasiconformal mappings*. J. Anal. Math. 85 (2001)

Further directions (global)

- Fractal behavior of processes in algebra and geometry and probabilistic approach to stability under Hölder continuous transformations (Gromov, Perelman).

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- Computational tools for natural sciences, such as geophysics, chemistry, biology etc.

Recent/current exciting new developments (local)

- Mario Bonk/Dimitrios Ntalampekos:
  Potential theory on Sierpiński carpets with applications to uniformization
  (compare to Koskela/Zhou **Geometry and analysis of Dirichlet forms**: Sierpiński gasket in harmonic coordinates)

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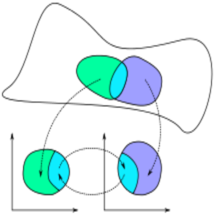
- Jun Kigami:
  Weighted partition of a compact metrizable space, its hyperbolicity, and Ahlfors regular conformal dimension

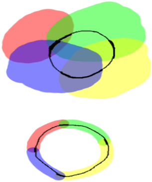
- **Long term goals:** Geometric analysis on **fractal** *Dirichlet metric measure spaces* and 'elements of intrinsic differential geometry', in particular **vector analysis and differential forms**.

- **Motivation:**
  - 'Items of Riemannian flavor already studied' (e.g. by Ambrosio, Bakry, Cheeger, Emery, Gigli, Hino, Kajino, Kigami, Koskela, Ledoux, Sturm, Zhou and others)
  - 'Items of deRham or Hodge type flavor hardly looked at', but in principle accessible using first order derivations (Cipriani, Sauvageot, Weaver and others)
  - Potential applications in physics (magnetic fields, fluid dynamics, optical waveguides) and data science.
  - A number of papers on sub-Riemannian and hypoelliptic setting, at UConn: Baudoin, Chousionis, Gordina.

Smooth manifold case

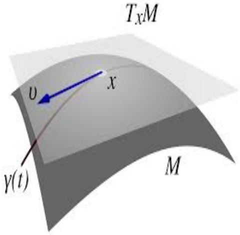
- dimension **dim** $\boldsymbol{M}$ of $\boldsymbol{M}$ defined as the dimension of the Euclidean space containing its charat images as open sets
- **dim** $\boldsymbol{M}$ equals its topological dimension $\mathbf{dim_{topo}}\ \boldsymbol{M}$





(Separable and metrizable X has $\mathbf{dim_{topo}}\ X = n$ if n is minimal value s.t. any finite open cover of X has refinement s.t. each $x \in X$ is contained in at most $n + 1$ sets of the refinement.)

- tangent space $T_x M$ at every $x \in M$ is n-dim vector space, similarly for cotangent space $T_x^* M$
- in particular, $\Lambda^k T_x^* M = \{0\}$ for $k > n$ (there are no nontrivial k-forms)
- **dim** $T_x M$ = **dim**$_{topo}$ M for all $x \in M$



Can talk about 'dimension of (co-)tangent spaces' using concepts of *AF-martingale dimension* $\mathbf{dim_{mart}}$ (Motoo, Watanabe, ...) resp. *index* of Dirichlet form (Hino):

- There is an equiv class of (mutually abs. cont.) minimal energy dominant measures m
- The index p of $(\mathcal{E}, \mathcal{F})$ is the smallest integer such that for any $N \in \mathbb{N}$ and any $f_1, ..., f_N \in \mathcal{F}$,

$$\mathbf{rank} \left(\frac{d\Gamma(f_i, f_j)}{dm}(x) \right)_{i,j=1}^{N} \leq p \quad \text{for } m\text{-a.e. } x \in X.$$

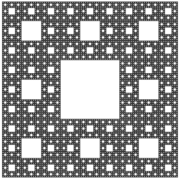
- Hino '08, '10: p well def (indep of choice of m) and $\mathbf{dim_{mart}} = p$

'Energy dominant measure': For any $\boldsymbol{f} \in \mathcal{F} \cap \boldsymbol{C_c(X)}$, energy measure $\boldsymbol{\Gamma(f)}$ satisfies $\boldsymbol{\Gamma(f)} << \boldsymbol{m}$; recall

$$\int_X \varphi \, d\Gamma(f) = \mathcal{E}(f\varphi, f) - \frac{1}{2}\mathcal{E}(f^2, \varphi), \ \ \varphi \in \mathcal{F} \cap C_c(X).$$

'Minimal': If $\boldsymbol{m'}$ has same property, then $\boldsymbol{m} << \boldsymbol{m'}$.

- Kusuoka '89: $\mathbf{dim}_{\mathbf{mart}} = \mathbf{1}$ for $\boldsymbol{d}$-dim standard Sierpinski gasket (although $\mathbf{dim}_{\boldsymbol{H}} = \frac{\log(d+1)}{\log 2}$ may be very large)
- Hino '08, '10, '13:
  - P.c.f. self-similar fractals: $\mathbf{dim}_{\mathbf{mart}} = \mathbf{1}$
  - Self-similar generalized Sierpinski carpets: $\mathbf{1} \leq \mathbf{dim}_{\mathbf{mart}} \leq \boldsymbol{d_s}$



- $\mathbf{dim_{mart}}$ may be interpreted as m-essential supremum of dimensions of tangent spaces in a measurable bundle sense (papers of Hino, also Eberle '99, H./Röckner/Teplyaev '13)

Examples

For $X = \mathbb{R}^n$ with $\mathcal{E}(f) = \int_{\mathbb{R}^n} (\nabla f)^2 dx$, $f \in H^1(\mathbb{R}^n)$, have $\mathbf{dim_{mart}} = n$.

Examples

For $X = M$ compact RMf with $\mathcal{E}(f) = \int_M (\nabla f)^2 \, \mathbf{dvol}$, $f \in H^1(M)$, have $\mathbf{dim_{mart}} = n$.

- Generally $\mathbf{dim_{topo}}$ and $\mathbf{dim_{mart}}$ may differ
- In particular: Topo one-dim spaces might carry nontrivial **2**-forms
  ... Somehow counterintuitive (would expect $\mathbf{dim_{mart}} \leq \mathbf{dim_{topo}}$)
  ... What happens ?
- Connected to behaviour of (analogs of) the exterior derivation

$$d : L^2(M, T^*M, \mathrm{dvol}) \to L^2(M, \Lambda^2 T^*M, \mathrm{dvol})$$

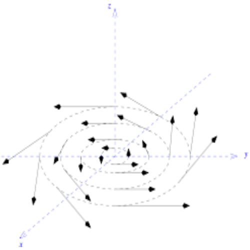
  taking **1**-forms into **2**-forms, $a_i\, dx^i \mapsto \frac{\partial a_i}{\partial x_j}\, dx^j \wedge dx^i$
- Phenomenon does not occur in classical theory
- For simplicity, illustrate issue for **curl**-operator

Curl of vector fields

- $U \subset \mathbb{R}^3$ open, connected, $v = (v_1, v_2, v_3) : U \to \mathbb{R}^3$ vector field
- **curl** $v = \nabla \times v = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}, \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}, \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right)$

If v velocity field of a fluid flow

- Small ball is made rotate by flow
- axis points in direction of vector field **curl** v (right hand rule)
- Angular speed is $\frac{1}{2}$ of length of **curl** v



- Connection to differential forms by duality argument: Given $v = (v_1, v_2, v_3)$, consider $\omega := v_i dx^i$. Then

$$
\begin{aligned}
d\omega &= \frac{\partial v_i}{\partial x^j}\, dx^j \wedge dx^i \\
&= \left(\frac{\partial v_3}{\partial x^2} - \frac{\partial v_2}{\partial x^3} \right) dx^2 \wedge dx^3 + \ldots
\end{aligned}
$$

- Two-dim curl: $U \subset \mathbb{R}^2$ open, connected and vector field $u = (u_1, u_2) : U \to \mathbb{R}^2$
- consider $v := (u_1, u_2, 0)$ then function $\mathbf{curl}\, u : U \to \mathbb{R}$,

$$
\mathbf{curl}\, u = \frac{\partial u_2}{\partial x}(x, y) - \frac{\partial u_1}{\partial y}(x, y),
$$

is third component of $\mathbf{curl}\, v = (0, 0, \mathbf{curl}\, u)$

- In terms of differential forms,

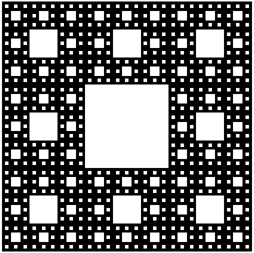
$$d(u_1(x,y)dx + u_2(x,y)dy) = \mathbf{curl}\, u(x,y)\; dx \wedge dy$$

- Can consider **curl** : $L^2(U, \mathbb{R}^2) \to L^2(U)$ as closed unbounded operator
- Next idea: Replace U by a generalized Sierpinski carpet with $2 = \dim_{\text{mart}} > \dim_{\text{topo}} = 1$

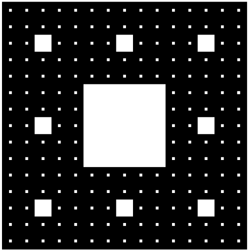
Sierpinski carpets

Consider non-self-similar generalized Sierpinski carpets studied by Mackay/Tyson/Wildrick '13.

- $\mathbf{a} = (a_1, a_2, \ldots)$ sequence of reals $a_i > 0$ s.t. $\frac{1}{a_i} > 1$ odd integer
- Rewrite $S_{\mathbf{a},0} := [0,1]^2$ as union of congruent closed subsquares of side lengths a_1, touching only at boundaries, remove middle one to get a set $S_{\mathbf{a},1}$
- Rewrite $S_{\mathbf{a},1}$ as union of congruent closed subsquares of side lengths $a_1 a_2$, touching only at boundaries, remove middle ones (w.r.t. the subsquares) to get a set $S_{\mathbf{a},2}$
- $S_{\mathbf{a}} := \bigcap_{m \geq 0} S_{\mathbf{a},m}$ generalized Sierpinski carpet associated with sequence $\mathbf{a}$



Standard self-similar carpet $S_{\mathbf{a}}$ with $\mathbf{a} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$



Non self-similar carpet $S_{\mathbf{a}}$ with $\mathbf{a} = (\frac{1}{3}, \frac{1}{5}, \frac{1}{7} \ldots)$

Proposition

(Mackay/Tyson/Wildrick'13)
If $\mathbf{a} \in l^2$ *then* $S_{\mathbf{a}}$ *has positive two dim Lebesgue measure and "all classical-type Sobolev inequalities".*

Examples

$a_n := \frac{1}{2n+1}$.

Let $\mathbf{a} \in l^2$ be fixed, write $S := S_{\mathbf{a}}$ and $L^2(S)$ for L^2-space on S w.r.t. two dim Lebesgue.

Energy form

Consider

$$\mathcal{E}_S(f) := \int_S (\nabla f(x,y))^2 \, d(x,y), \quad f \in C^1(\mathbb{R}^2).$$

Polarization yields bilinear form.

The form $(\mathcal{E}_S, C^1(\mathbb{R}^2))$ is closable, and its closure $(\mathcal{E}_S, \mathcal{D}_S)$ is a strongly local regular Dirichlet form on $L^2(S)$.

(Follows as in Koskela/Shanmugalingam/Tyson '04, Shanmugalingam '00; Newtonian Sobolev spaces; for $f \in C^1(\mathbb{R}^2)$ the function $|\nabla f|$ is minimal upper gradient of f.)

For a vector field $\boldsymbol{u} = (\boldsymbol{u_1}, \boldsymbol{u_2})$ with $\boldsymbol{u_1}, \boldsymbol{u_2} \in \boldsymbol{C^1}(\mathbb{R}^2)$, **curl** $\boldsymbol{u}$ is a continuous function and can be restricted to $\boldsymbol{S}$, and

$$(\mathbf{curl}\, \boldsymbol{u})|_{\boldsymbol{S}} \in \boldsymbol{L^2}(\boldsymbol{S})$$

Therefore: May view **curl** as densely defined unbounded operator

$$\mathbf{curl} : \boldsymbol{L^2}(\boldsymbol{S}, \mathbb{R}^2) \to \boldsymbol{L^2}(\boldsymbol{S})$$

with domain $\boldsymbol{C^1}(\mathbb{R}^2, \mathbb{R}^2)$

Slightly reformulated: Endow **curl** with abstract domain **dom curl** and let **curl*** be its adjoint with domain **dom(curl*****)**

Theorem (Hinz/T. '15)

1-dim Hodge-Helmholtz composition holds (despite that $\mathbf{dim}_H = 2$*).*

Theorem (Hinz/T. '17)

Let $\mathbf{a} \in l^2$ *be such that*

$$\lim_{n\to\infty} \frac{a_1 \cdots a_{n-1}}{a_n} = 0$$

If **dom curl** *contains all smooth vector fields, then* $\mathbf{dom(curl^*)} \subset L^2(S)$ *is* $\{0\}$ *and, in particular, the operator* **(curl, dom curl)** *is not closable.*

('**a** decays fast enough but not too fast'.)

Examples

$a_n := \frac{1}{2n+1}$.

Proof (by contradiction)

Suppose $\mathbf{0} \neq \boldsymbol{u} \in \mathbf{dom}(\mathbf{curl}^*) \subset \boldsymbol{L}^2(\boldsymbol{S})$ and $\mathbf{curl}^*\, \boldsymbol{u} = \boldsymbol{w} \in \boldsymbol{L}^2(\boldsymbol{S}, \mathbb{R}^2)$. Then ex. smooth function $\boldsymbol{f}$ such that $\langle \boldsymbol{u}, \boldsymbol{f} \rangle_{L^2(S)} > \mathbf{0}$.

Claim: Can construct sequence $(\boldsymbol{v_n})_n$ of smooth vector fields $\boldsymbol{v_n}$ s.t.

(a) $\lim_n \mathbf{curl}\, \boldsymbol{v_n} = \boldsymbol{f}$ in $\boldsymbol{L}^2(\boldsymbol{S})$

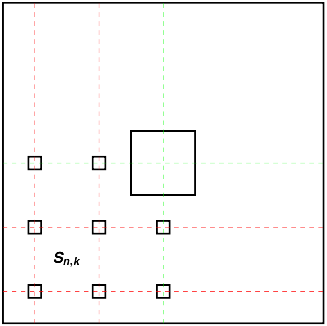
(b) $\lim_n \boldsymbol{v_n} = \mathbf{0}$ in $\boldsymbol{L}^2(\boldsymbol{S}, \mathbb{R}^2)$.

If so, then

$$\begin{aligned} \mathbf{0} = \lim_n \langle \boldsymbol{w}, \boldsymbol{v_n} \rangle_{L^2(S,\mathbb{R}^2)} = \lim_n \langle \mathbf{curl}^*\, \boldsymbol{u}, \boldsymbol{v_n} \rangle_{L^2(S,\mathbb{R}^2)} &= \lim_n \langle \boldsymbol{u}, \mathbf{curl}\, \boldsymbol{v_n} \rangle_{L^2(S)} \\ &= \langle \boldsymbol{u}, \boldsymbol{f} \rangle_{L^2(S)} \\ &> \mathbf{0}, \end{aligned}$$

what cannot be true. Suffices to show claim.

Cover S by compact subsets $S_{n,k}$ obtained by taking parallels to the axes through the midpoints of all holes of size $\delta_n = a_1 \cdots a_n$. Intersections are Cantor sets and $\mathbf{diam}\, S_{n,k} \le \sqrt{2}\delta_{n-1}$.

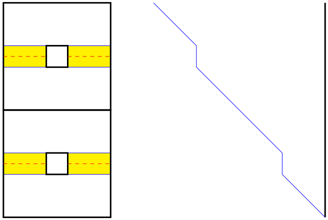


Step 1: We show how to choose small nbhs $U_{n,k}$ of the boundaries of the sets $S_{n,k}$ and construct sequence of energy finite functions g_n s.t.

(i) ∇g_n arbitrarily close to vector field $(0, 1)$ in $L^2(S, \mathbb{R}^2)$

(ii) each g_n is locally constant on each $U_{n,k}$.

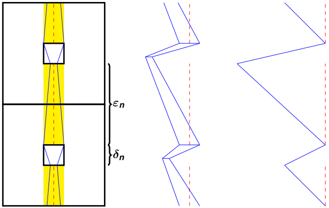
- For fixed n, consider Cantor-set parts of boundaries of the $S_{n,k}$ parallel to x-axis, let F_n be union of their vertical parallel sets
- Let φ_n be a continuous function that is constant in x on S, constant in y on F_n, and on each connected component of $S \setminus F_n$ differs from $g(x, y) := y$ by an additive constant.



Each φ_n is restriction to S of a Lipschitz function, hence of finite energy. Moreover

$$\lim_n \mathcal{E}(g - \varphi_n) = \lim_n \int_{F_n} (\nabla(g - \varphi_n))^2 \, d\lambda^2$$
$$= \lim_n \lambda^2(F_n) \leq \lim_n \frac{\delta_n}{a_1 \cdots a_{n-1}} = \lim_n a_n = 0.$$

Now consider vertical Cantor set parts of the boundaries of the sets $S_{n,k}$. Connect two vertically adjacent holes by rectangles with horizontal side length δ_n and vertical side length $\varepsilon_n := (1 - a_n)(a_1 \cdots a_{n-1})$. Inscribe trapezoids with lower edge length δ_n and upper edge length $\delta_{n/2}$ ...

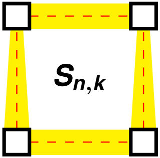


Let ψ_n be the function on $[0,1]^2$ created by putting little tents over each rectangle such that ψ_n is zero on left, right and lower edge of each rectangle, has value ε_n on the upper (short) edge of the trapezoid and is linear in between. For boundary pieces proceed similarly by 'mirroring to the outside'. Number of such tents is $\leq \frac{2}{a_1 \cdots a_{n-1}}$.

The functions $g_n := \varphi_n - \psi_n$ now satisfy

$$\lim_n \mathcal{E}_S(g - g_n)^{1/2} \leq \lim_n \mathcal{E}_S(g - \varphi_n)^{1/2} + \lim_n \mathcal{E}_S(\psi_n)^{1/2} = 0,$$

what is (i). Each g_n is locally constant on the neighborhood $U_{n,k}$ of $S_{n,k}$ consisting of two rectangles and two trapezoids (with modifications at the boundary of S), what shows (ii).



Step 2: Let $f_{n,k}$ be one of the values of the function f on $S_{n,k}$, and let $x_{n,k}$ be one of the values of the x coordinate on $S_{n,k}$. There exists a sequence of smooth functions h_n such that

$$\|h_n\|_{\sup} \leqslant a_1 \cdots a_{n-1} \|f\|_{\sup}$$

and on each set $S_{n,k} \setminus U_{n,k}$ we have

$$h_n(x, y) = f_{n,k}(x - x_{n,k}).$$

Then we define

$$v_n = h_n \nabla g_n.$$

Obviously (b) is satisfied, $\lim_n v_n = 0$ in $L^2(S, \mathbb{R}^2)$.

Strongly local forms on compact spaces

This is a part of the broader program to develop **probabilistic, spectral and vector analysis on singular spaces** by **carefully building approximations by graphs or manifolds**.

X compact metric space, μ finite Radon measure, full support, $(\mathcal{E}, \mathcal{F})$ strongly local regular Dirichlet form. We consider *differential forms with respect to a 'coordinate sequence' and an energy dominant measure*.

Theorem

(Hinz/T. '15)
Suppose that X is topologically one-dimensional. Then, under some natural conditions, either the martingale dimension of $(\mathcal{E}, \mathcal{F})$ is one or $(\partial_1, \mathcal{F} \otimes \mathcal{A}_{\mathrm{Lip}})$ is not closable.

Canonical diffusions on the pattern spaces of aperiodic Delone sets (Patricia Alonso-Ruiz, Michael Hinz, Rodrigo Trevino, T.)

A subset $\Lambda \subset \mathbb{R}^d$ is a **Delone set** if it is **uniformly discrete**:

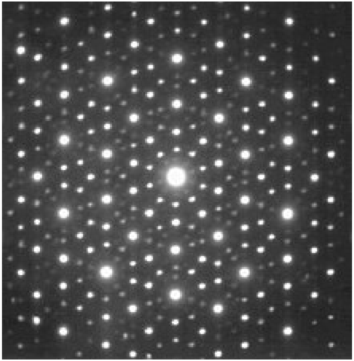
$$\exists \varepsilon > 0 : \; |\vec{x} - \vec{y}| > \varepsilon \quad \forall \vec{x}, \vec{y} \in \Lambda$$

and relatively dense:

$$\exists R > 0 : \; \Lambda \cap B_R(\vec{x}) \neq \varnothing \quad \forall \vec{x} \in \mathbb{R}^d.$$

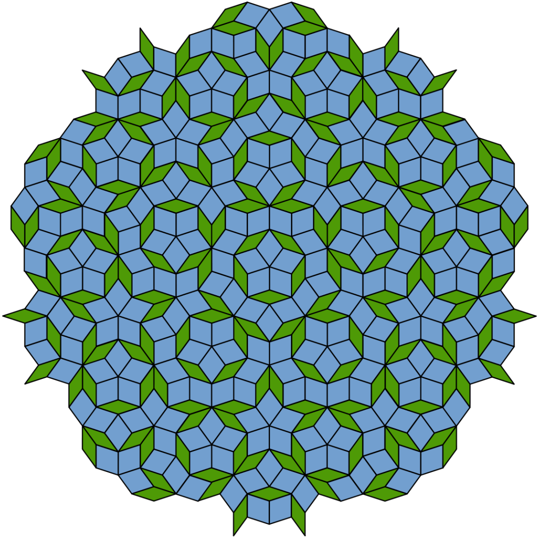
A Delone set has **finite local complexity** if $\forall R > 0 \exists$ finitely many clusters $P_1, \ldots, P_{n_R}$ such that for any $\vec{x} \in \mathbb{R}^d$ there is an i such that the set $B_R(\vec{x}) \cap \Lambda$ is translation-equivalent to P_i. A Delone set Λ is **aperiodic** if $\Lambda - \vec{t} = \Lambda$ implies $\vec{t} = \vec{0}$. It is **repetitive** if for any cluster $P \subset \Lambda$ there exists $R_P > 0$ such that for any $\vec{x} \in \mathbb{R}^d$ the cluster $B_{R_P}(\vec{x}) \cap \Lambda$ contains a cluster which is translation-equivalent to P. These sets have applications in crystallography (≈ 1920), coding theory, approximation algorithms, and the theory of quasicrystals.

Electron diffraction picture of a Zn-Mg-Ho quasicrystal



Aperiodic tilings were discovered by mathematicians in the early 1960s, and, some twenty years later, they were found to apply to the study of natural quasicrystals (1982 Dan Shechtman, 2011 Nobel Prize in Chemistry).

Penrose tiling



pattern space of a Delone set

Let $\Lambda_0 \subset \mathbb{R}^d$ be a **Delone set**. The **pattern space (hull)** of Λ_0 is the closure of the set of translates of Λ_0 with respect to the metric ϱ, i.e.

$$\Omega_{\Lambda_0} = \overline{\left\{\varphi_{\vec{t}}\,(\Lambda_0) : \vec{t} \in \mathbb{R}^d\right\}}.$$

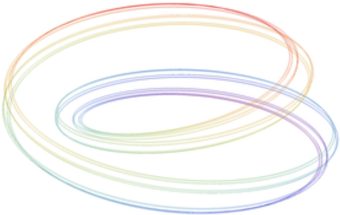
Definition

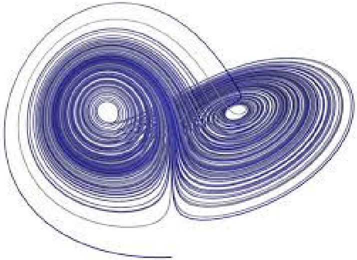
Let $\Lambda_0 \subset \mathbb{R}^d$ be a Delone set and denote by $\varphi_{\vec{t}}\,(\Lambda_0) = \Lambda_0 - \vec{t}$ its translation by the vector $\vec{t} \in \mathbb{R}^d$. For any two translates Λ_1 and Λ_2 of Λ_0 define $\varrho(\Lambda_1, \Lambda_2) = \inf\{\varepsilon > 0 : \exists\, \vec{s}, \vec{t} \in B_\varepsilon(\vec{0}) : B_{\frac{1}{\varepsilon}}(\vec{0}) \cap \varphi_{\vec{s}}(\Lambda_1) = B_{\frac{1}{\varepsilon}}(\vec{0}) \cap \varphi_{\vec{t}}(\Lambda_2)\} \wedge 2^{-1/2}$

Assumption

The action of $\mathbb{R}^d$ on Ω is uniquely ergodic:
Ω is a compact metric space with the unique $\mathbb{R}^d$-invariant probability measure μ.

Topological solenoids
(similar topological features as the pattern space Ω):





Theorem

(i) *If* $\vec{W} = (\vec{W}_t)_{t\geq 0}$ *is the standard Gaussian Brownian motion on* $\mathbb{R}^d$, *then for any* $\Lambda \in \Omega$ *the process* $X_t^\Lambda := \varphi_{\vec{W}_t}(\Lambda) = \Lambda - \vec{W}_t$ *is a conservative Feller diffusion on* (Ω, ϱ).

(ii) *The semigroup* $P_t f(\Lambda) = \mathbb{E}[f(X_t^\Lambda)]$ *is*

self-adjoint on L^2_μ, Feller but not strong Feller.

Its associated Dirichlet form is regular, strongly local, irreducible, recurrent, and has Kusuoka-Hino dimension d.

(iii) *The semigroup* $(P_t)_{t>0}$ **does not admit heat kernels with respect to** μ. *It does have Gaussian heat kernel with respect to the not-*σ*-finite (no Radon-Nykodim theorem) pushforward measure* λ^d_Ω

$$p_\Omega(t, \Lambda_1, \Lambda_2) = \begin{cases} p_{\mathbb{R}^d}(t, h_{\Lambda_1}^{-1}(\Lambda_2)) & \text{if } \Lambda_2 \in \mathrm{orb}(\Lambda_1), \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

(iv) **There are no semi-bounded or L^1 harmonic functions (Liouville-type)**.

no classical inequalities

Useful versions of the Poincare, Nash, Sobolev, Harnack inequalities DO NOT HOLD,
except in orbit-wise sense.

spectral properties

Theorem

*The unitary **Koopman operators** $U_{\vec{t}}$ on $L^2(\Omega, \mu)$ defined by $U_{\vec{t}} f = f \circ \varphi_{\vec{t}}$ commute with the heat semigroup*

$$\boxed{U_{\vec{t}} P_t = P_t U_{\vec{t}}}$$

hence commute with the Laplacian $\mathbf{\Delta}$, and all spectral operators, such as the unitary Schrödinger semigroup.

... hence we may have continuous spectrum (no eigenvalues) under some assumptions even though μ is a probability measure on the compact set Ω.

Under special conditions P_t is connected to the evolution of a **Phason**:

"Phason is a quasiparticle existing in quasicrystals due to their specific, quasiperiodic lattice structure. Similar to phonon, phason is associated with atomic motion. However, whereas phonons are related to translation of atoms, phasons are associated with atomic rearrangements. As a result of these rearrangements, waves, describing the position of atoms in crystal, change phase, thus the term "phason" (from the wikipedia)".

Phason evolution

Corollary

The unitary **Koopman operators** $U_{\vec{t}}$ *on* $L^2(\Omega, \mu)$ *defined by* $U_{\vec{t}} f = f \circ \varphi_{\vec{t}}$ *commute with the heat semigroup*

$$\boxed{U_{\vec{t}} P_t = P_t U_{\vec{t}}}$$

hence commute with the Laplacian Δ*, and all spectral operators, including the unitary* **Schrödinger semigroup** $e^{i\Delta t}$

$$\boxed{U_{\vec{t}} e^{i\Delta t} = e^{i\Delta t} U_{\vec{t}}}$$

Recent physics work on phason ("accounts for the freedom to choose the origin"): Topological Properties of Quasiperiodic Tilings (Yaroslav Don, Dor Gitelman, Eli Levy and Eric Akkermans Technion Department of Physics)
`https://phsites.technion.ac.il/eric/talks/`
J. Bellissard, A. Bovier, and J.-M. Chez, Rev. Math. Phys. 04, 1 (1992).

Helmholtz, Hodge and de Rham

Theorem

Assume $d = 1$. Then the space $L^2(\Omega, \mu, \mathbb{R}^1)$ admits the orthogonal decomposition

$$L^2(\Omega, \mu, \mathbb{R}^1) = \operatorname{Im} \nabla \oplus \mathbb{R}(dx). \tag{2}$$

In other words, the L^2-cohomology is 1-dimensional, which is surprising because the **de Rham cohomology is not one dimensional**.

M. Hinz, M. Röckner, T., Vector analysis for Dirichlet forms and quasilinear PDE and SPDE on fractals, Stoch. Proc. Appl. (2013). M. Hinz, T., Local Dirichlet forms, Hodge theory, and the Navier-Stokes equation on topologically one-dimensional fractals, Trans. Amer. Math. Soc. (2015,2017).

Lorenzo Sadun. Topology of tiling spaces, volume 46 of University Lecture Series. American Mathematical Society, Providence, RI, 2008. Johannes Kellendonk, Daniel Lenz, and Jean Savinien. Mathematics of aperiodic order, volume 309. Springer, 2015.

BV and Besov spaces on fractals with Dirichlet forms (Patricia Alonso-Ruiz, Fabrice Baudoin, Li Chen, Luke Rogers, Nages Shanmugalingam, T.)

Open question: on the Sierpinski carpet

$$\kappa = d_W - d_H + d_{tH} - 1 = d_W - d_H + \frac{\log 2}{\log 3}$$

would give the optimal Hölder exponent for harmonic functions?

[*Strongly supported by numerical results: L.Rogers et al*]

References: **Besov class via heat semigroup on Dirichlet spaces**
I: Sobolev type inequalities
arXiv:1811.04267
II: BV functions and Gaussian heat kernel estimates
arXiv:1811.11010
III: BV functions and sub-Gaussian heat kernel estimates
arXiv:1903.10078

For nested fractals we do have $\kappa = d_W - d_H > 0$. Moreover, a set has finite perimeter if and only if it has finite boundary, $P(E) \sim \#(\partial E)$.

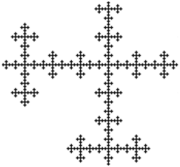
Theorem (research in progress)

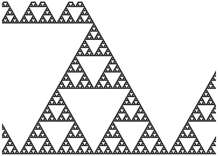
$f \in BV$ iff ∇f is a "vector valued Radon measure".

This is understood in the distributional sense (Hinz, Rogers, Strichartz et al)

Corollary

1. *on the Vicsek set, any BV function is $\mathbb{R}^1$-BV along each geodesic path.*
2. *on the Sierpiński gasket, any BV function is discontinuous.*





end of the talk :-)

Thank you!

reminder: 7th Cornell Conference on Analysis, Probability, and Mathematical Physics on Fractals: June 9–13, 2020