Please write Your name: ______________________

Show all work. You should either write at a sentence explaining your reasoning, or annotate your math work with brief explanations. There is no need to simplify, and no calculators are needed.

In this quiz use the notation $\Phi(x)$ for the distribution function for $\mathcal{N}(0,1)$, that is

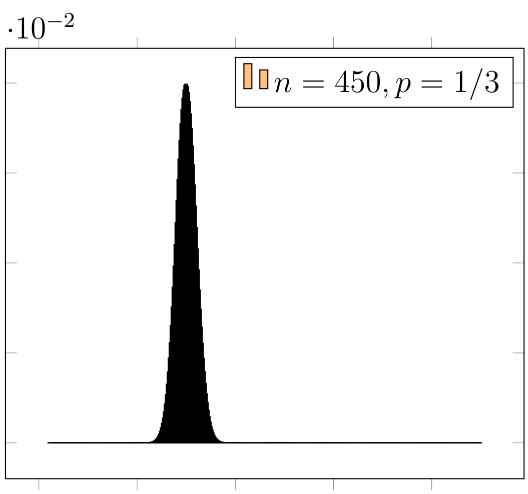
$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy = \mathbb{P}(Z < x)$$

where Z is the standard normal random variable. You do not need a table of values of Φ.

(1) When a die is rolled, you win \$1 if the outcome is divisible by 3, or \$0 otherwise. This means that you win \$1 for each roll which is a 3, and for each roll which is a 6, but win nothing for other results. Let X be the number of dollars you win after 450 rolls. What is the mean and the standard deviation of X?
Answer: $\mathbb{E}X = \mu = 450 \cdot 1/3 = 150 \quad \mathrm{SD}(X) = \sigma = \sqrt{np(1-p)} = \sqrt{450 \cdot \frac{1}{3} \cdot \frac{2}{3}} = \sqrt{\frac{900}{3}} = \sqrt{100} = 10$

(2) Estimate the probability that $X > 150$ using the normal approximation.
Answer: $\mathbb{P}(X > 150) \approx \mathbb{P}(150 + 10Z > 150) = \mathbb{P}(Z > 0) = 1/2$
A more accurate answer: $\mathbb{P}(X > 150) \approx \mathbb{P}(150 + 10Z > 150.5) = \mathbb{P}(Z > 0.05) = 1 - \Phi(0.05)$ In the table this is $1 - 0.51994 = 0.48006$

(3) Estimate the probability that $X > 250$ using the normal approximation.
Answer:
$\mathbb{P}(X > 150) \approx \mathbb{P}(150 + 10Z > 250) = \mathbb{P}(Z > 10)$
In the wikipedia table
https://en.wikipedia.org/wiki/Standard_normal_table#Complementary_cumulative
this is $7.61985E{-24} = 0.00000000000000000000000761985$
A more accurate answer: $\mathbb{P}(X > 250) \approx \mathbb{P}(150 + 10Z > 250.5) = \mathbb{P}(Z > 10.05) = 1 - \Phi(10.05)$ which is an even smaller number.



(4) Find a formula for $\mathbb{P}(0 \le X \le 3)$ if X is $\mathcal{N}(-1, 4)$. Your answer should include Φ twice.
Answer: $\mathbb{P}(0 \le X \le 3) = \mathbb{P}(0 \le -1 + 2Z \le 3) = \mathbb{P}(1/2 \le Z \le 2) = \Phi(2) - \Phi(1/2)$
In the table this is $0.97725 - 0.69146 = 0.28579$

(5) If a coin is tosses 16 times, and X is the number of heads, what is the normal approximation for $\mathbb{P}(X > 12)$ using the normal approximation. Your answer should include Φ.
Answer: Here $n = 16, p = 1/2, \mu = 8, \sigma = \sqrt{np(1-p)} = 2$
$\mathbb{P}(X > 12) \approx \mathbb{P}(8 + 2Z > 12) = \mathbb{P}(Z > 2) = 1 - \Phi(2)$
which is $1 - 0.97725 = 0.02275$ from the table.
A more accurate answer: $\mathbb{P}(X > 12) \approx \Phi(8 + 2Z > 12.5) = \Phi(Z > 2.25) = 1 - \Phi(2.25)$
In the table this is $1 - 0.98778 = 0.01222$

The exact number using binomial distribution is

$$\sum_{k=13}^{16} \frac{16!}{(k)!(16-k)!} 2^{-16} = \frac{697}{65536} \approx 0.010635...$$

This was not on the quiz, but the Poisson approximation for counting tails with $\lambda = 8$ would give $e^{-8} \sum_{k=0}^{3} \frac{8^k}{k!} \approx 0.04238...$

[(*optional question for extra credit*)]:
In the same situation, estimate the probability that $\mathbb{P}(X = 12)$ using the normal approximation.
Answer: $\mathbb{P}(X = 12) \approx \mathbb{P}(11.5 < 8 + 2Z < 12.5) = \mathbb{P}(1.75 < Z < 2.25) = \Phi(2.25) - \Phi(1.75)$
which is $0.98778 - 0.95994 = 0.02784$ from the table.

The exact number using binomial distribution is

$$\frac{16!}{(12)!(16-12)!} 2^{-16} = \frac{455}{16384} \approx 0.027771...$$