Sample Test 2 questions:

(1) Let X be a Poisson random variable with $\mathbb{E}X = 4$. Find the formula for $\mathbb{P}(2 \leqslant X \leqslant 5)$.

$$\sum_{k=2}^{5} e^{-4} \frac{4^k}{k!}$$

(2) Let X be a binomial random variable with $\mathbb{E}X = 4$ and $n = 10$. Find the formula for $\mathbb{P}(2 \leqslant X \leqslant 5)$.

$$p = \frac{EX}{10} = \frac{2}{5} \qquad \sum_{k=2}^{5} \binom{10}{k} \left(\frac{2}{5}\right)^k \left(\frac{3}{5}\right)^{10-k}$$

(3) Let X be the number of coin tosses until we have at one head. Find $\mathbb{E}X$ and $\text{Var}X$. Find $\mathbb{P}(X \geqslant 3)$

geometric distribution from the table $\text{Geo}(p = \frac{1}{2})$

$$\mathbb{E}X = \frac{1}{p} = 2 \qquad \text{Var}(X) = \frac{\frac{1}{2}}{\frac{1}{2^2}} = 2 \qquad P(X \geqslant 3) = \sum_{k=3}^{\infty} 2^{-k} = \frac{2^{-3}}{1 - \frac{1}{2}} = \frac{1}{4}$$

or can compute $P(X \geqslant 3) = 1 - P(X=1) - P(X=2) = 1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

(4) Suppose X has the following p.d.f.

$$f(x) = \begin{cases} ax^2 & \text{if } 1 \leqslant x \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

Find a, $\mathbb{E}X$, $\text{Var}X$, and the cdf $F(x)$. Make a picture of the p.d.f. and c.d.f.

$a = \frac{3}{7}$ because $\int_1^2 x^2 \, dx = \frac{7}{3}$

$EX = \frac{3}{7}\int_1^2 x^3 \, dx = \frac{3}{7}\left(\frac{16}{4} - \frac{1}{4}\right) = \frac{45}{28}$

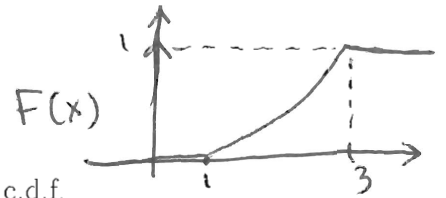$EX^2 = \frac{3}{7}\int_1^2 x^4 \, dx = \frac{3}{7}\left(\frac{2^5}{5} - \frac{1}{5}\right)$

$\text{Var}X = \left(EX^2\right) - \left(EX\right)^2$

do not simplify

(5) Suppose X has the following p.d.f.

$$f(x) = \begin{cases} \dfrac{a}{x^2} & \text{if } 1 \leqslant x \leqslant 3 \\ 0 & \text{otherwise} \end{cases}$$

$F(x)$

Find a, $\mathbb{E}X$, $\text{Var}X$, and the cdf $F(x)$. Make a picture of the p.d.f. and c.d.f.

$a = \dfrac{3}{2}$ because $\displaystyle\int_1^3 \dfrac{1}{x^2}dx = -\dfrac{1}{x}\Big|_1^3 = 1 - \dfrac{1}{3} = \dfrac{2}{3}$

$\mathbb{E}X = \dfrac{3}{2}\displaystyle\int_1^3 \dfrac{1}{x}dx = \dfrac{3}{2}\log 3$

$\mathbb{E}X^2 = \dfrac{3}{2}\displaystyle\int_1^3 1\, dx = 3$

$\text{Var}\, X = \left(\dfrac{3}{2}\log 3\right)^2 + 3$

$$F(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ \dfrac{3}{2}\left(1 - \dfrac{1}{x}\right) & \text{if } 1 \leq x \leq 3 \\ 1 & \text{if } x \geqslant 3 \end{cases}$$

(6) Find the numerical value for $\mathbb{P}(-2 \leq X \leq 3)$ if X is $\mathcal{N}(3, 4)$. Your answer should include Φ twice. After that, use the normal table attached in the end of the quiz find the approximate answer.

$X = 3 + 2Z$

$P(-2 < 3 + 2Z < 3) =$

$= P\left(-\dfrac{5}{2} < Z < 0\right) = \Phi(0) - \Phi\left(-\dfrac{5}{2}\right) =$

$= \Phi(0) - \left(1 - \Phi\left(\dfrac{5}{2}\right)\right) = \Phi\left(\dfrac{5}{2}\right) - \dfrac{1}{2}$

(7) Find a formula for $\mathbb{P}(-2 \leq X \leq 3)$ if X is $\mathcal{N}(-3, 4)$. Your answer should include Φ twice. After that, use the normal table attached in the end of the quiz find the approximate answer.

$X = -3 + 2Z$

$P(-2 < -3 + 2Z < 3) =$

$= \left(\dfrac{1}{2} < Z < 3\right) = \Phi(3) - \Phi\left(\dfrac{1}{2}\right)$

(8) If Z_1 and Z_1 are standard normal independent random variables, and $X = 3Z_1 + 4Z_2$, find $\mathbb{P}(-2 \leq X \leq 3)$
Your answer should include Φ twice. After that, use the normal table attached in the end of the quiz find the approximate answer.

$$\Phi(0.6) + \Phi(0.4) - 1$$

this was done in class

(9) Let X be a binomial random variable with $\mathbb{E}X = 12$ and $n = 20$. Find the normal approximation for
$\mathbb{P}(54 \leq X \leq 72)$

$EX = 60 \qquad n = 150$

$p = \dfrac{60}{150} = \dfrac{2}{5}$

$\mu = EX = 60 \qquad \sigma^2 = 150 \cdot \dfrac{2}{5} \cdot \dfrac{3}{5} = 36 \qquad \sigma = 6$

$P(54 \leq X \leq 72) \approx P(54 \leq 60 + 6z \leq 72) = P(-1 < z < 2)$

$= \Phi(2) - \Phi(-1) = \Phi(2) + \Phi(1) - 1$

(10) Let X be the number of tails in 25 fair coin tosses. Find the best normal approximation for $\mathbb{P}(X \leqslant 15)$.

$n = 25 \qquad p = \dfrac{1}{2} \qquad \mu = \dfrac{25}{2} \qquad \sigma^2 = 25 \cdot \dfrac{1}{4} \qquad \sigma = \dfrac{5}{2}$

$P(X \leq 15) \approx P\left(12.5 + \dfrac{5}{2}z \leq 15.5\right) \approx P(z \leq 1.2) = \Phi(1.2)$

In the table $\Phi(1.2) \approx 0.88493$

4

(11) Let a fair dice be thrown 2000 times. Find the normal approximation of the probability that 6 appears at least 300 times.

$$\Phi(2)$$

This was done in class

(12) Let X be an exponential random variable with parameter $\lambda = 2$. Find
  (a) Find $P(X > 3)$
  (b) Find $P(X > 3 | X > 2)$
  (c) Find $P(4 < X < 6 | X > 2)$

(a) e^{-6}

(b) e^{-2}

(c) $e^{-4} - e^{-8}$

use that exponential r.v. is memoryless

(13) Let X be a uniform random variable on $[1, 5]$. Find
  (a) Find $P(X > 3)$
  (b) Find $P(X > 3 | X > 2)$
  (c) Find $P(4 < X < 6 | X > 2)$
  (d) Find $\mathbb{E}e^{2X}$.
  (e) Find $\mathbb{E}\dfrac{1}{X^2}$.

$$P(X > 3) = \frac{1}{2}$$

$$P(X > 3 | P > 2) = \frac{2}{3}$$

$$P(4 < X < 6 | P > 2) = \frac{1}{3}$$

$$\mathbb{E}e^{2X} = \int_1^5 \frac{1}{4} e^{2x} \, dx = \frac{1}{4}\frac{1}{2} e^{2x} \Big|_1^5 = \frac{1}{8}\left(e^{10} - e^2\right)$$

$$\mathbb{E}\frac{1}{X^2} = \int_1^5 \frac{1}{4} \frac{1}{x^2} \, dx = \frac{1}{4}\left(-\frac{1}{x}\right)\Big|_1^5 = \frac{1}{4}\frac{4}{5} = \frac{1}{5}$$

(14) Consider random variables X and Y with the joint probability density function
$$f(x,y) = \begin{cases} axy^2 & \text{if } 0 \le x \le y \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find a
(b) Find the marginal p.d.f. $f_X(x)$.
(c) Find the marginal p.d.f. $f_Y(y)$.

$a = \dfrac{5}{16}$ because $\displaystyle\int_0^2 \int_0^y xy^2\, dx\, dy = \frac{1}{2}\int_0^2 y^4\, dy = \frac{2^5}{10}$

$f_X(x) = a\displaystyle\int_x^2 xy^2\, dy = \frac{a}{3}(8-x^3)x \qquad f_Y(y) = a\displaystyle\int_0^y xy^2\, dx = \frac{a}{2}y^4$

(15) Consider random variables X and Y with the joint probability density function
$$f(x,y) = \begin{cases} axy^2 & \text{if } 0 \le y \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find a
(b) Find the marginal p.d.f. $f_X(x)$.
(c) Find the marginal p.d.f. $f_Y(y)$.

$a = \dfrac{15}{32}$ because $\displaystyle\int_0^2 \int_0^x xy^2\, dy\, dx = \frac{1}{3}\int_0^2 x^4\, dx = \frac{2^5}{15}$

$f_X(x) = a\displaystyle\int_0^x xy^2\, dy = \frac{a}{3}x^4 \qquad f_Y(y) = a\displaystyle\int_y^2 xy^2\, dx = \frac{a}{2}y^2(4-y^2)$

(16) Consider random variables X and Y with the joint probability density function
$$f(x,y) = \begin{cases} axy^2 & \text{if } 0 \le x,\ 0 \le y,\ x+y < 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find a
(b) Find the marginal p.d.f. $f_X(x)$.
(c) Find the marginal p.d.f. $f_Y(y)$.

In class!

$1 = \displaystyle\int_0^2\int_0^{2-y} axy^2\, dx\, dy =$

$= \displaystyle\int_0^2 \frac{a}{2}y^2(y-2)^2\, dy = \frac{a}{2}\left(\frac{y^5}{5} - \frac{4}{4}y^4 + \frac{4}{3}y^3\right)\Big|_0^2$

$a = \dfrac{2}{\frac{1}{5}2^5 - 2^4 + \frac{4}{3}2^3}$

$f_X(x) = \frac{a}{3}x(2-x)^3$

$f_Y(y) = \frac{a}{2}y^2(2-y)^2$

was done in class

(17) In the previous problem. let $Y_1 = X + Y$ and $Y_2 = X - Y$. Find the joint probability density function $f_{Y_1,Y_2}(y_1. y_2)$

$$|J| = \det \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1 - 1 = -2$$

$$\begin{cases} y_1 = x + y \\ y_2 = x - y \end{cases} \implies \begin{array}{l} x = \frac{1}{2}(y_1 + y_2) \\ y = \frac{1}{2}(y_1 - y_2) \end{array}$$

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \dfrac{a}{16}(y_1 + y_2)(y_1 - y_2)^2 & \text{if} \quad \boxed{\begin{array}{l} y_1 + y_2 \geqslant 0 \\ y_1 - y_2 \geqslant 0 \\ y_1 \leqslant 2 \end{array}} \\ \\ 0 & \text{otherwise} \end{cases}$$