Please write Your name: ______________________________

Show all work. You should either write at a sentence explaining your reasoning, or annotate your math work with brief explanations. There is no need to simplify, and no calculators are needed. Two two-sided pages of handwritten notes are allowed.

. . . .

(1) Let X_1, X_2 be independent $\mathrm{Bin}(2, \frac{1}{2})$ random variables.

In this table, fill the values of the probability mass function of X_1:

i	0	1	2
$P(X_1 = i)$	1/4	1/2	1/4

Find the joint probability mass function of (X_1, X_2), that is, $P(X_1 = i, X_2 = j)$:

$i \backslash j$	0	1	2
0	1/16	1/8	1/16
1	1/8	1/4	1/8
2	1/16	1/8	1/16

Let $Y_1 = \min(X_1, X_2)$ and $Y_2 = \max(X_1, X_2)$.
Find the joint probability mass function of (Y_1, Y_2), that is, $P(Y_1 = i, Y_2 = j)$:

$i \backslash j$	0	1	2
0	1/16	1/4	1/8
1	0	1/4	1/4
2	0	0	1/16

In this table, fill the values of the probability mass function of Y_1:

i	0	1	2
$P(Y_1 = i)$	1/16	1/2	7/16

In this table, fill the values of the probability mass function of Y_2:

j	0	1	2
$P(Y_2 = j)$	7/16	1/2	1/16

(2) If X is an exponential random variable with $\mathbb{E}X = 2$, and $Y = 1 + 3X$, find $F_Y(y)$ and $f_Y(y)$.
Answer: $X = (Y-1)/3$

$$F_Y(y) = \begin{cases} 1 - e^{(1-y)/6} & \text{if } 1 \le y \\ 0 & \text{otherwise.} \end{cases} \qquad f_Y(y) = \begin{cases} \frac{1}{6}e^{(1-y)/6} & \text{if } 1 \le y \\ 0 & \text{otherwise.} \end{cases}$$

(3) If Z_1, Z_2, Z_3 are independent standard normal random variables, and $X = Z_1 + Z_2 + Z_3$, find $\mathbb{P}(|X| < 1)$. Your answer should include Φ. ***Answer:*** $2\Phi(1/\sqrt{3}) - 1$

(4) Consider random variables X and Y with the joint probability density function

$$f(x,y) = \begin{cases} a(2x+3y) & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find a ***Answer:*** $a = 2/5$

(b) Find the marginal p.d.f. $f_X(x)$. ***Answer:***

$$f_X(x) = \begin{cases} 2(2x+3/2)/5 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(c) Find the marginal p.d.f. $f_Y(y)$. ***Answer:***

$$f_Y(y) = \begin{cases} 2(1+3y)/5 & \text{if } 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(5) If X_1, X_2 are independent exponential random variables with $\lambda = 1$, let $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$. Find the joint probability density function $f_{Y_1,Y_2}(y_1, y_2)$ ***Answer:***

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} \frac{1}{2}e^{-y_1} & \text{if } 0 \le y_1 + y_2 \text{ and } 0 \le y_1 - y_2 \\ 0 & \text{otherwise.} \end{cases}$$

[(*optional questions for extra credit*)]:

Let X_1, X_2 be independent continuous random variables, uniform on the interval $[0,1]$.
Let $Y_1 = \min(X_1, X_2)$ and $Y_2 = \max(X_1, X_2)$. Find the joint probability density function $f_{Y_1,Y_2}(y_1, y_2)$
Answer:

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} 2 & \text{if } 0 \le y_1 \le y_2 \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find the marginal probability density function $f_{Y_1}(y_1)$ ***Answer:***

$$f_{Y_1}(y_1) = \begin{cases} 2(1-y_1) & \text{if } 0 \le y_1 \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find the marginal probability density function $f_{Y_2}(y_2)$ ***Answer:***

$$f_{Y_2}(y_2) = \begin{cases} 2y_2 & \text{if } 0 \le y_2 \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Use either calculus or geometry to find $\mathbb{E}Y_1$ and $\mathbb{E}Y_2$ ***Answer:*** $\mathbb{E}Y_1 = 1/3$ and $\mathbb{E}Y_2 = 2/3$