Please write ***Your name:*** ________________________________

Show all work. You should either write at a sentence explaining your reasoning, or annotate your math work with brief explanations. There is no need to simplify, and no calculators are needed. Two two-sided pages of handwritten notes are allowed.

. . . .

(1) Let X_1, X_2 be independent $\text{Bin}(2, \frac{1}{2})$ random variables.

In this table, fill the values of the probability mass function of X_1:

i	0	1	2
$P(X_1 = i)$			

Find the joint probability mass function of (X_1, X_2), that is, $P(X_1 = i, X_2 = j)$:

$i \backslash j$	0	1	2
0			
1			
2			

Let $Y_1 = \min(X_1, X_2)$ and $Y_2 = \max(X_1, X_2)$.
Find the joint probability mass function of (Y_1, Y_2), that is, $P(Y_1 = i, Y_2 = j)$:

$i \backslash j$	0	1	2
0			
1			
2			

In this table, fill the values of the probability mass function of Y_1:

i	0	1	2
$P(Y_1 = i)$			

In this table, fill the values of the probability mass function of Y_2:

j	0	1	2
$P(Y_2 = j)$			

(2) If X is an exponential random variable with $\mathbb{E}X = 2$, and $Y = 1 + 3X$, find $F_Y(y)$ and $f_Y(y)$.

(3) If Z_1, Z_2, Z_3 are an independent standard normal random variables, and $X = Z_1 + Z_2 + Z_3$, find $\mathbb{P}(|X| < 1)$. Your answer should include Φ.

(4) Consider random variables X and Y with the joint probability density function

$$f(x, y) = \begin{cases} a(2x + 3y) & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find a

(b) Find the marginal p.d.f. $f_X(x)$.

(c) Find the marginal p.d.f. $f_Y(y)$.

(5) If X_1, X_2 are independent exponential random variables with $\lambda = 1$, let $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$. Find the joint probability density function $f_{Y_1,Y_2}(y_1, y_2)$

[(*optional questions for extra credit*)]:

Let X_1, X_2 be independent continuous random variables, uniform on the interval $[0, 1]$.
Let $Y_1 = \min(X_1, X_2)$ and $Y_2 = \max(X_1, X_2)$. Find the joint probability density function $f_{Y_1,Y_2}(y_1, y_2)$

Find the marginal probability density function $f_{Y_1}(y_1)$

Find the marginal probability density function $f_{Y_2}(y_2)$

Use either calculus or geometry to find $\mathbb{E}Y_1$ and $\mathbb{E}Y_2$

— end of the test —

(the table of distributions is on the next page)

Table of probability distributions

Discrete random variables

Name	Abbrev.	Parameters	p.m.f.: $\mathbb{P}[X=k],\ k \in \mathbb{N}_0$	$\mathbb{E}[X]$	$\mathrm{Var}(X)$	m.g.f.: $\mathbb{E}[e^{tX}],\ t \in \mathbb{R}$
Bernoulli	Bern(p)	$p \in [0,1]$	$\binom{1}{k} p^k (1-p)^{1-k}$	p	$p(1-p)$	$(1-p)+pe^t$
Binomial	Bin(n,p)	$n \in \mathbb{N}, p \in [0,1]$	$\binom{n}{k} p^k (1-p)^{n-k}$	np	$np(1-p)$	$[(1-p)+pe^t]^n$
Poisson	Pois(λ)	$\lambda > 0$	$e^{-\lambda}\frac{\lambda^k}{k!}$	λ	λ	$\exp(\lambda(e^t-1))$
Geometric	Geo(p)	$p \in (0,1)$	$\begin{cases} (1-p)^{k-1}p, & \text{for } k \geq 1, \\ 0, & \text{else.} \end{cases}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$, for $t < -\log(1-p)$
Negative binomial	NB(r,p)	$r \in \mathbb{N}, p \in (0,1)$	$\begin{cases} \binom{k-1}{r-1} p^r (1-p)^{k-r}, & \text{if } k \geq r, \\ 0, & \text{else.} \end{cases}$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$	$\left(\frac{pe^t}{1-(1-p)e^t}\right)^r$, for $t < -\log(1-p)$
Hypergeometric		$n, m, N \in \mathbb{N}_0$	$\frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}$	$\frac{nm}{N}$	(not tested)	(not tested)

Continuous random variables

Name	Abbrev.	Parameters	p.d.f.: $f(x),\ x \in \mathbb{R}$	$\mathbb{E}[X]$	$\mathrm{Var}(X)$	m.g.f.: $\mathbb{E}[e^{tX}],\ t \in \mathbb{R}$
Uniform	$\mathcal{U}(a,b)$	$a, b \in \mathbb{R}, a < b$	$\begin{cases} \frac{1}{b-a}, & \text{if } x \in [a,b], \\ 0, & \text{if } x \notin [a,b]. \end{cases}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{tb}-e^{ta}}{t(b-a)}$
Normal	$\mathcal{N}(\mu,\sigma^2)$	$\mu, \sigma \in \mathbb{R}$	$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$	μ	σ^2	$e^{\mu t} e^{\sigma^2 t^2/2}$
Exponential	Exp(λ)	$\lambda > 0$	$\begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda - t}$, for $t < \lambda$