BV functions and derivatives on fractals

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Joint work with
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- Besov class via heat semigroup on Dirichlet spaces I: **Sobolev type inequalities**, arXiv:1811.04267
- Besov class via heat semigroup on Dirichlet spaces II: **BV functions and Gaussian heat kernel estimates**, arXiv:1811.11010
- Besov class via heat semigroup on Dirichlet spaces III: **BV functions and sub-Gaussian heat kernel estimates**, arXiv:1903.10078
  - New conjectures about Hölder continuity on the Sierpinski carpet
  - BV functions on nested fractals (research in progress).
- **BV functions and fractional Laplacians on Dirichlet spaces**, arXiv:1910.13330
- **BV functions on finitely ramified fractals**, in preparation

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Abstract: we introduce heat semigroup-based Besov classes for general Dirichlet spaces, study quantitative regularization estimates for the heat semigroup in this scale of spaces, and obtain a far reaching $\boldsymbol{L^p}$-analogue, $\boldsymbol{p \geq 1}$, of the Sobolev inequality that was proved for $\boldsymbol{p = 2}$ by N. Varopoulos under the assumption of ultracontractivity for the heat semigroup. The case $\boldsymbol{p = 1}$ may yield isoperimetric type inequalities and Bounded Variation (BV) function spaces.

Motivation

Isoperimetric inequalities and BV functions, for $\mathbb{R}^n$ and manifolds, were studied by **Caccioppoli, De Giorgi, Federer, Ledoux, Miranda** et al and more recently by **Ambrosio (related to Cheeger, Hajłasz, Heinonen, Koskela)** et al in non-smooth setting.

Besov and related spaces in DMMS and heat semi-group setting were studied by **Barlow, Bass, Hambly, Hinz, Hu, Jonsson, Grigoryan, Kumagai, Lau, Pietruska-Pałuba, Triebel, Wallin, Zähle**.

Bakry, Coulhon, Ledoux, Saloff-Coste: Sobolev inequalities in disguise, 1995

Contents of paper I

Sobolev type inequalities for general Dirichlet forms

In $\mathbb{R}^n$, $n \geq 2$,

$$\|f\|_{L^q(\mathbb{R}^n)} \leq C \| \, |\nabla f| \, \|_{L^p(\mathbb{R}^n)}, \quad f \in C_0^\infty(\mathbb{R}^n) \tag{1}$$

$1 \leq p < n$, $q = \frac{np}{n-p}$, and C depends on n and p.

For a measure space (X, μ) with a symmetric Dirichlet form $\mathcal{E}$ with domain $\mathcal{F}$, N. Varopoulos proved in 1985 that a heat kernel upper bound of the type $p_t(x, y) \leq \frac{C}{t^{n/2}}$, $n > 2$, implies the following Sobolev inequality

$$\|f\|_{L^q(X,\mu)} \leq C \sqrt{\mathcal{E}(f, f)}, \quad f \in \mathcal{F}, \tag{2}$$

where $q = \frac{2n}{n-2}$.

Note that the condition

$$p_t(x, y) \leq Ct^{-\beta}$$

is possible to verify in many classical and fractal examples because it is equivalent to the **Nash inequality**

$$\|f\|_{L^2(X,\mu)}^{2+2/\beta} \leqslant C\mathcal{E}(f, f)\|f\|_{L^1(X,\mu)}^{2/\beta} \quad (3)$$

by Carlen-Kusuoka-Stroock 1987.

Theorem (Weak Sobolev and isoperimetric inequalities):
Let $(X, \mu, \mathcal{E}, \mathcal{F})$ be a Dirichlet space. Let $\{P_t\}_{t\in[0,\infty)}$ denote the Markovian semigroup associated with $(X, \mu, \mathcal{E}, \mathcal{F})$. Let $p \geq 1$. Assume that P_t admits a measurable heat kernel $p_t(x, y)$ satisfying, for some $C > 0$ and $\beta > 0$, $p_t(x, y) \leq Ct^{-\beta}$ for $\mu \times \mu$-a.e. $(x, y) \in X \times X$, and for each $t \in (0, +\infty)$. Let $0 < \alpha < \beta$. Let $1 \leq p < \frac{\beta}{\alpha}$. There exists a constant $C_{p,\alpha}0$ such that

$$\sup_{s\geq 0} s\,\mu\left(\{x \in X \,:\, |f(x)| \geq s\}\right)^{\frac{1}{q}} \leq C_{p,\alpha}\|f\|_{p,\alpha},$$

$$\|f\|_{p,\alpha} := \sup_{t>0} t^{-\alpha}\left(\int_X\int_X |f(x) - f(y)|^p p_t(x, y)d\mu(x)d\mu(y)\right)^{1/p}.$$

where $q = \frac{p\beta}{\beta - p\alpha}$. Therefore, there exists a constant

$$C_{iso} = \frac{C^{\frac{\alpha}{\beta}}(\alpha+\beta)^{\frac{\alpha+\beta}{\beta}}}{2\beta\alpha^{\frac{\alpha}{\beta}}}$$

such that for every subset set $E \subset X$ with $1_E \in \mathrm{B}^{1,\alpha}(X)$

$$\mu(E)^{\frac{\beta-\alpha}{\beta}} \leq C_{iso}\|1_E\|_{1,\alpha} := C_{iso}P_\alpha(E).$$

Theorem (Strong Sobolev inequality):
Let $(X, \mu, \mathcal{E}, \mathcal{F})$ be a Dirichlet space. Let $\{P_t\}_{t\in[0,\infty)}$ denote the Markovian semigroup associated with $(X, \mu, \mathcal{E}, \mathcal{F})$. Let $p \geq 1$. Assume that P_t admits a measurable heat kernel $p_t(x, y)$ satisfying, for some $C > 0$ and $\beta > 0$, $p_t(x, y) \leq Ct^{-\beta}$ for $\mu \times \mu$-a.e. $(x, y) \in X \times X$, and for each $t \in (0, +\infty)$. Assume that there exist $\alpha > 0$ and $C > 0$ such that for every $f \in L^p(X, \mu)$

$$\|f\|_{p,\alpha} \leq C \liminf_{t\to 0} t^{-\alpha} \left(\int_X \int_X |f(x) - f(y)|^p p_t(x, y) d\mu(x) d\mu(y) \right)^{1/p},$$

where

$$\|f\|_{p,\alpha} := \sup_{t>0} t^{-\alpha} \left(\int_X \int_X |f(x) - f(y)|^p p_t(x, y) d\mu(x) d\mu(y) \right)^{1/p}.$$

Then, if $0 < \alpha < \beta$ and $p < \frac{\beta}{\alpha}$, there exists a constant $C_{p,\alpha,\beta} > 0$ such that for every $f \in L^p(X, \mu)$,

$$\|f\|_{L^q(X,\mu)} \leq C_{p,\alpha,\beta} \|f\|_{p,\alpha},$$

where $q = \frac{p\beta}{\beta - p\alpha}$.

Locality in time

Lemma

Let $p \geq 1$ *and* $\alpha \geq 0$. *Then* $\mathrm{B}^{p,\alpha}(X) =$

$$\left\{ f \in L^p(X,\mu) \ : \ \limsup_{t\to 0} t^{-\alpha} \left(\int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} < +\infty \right\}.$$

Moreover, if $\beta > \alpha$, *then* $\mathrm{B}^{p,\beta}(X) \subset \mathrm{B}^{p,\alpha}(X)$. *Furthermore, for* $f \in \mathrm{B}^{p,\alpha}(X)$, *and for every* $t > 0$, *we have*

$$\|f\|_{p,\alpha} \leq \frac{2}{t^\alpha} \|f\|_{L^p(X,\mu)} + \sup_{s\in(0,t]} s^{-\alpha} \left(\int_X P_s(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p}.$$

Triviality of some of the spaces $\mathbf{B}^{p,\alpha}(X)$

$\mathcal{F} = \mathbf{B}^{2,1/2}(X)$ is dense in L^2. Assume $\mathcal{E}(f,f) = 0$ only for constant f. Then, any $f \in \mathbf{B}^{p,\alpha}(X)$ with $1 \le p \le 2$ and $\alpha > 1/p$ is constant.

Theorem

Let $p > 2$. If $f \in \mathbf{B}^{p,1/2}(X) \cap \mathcal{F}$ then there is $\Gamma(f) \in L^1(X,\mu)$ such that for all $g \in L^\infty(X,\mu) \cap \mathcal{F}$,

$$\int_X g\Gamma(f)d\mu = 2\mathcal{E}(gf,f) - \mathcal{E}(f^2,g).$$

Corollary

If $\mathbf{B}^{p,1/2}(X) \cap \mathcal{F}$ is dense in $\mathcal{F}$ for some $p > 2$, then $\mathcal{E}$ admits a carré du champ operator.
Suppose that for all $f \in \mathcal{F}$ we have that f is constant whenever $\mathcal{E}(f,f) = 0$. If $\mathcal{E}$ is regular and the energy measure ν_f is singular to μ for any non-constant $f \in \mathcal{F}$. Then $\mathbf{B}^{p,1/2}(X)$ contains only constant functions when $p > 2$.

Continuity of P_t on the Besov spaces

Theorem

Let $1 < p \leq 2$. *There exists a constant* $C_p > 0$ *such that for every* $f \in L^p(X, \mu)$ *and* $t \geq 0$

$$\|P_t f\|_{p,1/2} \leq \frac{C_p}{t^{1/2}} \|f\|_{L^p(X,\mu)}.$$

In particular $P_t : L^p(X, \mu) \to \mathrm{B}^{p,1/2}(X)$ *is bounded for* $t > 0$.

Corollary

Let $2 \leq p < +\infty$ *and* $\alpha > 1/2$. *If* $f \in \mathrm{B}^{p,\alpha}(X)$ *then* $\mathcal{E}(f, f) = 0$.
Let $1 < p \leq 2$. *Let* L *be the generator of* $\mathcal{E}$ *and* $\mathcal{L}_p$ *be the domain of* L *in* $L^p(X, \mu)$. *Then*

$$\mathcal{L}_p \subset \mathrm{B}^{p,1/2}(X)$$

and for every $f \in \mathcal{L}_p$, $\|f\|^2_{p,1/2} \leq C \|Lf\|_{L^p(X,\mu)} \|f\|_{L^p(X,\mu)}$.

Besov spaces and critical exponents

$$\alpha_p^*(X) = \sup\{\alpha > 0 \,:\, \mathrm{B}^{p,\alpha}(X) \text{ is dense in } L^p(X,\mu).\}$$

$$\alpha_p^\#(X) = \sup\{\alpha > 0 \,:\, \mathrm{B}^{p,\alpha}(X) \text{ contains non-constant functions}\}.$$

1. Both $p \mapsto \alpha_p^*(X)$ and $p \mapsto \alpha_p^\#(X)$ are non-increasing;
2. For $1 \le p \le 2$ we have $\alpha_p^\#(X) \ge \alpha_p^*(X) \ge \frac{1}{2}$;

If we assume that $\mathcal{E}(f,f) = 0$ implies f constant, then we have in addition

3. If $1 \le p \le 2$ then $\alpha_p^*(X) \le \alpha_p^\#(X) \le \frac{1}{p}$;
4. $\alpha_2^*(X) = \alpha_2^\#(X) = \frac{1}{2}$;
5. For $2 \le p < \infty$ one has $\alpha_p^*(X) \le \alpha_p^\#(X) \le \frac{1}{2}$;

Furthermore if $\mathcal{E}$ is regular and the energy measure ν_f for each non-constant $f \in \mathcal{F}$ is singular to μ (as is the case on some fractals) we obtain

6. For $p > 2$ one has $\alpha_p^*(X) \le \alpha_p^\#(X) < \frac{1}{2}$.

Theorem

For Riemannian manifolds, sub-Riemannian manifolds, or metric graphs one has $\alpha_1^*(X) = \alpha_1^\#(X) = \frac{1}{2}$, *but for nested fractals or their products* $\alpha_1^*(X) = \alpha_1^\#(X) = \frac{d_H - d_{tH} + 1}{d_W}$

Topological-Hausdorff dimension:

$$d_{tH} := 1 + \inf(d_H(\partial O))$$

where the **inf** is taken over collections of open sets O that form a base of the topology.

Compare to the topological dimension $d_t := 1 + \inf(d_t(\partial O))$ where the **inf** is taken over collections of open sets O that form a base of the topology.

Main examples

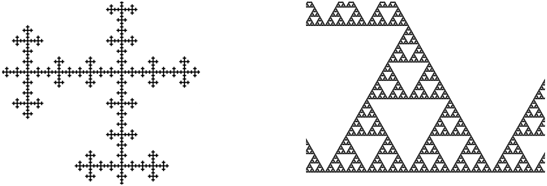


Figure: A part of an infinite, or unbounded, Vicsek set and a Sierpinski gasket.
$$d_H = \frac{\log N}{\log c}, \quad d_{tH} = 1, \quad \alpha_1^*(X) = \alpha_1^\#(X) = \frac{d_H - d_{tH} + 1}{d_W} = \frac{d_H}{d_W}$$

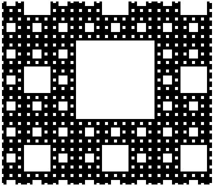


Figure: A part of an infinite, or unbounded, Sierpinski carpet, $d_{tH} - 1 = \frac{\log 2}{\log 3}$

Besov class via heat semigroup on Dirichlet spaces III: BV functions and sub-Gaussian heat kernel estimates.

Let (X, d, μ) be a locally compact complete metric Radon measure space and

$$\|f\|^p_{KS^{\lambda,p}(X)} := \limsup_{r\to 0^+} \int_X \int_{B(x,r)} \frac{|f(y)-f(x)|^p}{r^{\lambda p}\mu(B(x,r))}\, d\mu(y)\, d\mu(x) < +\infty.$$

The L^p–Korevaar-Schoen critical exponent is

$$\lambda_p^\# = \sup\{\lambda > 0 \,:\, KS^{\lambda,p}(X) \text{ contains non-constant functions}\}.$$

GHKE

With a $\mathbf{1}$-Poincaré inequality and doubling, one has $\lambda_p^\# = 1$ for every $p \geq 1$.
Note that, at the critical exponent $\lambda_2^\# = 1$, one can construct a Dirichlet form

$$\mathcal{E}(f) \simeq \|f\|^2_{KS^{1,2}(X)}$$

with domain $KS^{1,2}(X)$ by using a choice of a Cheeger differential structure. This Dirichlet form is then strictly local and the intrinsic distance $d_{\mathcal{E}}$ associated to $\mathcal{E}$ is bi-Lipschitz equivalent to the original metric d.
At the critical exponent $\lambda_1^\# = 1$, one has $KS^{1,1}(X) = BV(X)$ and

$$\mathrm{Var}(f) \simeq \|f\|_{KS^{1,1}(X)}.$$

sGHKE

$$p_t(x,y) \simeq t^{-d_H/d_W} \exp\left(-c\left(\frac{d(x,y)^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right)$$

$$\lambda_2^\# = \frac{d_W}{2}$$

$$\mathcal{E}(f) \simeq \|f\|^2_{KS^{\frac{d_W}{2},2}(X)}$$

For nested fractals

$$\lambda_1^\# = d_H$$

is the Hausdorff dimension.
For the Sierpinski carpet we prove that

$$\lambda_1^\# \geqslant d_H - d_{tH} + 1$$

and conjecture that in fact there is an equality. Here d_{tH} is the topological-Hausdorff dimension.

Weak Bakry-Émery nonnegative curvature condition

We say that the weak Bakry-Émery non-negative curvature condition $wBE(\kappa)$ is satisfied if there exist a constant $C > 0$ and a parameter $0 < \kappa < d_W$ such that for every $t > 0$, $g \in L^\infty(X, \mu)$ and $x, y \in X$,

$$|P_t g(x) - P_t g(y)| \leq C \frac{d(x,y)^\kappa}{t^{\kappa/d_W}} \|g\|_{L^\infty(X,\mu)}.$$

This inequality is proved by Barlow for resistance spaces under the sGHKE.

For nested fractals, $wBE(\kappa)$ is satisfied with $\kappa = d_W - d_H$. In that case the value $d_W - d_H$ is optimal.

For the Sierpinski carpet satisfies $wBE(\kappa)$ with $\kappa = d_W - d_H$, however we conjecture that in fact the Sierpinski carpet satisfies $wBE(\kappa)$ with $\kappa > d_W - d_H$.

It will be a subject of future work to investigate whether the Sierpinski carpet has

$$\kappa = d_W - d_H + d_{tH} - 1 = d_W - d_H + \frac{\log 2}{\log 3}$$

[*Strongly supported by numerical results: L.Rogers et al*]

Main results on BV functions under the $wBE(\kappa)$ condition

Define

$$BV(X) := KS^{\lambda_1^\#,1}(X) = \mathrm{B}^{1,\alpha_1^\#}(X)$$

and for $f \in BV(X)$,

$$\mathrm{Var}(f) := \liminf_{r\to 0^+} \int_X \int_{B(x,r)} \frac{|f(y) - f(x)|}{r^{\lambda_1^\#}\mu(B(x,r))}\, d\mu(y)\, d\mu(x).$$

We show that for nested fractals and their products, $BV(X)$ is dense in $L^1(X,\mu)$.

Locality property There is a constant $C > 0$ such that for every $f \in BV(X)$,

$$\sup_{r>0} \frac{1}{r^{d_H+d_W-\kappa}} \int_X \int_{B(y,r)} |f(x) - f(y)| d\mu(x)\, d\mu(y) \leq C\mathrm{Var}(f).$$

Co-area estimate

There exist constants $c, C > 0$ such that for every non-negative $f \in BV(X)$,

$$c \int_0^\infty \mathrm{Var}(1_{E_t(f)})dt \leq \mathrm{Var}(f) \leq C \int_0^\infty \mathrm{Var}(1_{E_t(f)})dt,$$

where $E_t(f) = \{x \in X \,:\, f(x) > t\}$. In particular, for $f \in BV(X)$ the sets $E_t(f) = \{x \in X \,:\, f(x) > t\}$ are of finite perimeter for almost every $t > 0$.

Minkowski There exists a constant $C > 0$ such that for every Borel set $E \subset X$,

$$P(E) \leq C\mathcal{C}^*_{d_W-\kappa}(E),$$

where $\mathcal{C}^*_{d_W-\kappa}(E)$ denotes the $(d_W - \kappa)$-codimensional lower Minskowski content of E. In particular, any set whose measure-theoretic boundary has finite $(d_W - \kappa)$-codimensional lower Minskowski content has finite perimeter

Sobolev inequality I Assume $d_W - \kappa < d_H$. Then $BV(X) \subset L^{1^*}(X, \mu)$ and there is $C > 0$ such that for every $f \in BV(X)$,

$$\|f\|_{L^{1^*}(X,\mu)} \leq C\mathrm{Var}(f),$$

where the critical Sobolev exponent 1^* is given by the formula

$$\frac{1}{1^*} = 1 - \frac{d_W - \kappa}{d_H} \qquad 1^* = \frac{d_H - d_W + \kappa}{d_H}.$$

Isoperimetric inequality

$$\mu(E)^{\frac{d_H - d_W + \kappa}{d_H}} \leq CP(E).$$

Sobolev inequality II
Assume $\kappa = d_W - d_H > 0$. Then $BV(X) \subset L^\infty(X, \mu)$ and there exists a constant $C > 0$ such that for every $f \in BV(X)$, and a.e. $x, y \in X$

$$|f(x) - f(y)| \leq C\mathrm{Var}(f).$$

For nested fractals we do have $\kappa = d_W - d_H > 0$. Moreover, a set has finite perimeter if and only if it has finite boundary, $P(E) \sim \#(\partial E)$.

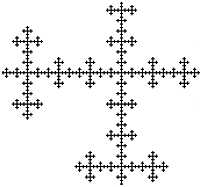
Theorem (research in progress)

$f \in BV$ iff ∇f is a "vector valued Radon measure".

This is understood in the distributional sense (Hinz, Rogers, Strichartz et al)

Corollary

1. *on the Vicsek set, any BV function is $\mathbb{R}^1$-BV along each geodesic path.*
2. *on the Sierpiński gasket, any BV function is discontinuous.*



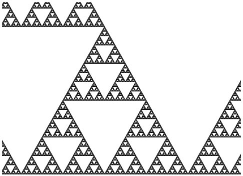


Figure: A part of an infinite, or unbounded, Vicsek set and a Sierpinski gasket.