Use the notation $\Phi(x)$ for the $\mathcal{N}(0, 1)$ distribution function, that is $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy = P(Z < x)$ where $Z$ is the standard normal random variable.

1. Let $X$ be a uniform random variable on the interval $[0, 4]$, and $Y = \sqrt{X}$. Find the c.d.f. $F_Y(y)$ of $Y$. Hint: it may be useful if first you find the range of $Y$, and use cases to define the c.d.f.

Solution: we know from the text that $F_X(x) = \frac{x}{4}$ if $x \in [0, 4]$. Let $y = \sqrt{x}$, and so $x = y^2$. Hence $F_Y(y) = P(Y < y) = P(\sqrt{X} < \sqrt{y}) = P(X < x) = F_X(x) = F_X(y^2) = \frac{y^2}{4}$

Answer: $F_Y(y) = \frac{y^2}{4}$ when $0 < y < 2$; $F_Y(y) = 0$ when $y \leq 0$; $F_Y(y) = 1$ when $2 \leq y$.

2. Find the p.d.f. $f_Y(y)$ of $Y$.

Solution: since $f = F'$, we have that $f_Y(y) = \frac{d}{dy} \frac{y^2}{4} = \frac{y}{2}$ when $0 < y < 2$.

Answer: $f_Y(y) = \frac{y}{2}$ when $0 < y < 2$ and zero otherwise.

3. Find $EY$

Solution: $EY = \int yf(y)dy = \int_0^2 y \frac{y}{2} dy = \int_0^2 \frac{y^2}{2} dy = \frac{y^3}{6} \bigg|_0^2 = \frac{8}{6} = \frac{4}{3}$

Answer: $EY = \frac{4}{3}$

4. Find $EY^2$

Solution: $EY^2 = \int y^2 f(y)dy = \int_0^2 y^2 \frac{y}{2} dy = \int_0^2 \frac{y^3}{2} dy = \frac{y^4}{8} \bigg|_0^2 = \frac{16}{8} = 2$

Answer: $EY^2 = 2$

5. If $Z$ is the standard normal $\mathcal{N}(0, 1)$ random variable, find the c.d.f. and the p.d.f. of $|Z|$.

Hint: you can use function $\Phi(x)$ and cases.

Solution: $P(|Z| < x) = P(-x < Z < x) = \Phi(x) - \Phi(-x)$

Answer: $\Phi(x) - \Phi(-x) = 2\Phi(x) - 1$ if $x \geq 0$, and zero otherwise.

Extra credit question: what is $E|Z|$ and $\text{Var } |Z|$? Answer: $E|Z| = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} xe^{-x^2/2} dx = \sqrt{\frac{2}{\pi}}$ by the substitution $u = x^2/2$, $du = x dx$, $E|Z|^2 = 1$, and so $\text{Var } |Z| = E|Z|^2 - (E|Z|)^2 = 1 - \frac{2}{\pi}$. 