MATH 3160 - Probability - Fall 2017
Test 2, Wednesday November 15

(1) Two balls are withdrawn randomly without replacement from a bowl containing **3** white and **3** black balls. Let $\boldsymbol{X}$ be the number of white balls among the withdrawn balls. What are the probability mass function of $\boldsymbol{X}$, $\mathbb{E}\boldsymbol{X}$ and $\mathbf{Var}\,(\boldsymbol{X})$?

Solution: $\mathbb{P}(X=0) = \binom{3}{2} / \binom{6}{2} = 3/15 = 1/5$

$\mathbb{P}(X=1) = \binom{3}{1} \cdot \binom{3}{1} / \binom{6}{2} = 9/15 = 3/5$, $\mathbb{P}(X=2) = \binom{3}{2} / \binom{6}{2} = 3/15 = 1/5$

Answer:

p.m.f.: $p(0) = p(2) = 1/5$, $p(1) = 3/5$

$\mathbb{E}X = 0 + 2/5 + 3/5 = 1$

$\mathbf{Var}\,(X) = \mathbb{E}(X - EX)^2 = 1/5 + 0 + 1/5 = 2/5$

(2) Suppose that earthquakes occur on the West coast of the U.S. on average at a rate of 3 per week (including very mild ones) and follow Poisson probability distribution. What is the probability that there will be 2 earthquakes next week, if we suppose that at least one will happen? *(Hint: use conditional probability).*

Solution: $P(X=2) = 3^2 e^{-3}/2$, $P(X \geqslant 1) = 1 - e^{-3}$

$e^{-\lambda}\lambda^k/k!$

Answer:

$$P(X = 2 | X \geqslant 1) = \frac{3^2 e^{-3}}{2(1 - e^{-3})}$$

(3) Suppose X is exponentially distributed with the mean $\mathbb{E}X = 2$. What is the probability $3 < X < 5$ if we know that $X > 2$? *(Hint: use conditional probability and the basic properties of the exponentially distribution).*

Solution: Exponentials are memory-less: $\mathbb{P}(X > s + t \mid X > t) = \mathbb{P}(X > s)$.
Hence $P(3 < X < 5 | X > 2) = P(3 < X | X > 2) - 1 + P(X > 5 | X > 2) = P(1 < X < 3)$

Here $\lambda = \frac{1}{2}$

Answer:

$P(3 < X < 5 | X > 2) = e^{-1/2} - e^{-3/2}$

$P(X > t) = e^{-t\lambda}$

(4) Suppose $X = \mathcal{N}(\mu, \sigma^2)$, $P(X < 0) = 0.15866 = \Phi(-1)$ and $P(X < 5) = 0.97725 = \Phi(2)$. Find μ and σ.

Z

Solution: $\mu - \sigma = 0$, $\mu + 2\sigma = 5$. This implies

$X = \mu + \sigma Z$

Answer:

$\mu = \sigma = 5/3$

$X = Bin(16, \frac{1}{2})$

(5) Suppose we toss a fair coin **16** times. Find the formula for the best possible normal approximation of the probability that there are at least **9** heads. You do not have to evaluate the numeral value but your answer should include $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy = \mathbb{P}(Z < x)$, where Z is the standard normal random variable.

$\mu = np \qquad \sigma^2 = np(1-p)$

Solution: $\mu = 16/2 = 8$, $\sigma = \sqrt{16/4} = 2$. This implies $\mathbb{P}(X \geqslant 9) \approx \mathbb{P}(8 + 2Z > 8.5) = \mathbb{P}(Z > 0.25)$

Answer:

$\mathbb{P}(X \geqslant 9) \approx 1 - \Phi(0.25)$

This is approximately equal to 0.40129 but this was not part of the test.

A different but also correct solution is

$\mathbb{P}(X \geqslant 9) = \mathbb{P}(16 \geqslant X \geqslant 9) \approx \mathbb{P}(16.5 > 8 + 2Z > 8.5) = \mathbb{P}(4.25 > Z > 0.25) = \Phi(4.25) - \Phi(0.25)$, which numerically is about 0.40128

The exact probability, using the binomial distribution, is $\frac{26333}{65536} \approx 0.40180969$

$f_X = 1/2 = \frac{1}{\beta - \alpha}$

(6) Suppose the random variable X is uniformly distributed in the interval $[0, 2]$ and $Y = X^3$. Find the c.d.f. $F_Y(y)$ and $\mathbb{E}Y$.

Solution: Let $y = x^3$, $x = \sqrt[3]{y}$. We have $0 < Y < 8$. Then $F_Y(y) = \mathbb{P}(Y < y) = \mathbb{P}(X < x) = F_X(\sqrt[3]{y}) = \frac{1}{2}\sqrt[3]{y}$ when $0 < y < 8$.

$$\mathbb{E}Y = \frac{1}{2}\int_0^2 x^3 dx = \frac{1}{2}\frac{1}{4}x^4\Big|_0^2 = 2$$

$EY = \int g(x) f(x) dx$

Answer:

$F_Y(y) = 0$ when $y \le 0$, $F_Y(y) = \frac{1}{2}\sqrt[3]{y}$ when $0 < y < 8$, $F_Y(y) = 1$ when $y \geq 8$

$\mathbb{E}Y = 2$