(1) Two balls are withdrawn randomly without replacement from a bowl containing 3 white and 3 black balls. Let \( X \) be the number of white balls among the withdrawn balls. What are the probability mass function of \( X \), \( \mathbb{E}X \) and \( \text{Var}(X) \)?

Solution: \( P(X=0) = \binom{3}{2} / \binom{6}{2} = 3/15 = 1/5 \)
\( P(X=1) = \binom{3}{1} \cdot \binom{3}{1} / \binom{6}{2} = 9/15 = 3/5 \)
\( P(X=2) = \binom{3}{2} / \binom{6}{2} = 3/15 = 1/5 \)

Answer:

p.m.f.: \( p(0) = p(2) = 1/5 \)
\( p(1) = 3/5 \)
\( \mathbb{E}X = 0 + 2/5 + 3/5 = 1 \)
\( \text{Var}(X) = \mathbb{E}(X - \mathbb{E}X)^2 = 1/5 + 0 + 1/5 = 2/5 \)

(2) Suppose that earthquakes occur on the West coast of the U.S. on average at a rate of 3 per week (including very mild ones) and follow Poisson probability distribution. What is the probability that there will be 2 earthquakes next week, if we suppose that at least one will happen? (Hint: use conditional probability).

Solution: \( P(X=2) = 3^2 e^{-3}/2, P(X \geq 1) = 1 - e^{-3} \)

Answer:

\( P(X=2|X \geq 1) = \frac{3^2 e^{-3}}{2(1 - e^{-3})} \)

(3) Suppose \( X \) is exponentially distributed with the mean \( \mathbb{E}X = 2 \). What is the probability \( 3 < X < 5 \) if we know that \( X > 2 \)? (Hint: use conditional probability and the basic properties of the exponentially distribution).

Solution: Exponentials are memory-less: \( P(X > s + t | X > t) = P(X > s) \).
Hence \( P(3 < X < 5|X > 2) = P(3 < X|X > 2) - 1 + P(X > 5|X > 2) = P(1 < X < 3) \)

Here \( \lambda = \frac{1}{2} \)

Answer:

\( P(3 < X < 5|X > 2) = e^{-1/2} - e^{-3/2} \)
(4) Suppose $X = \mathcal{N}(\mu, \sigma^2)$, $P(X < 0) = 0.15866 = \Phi(-1)$ and $P(X = 5) = 0.97725 = \Phi(2)$.

Find $\mu$ and $\sigma$.

Solution: $\mu - \sigma = 0$, $\mu + 2\sigma = 5$. This implies

$$X = M + 6\sigma$$

Answer:

$$\mu = \sigma = 5/3$$

(5) Suppose we toss a fair coin 16 times. Find the formula for the best possible normal approximation of the probability that there are at least 9 heads. You do not have to evaluate the numeral value but your answer should include $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy = P(Z < x)$, where $Z$ is the standard normal random variable.

Solution: $\mu = 16/2 = 8$, $\sigma = \sqrt{16/4} = 2$. This implies $P(X \geq 9) \approx P(8 + 2Z > 8.5) = P(Z > 0.25)$

Answer:

$$P(X \geq 9) \approx 1 - \Phi(0.25)$$

This is approximately equal to 0.40129 but this was not part of the test.

A different but also correct solution is

$$P(X \geq 9) = P(16 \geq X \geq 9) \approx P(16.5 > 8 + 2Z > 8.5) = P(4.25 > Z > 0.25) = \Phi(4.25) - \Phi(0.25),$$

which numerically is about 0.40128.

The exact probability, using the binomial distribution, is

$$\frac{26333}{65536} \approx 0.40180969$$

(6) Suppose the random variable $X$ is uniformly distributed in the interval $[0, 2]$ and $Y = X^3$. Find the c.d.f. $F_Y(y)$ and $EY$.

Solution: Let $y = x^3$, $x = \sqrt[3]{y}$. We have $0 < Y < 8$. Then $F_Y(y) = P(Y < y) = P(X < x) = F_X(\sqrt[3]{y}) = \frac{1}{2} \sqrt[3]{y}$ when $0 < y < 8$.

$$EY = \int_0^2 x^3 dx = \frac{11}{24} x^4 \bigg|_0^2 = 2$$

Answer:

$$F_Y(y) = 0 \text{ when } y \leq 0, \quad F_Y(y) = \frac{1}{2} \sqrt[3]{y} \text{ when } 0 < y < 8, \quad F_Y(y) = 1 \text{ when } y \geq 8$$

$$EY = 2$$