Show all work.

Write each problem on a separate page. Each answer should be clearly written in the end of the page. Preferably, make a 7-page pdf file and submit in HuskyCT.

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(1a) Let X, Y be two independent geometric random variables with $p = 1/2$. Find $\mathbb{P}(\max(X,Y) \leqslant 2)$ and $\mathbb{P}(\min(X,Y) \leqslant 2)$.

Answer: $\mathbb{P}(\max(X,Y) \leqslant 2) = 9/16$ from the joint probability mass function of X and Y. Also, $\mathbb{P}(\max(X,Y) \leqslant 2) = \mathbb{P}(X \leqslant 2, Y \leqslant 2) = \mathbb{P}(X \leqslant 2) \cdot \mathbb{P}(Y \leqslant 2) = \frac{3}{4} \cdot \frac{3}{4} = 9/16$

$\mathbb{P}(\min(X,Y) \leqslant 2) = 1 - \mathbb{P}(X \geqslant 3, Y \geqslant 3) = 1 - \mathbb{P}(X \geqslant 3) \cdot \mathbb{P}(Y \geqslant 3) = 1 - \frac{1}{4} \cdot \frac{1}{4} = 15/16$

(1b) In the same situation find let $U = \max(X,Y)$ and $V = \min(X,Y)$. Find the joint probability mass function of U and V.

Answer:

$$\mathbb{P}(U = u, V = v) \begin{cases} 2^{-u-v+1} & \text{if } u > v \\ 2^{-2u} & \text{if } u = v \\ 0 & \text{if } u < v \end{cases}$$

(2a) Let X, Y be two independent exponential random variables with $\lambda = 1$. Find $\mathbb{P}(X + Y < 1)$.

Answer: $\mathbb{P}(X + Y < 1) = 1 - 2/e$

(2b) Find $\mathbb{P}(X - Y < 1)$.

Answer: $\mathbb{P}(X - Y < 1) = 1 - 1/2e$

(2c) Let $U = X + Y$ and $V = X - Y$. Find the joint probability density function of U and V.

Answer: $f_{U,V}(u,v) = \frac{1}{2}e^{-u}$ if $u > 0, -u < v < u$, and zero otherwise. Note that this can be written $u > |v|$.

(2d) In the same situation find the conditional probability density function of U given V: find $f_{U|V}(u,v)$.

Answer: $f_V(v) = \frac{1}{2}e^{-|v|}$. Therefore $f_{U|V}(u,v) = e^{-u+|v|}$ if $u > 0, -u < v < u$, and undefined otherwise.

(2e) In the same situation find the conditional probability density function of V given U: find $f_{V|U}(v,u)$.

Answer: $f_U(u) = ue^{-u}$ if $u > 0$, and zero otherwise. Therefore $f_{V|U}(u,v) = 1/2u$ if $u > 0, -u < v < u$, and undefined otherwise.

end of the quiz