Show all work.

Write each problem on a separate page. Each answer should be clearly written in the end of the page. Preferably, make a single pdf file and submit in HuskyCT.

Required problem:

(1a) Let X, Y be uniformly distributed in the triangle defined by $x < 2$, $y < 1$, $x + 2y > 2$. Find the marginal densities $f_X(x)$ and $f_Y(y)$.
Answer:
$f_X(x) = x/2$ when $0 < x < 2$ and zero otherwise;
$f_Y(y) = 2y$ when $0 < y < 1$ and zero otherwise.

(1b) In the same situation find $\mathbb{E}(X|Y)$ and $\mathbb{E}(Y|X)$.
Answer:
$\mathbb{E}(X|Y) = 2 - y$ when $0 < y < 1$ and undefined otherwise;
$\mathbb{E}(Y|X) = 1 - x/4$ when $0 < x < 2$ and undefined otherwise.

Extra credit problems:

(1c) In the same situation find $\rho(X, Y)$. ***Answer***:
$\mathbb{E}X = 4/3$, $\mathbb{E}X^2 = 2$, $\operatorname{Var} X = 2/9$,
$\mathbb{E}Y = 2/3$, $\mathbb{E}Y^2 = 1/2$, $\operatorname{Var} Y = 1/18$,
$\mathbb{E}XY = 5/6$, $\operatorname{Cov}(X, Y) = -1/18$,

$$\rho(X, Y) = \frac{-1/18}{\sqrt{\frac{2}{9} \cdot \frac{1}{18}}} = -\frac{1}{2}$$

(2a) Let $U = e^{X+Y}$ and $V = e^X$ where X, Y are two independent exponential random variables with parameter $\lambda > 0$. Find $\operatorname{Cov}(U, V)$ if $\lambda = 3$. ***Answer***: $9/8$, see the next question.

(2b) In the same situation find how $\operatorname{Cov}(U, V)$ depends on λ. Is there λ for which $\operatorname{Cov}(U, V) = 0$?

$\lambda^2 \int_0^\infty \int_0^\infty e^x e^{-\lambda x - \lambda y} dx dy = \dfrac{\lambda}{\lambda - 1}$ when $\lambda > 1$

$\lambda^2 \int_0^\infty \int_0^\infty e^{x+y} e^{-\lambda x - \lambda y} dx dy = \dfrac{\lambda^2}{(\lambda - 1)^2}$ when $\lambda > 1$

$\lambda^2 \int_0^\infty \int_0^\infty e^x e^{x+y} e^{-\lambda x - \lambda y} dx dy = \dfrac{\lambda^2}{(\lambda - 1)(\lambda - 2)}$ when $\lambda > 2$

Therefore, $\operatorname{Cov}(U, V) = \dfrac{\lambda^2}{(\lambda - 1)(\lambda - 2)} - \dfrac{\lambda}{\lambda - 1} \cdot \dfrac{\lambda^2}{(\lambda - 1)^2} = \dfrac{\lambda^2}{(\lambda - 1)^3(\lambda - 2)}$ when $\lambda > 2$ and undefined otherwise.

There is no λ for which $\operatorname{Cov}(U, V) = 0$.

End of the quiz