Show all work. A correct answer with no solution will give only a partial credit.
Write each problem on a separate page. Each answer should be clearly written in the end of the page.
Preferably, make a single pdf file and submit in HuskyCT.

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(1) Find the moment generation function of $m_X(t) = \mathbb{E}e^{tX}$ for a random variable X with the probability density $f(x) = (x-1)/2$ when $1 < x < 3$ and zero otherwise. Also, find $m_X(0)$, $m'_X(0)$, $m''_X(0)$ by computing the moments of X. ***Solution***:

$$m_X(0) = \frac{1}{2}\int_1^3 (x-1)dx = 1$$

$$m'_X(0) = \mathbb{E}X = \frac{1}{2}\int_1^3 x(x-1)dx = ... = \frac{7}{3}$$

$$m''_X(0) = \mathbb{E}X^2 = \frac{1}{2}\int_1^3 x^2(x-1)dx = ... = \frac{17}{3}$$

Using integration by parts $\int xe^{tx}dx = \frac{1}{t}xe^{tx} - \frac{1}{t^2}e^{tx} + c$:

$$m_X(t) = \mathbb{E}e^{tX} = \int_1^3 e^{tx}(x-1)/2dx = ... = \frac{1}{2t^2}\left((2t-1)e^{3t} + e^t\right)$$

$$= \frac{1}{2t^2}\left((2t-1)\left(1 + 3t + \frac{9t^2}{2} + \frac{9t^3}{2} + \frac{27t^4}{8} + \frac{81t^5}{40} + ...\right) + 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} + ...\right)$$

Using $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + ... = \sum_{k=0}^{\infty}\frac{x^k}{k!}$

$$m_X(t) = 1 \quad + \frac{7}{3}t \quad + \frac{17}{6}t^2 \quad + \frac{71}{30}t^3 \quad + ...$$

$$m_X(t) = 1 \quad + m'_X(0)t \quad + \frac{m''_X(0)}{2}t^2 \quad + \frac{m'''_X(0)}{6}t^3 \quad + ...$$

$$m'''_X(0) = \mathbb{E}X^3 = \frac{1}{2}\int_1^3 x^3(x-1)dx = ... = \frac{71}{5}$$

(2) Look at problems $\mathbf{2(a,b,c)}$ in Test 2 given last week: find the moment generating function $m_X(t) = \mathbb{E}e^{tX}$ and the joint moment generating function $m_{X,Y}(s,t) = \mathbb{E}e^{tX+sY}$.

Extended Hint: if you use the change of variables $u = x + y,\ v = x - y$, you can solve this problem using the table of distributions, without computing any integrals.

$$\mathbb{E}e^{tX+sY} = \mathbb{E}e^{t(U+V)/2+s(U-V)/2} = \mathbb{E}e^{\alpha U+\beta V}$$

To finish this solution you need to do the following:

- compute α, β:

$$\alpha = (t+s)/2, \qquad \beta = (t-s)/2$$

- justify why $\mathbb{E}e^{\alpha U+\beta V} = \mathbb{E}e^{\alpha U}\mathbb{E}e^{\beta V}$ using the solution for Test 2:

  U and V are independent and unifomly distributed in the rectangle $[1,4] \times [-1,1]$

- find formulas for $\mathbb{E}e^{\alpha U}$ and $\mathbb{E}e^{\beta V}$ using the solution for Test 2, and the table of distributions

$$\mathbb{E}e^{\alpha U} = m_U(\alpha) = \frac{e^{4\alpha} - e^{\alpha}}{3\alpha} \qquad \mathbb{E}e^{\beta V} = m_V(\beta) = \frac{e^{\beta} - e^{-\beta}}{2\beta}$$

- use this to obtain the formula for $m_{X,Y}(s,t)$

$$m_{X,Y}(s,t) = \frac{e^{2(t+s)} - e^{(t+s)/2}}{3(t+s)/2} \cdot \frac{e^{(t-s)/2} - e^{(s-t)/2}}{t-s}$$

- use this to obtain the formula for $m_X(t)$: we can re-trace all steps for e^{tX}, dropping Y. In another solution, use $m_X(t) = m_{X,Y}(t,0)$. This means take the formula for $m_{X,Y}(t,s)$ and substitute $s = 0$.

$$m_X(t) = \frac{2\left(e^{2t} - e^{t/2}\right)\left(e^{t/2} - e^{-t/2}\right)}{3t^2}$$

Extra credit question: in the situation of problems $\boldsymbol{2(a,b,c)}$ in Test 2 given last week, use moment generating functions to confirm that $\mathbb{E}\boldsymbol{X} = \boldsymbol{5/4}$ and $\mathbb{E}\boldsymbol{XY} = \boldsymbol{5/3}$, and to compute $\mathbb{E}\boldsymbol{X^2}$ and $\mathbb{E}\boldsymbol{X^3}$.

Hint: it may be helpful to use Taylor expansions.

- The first two non-zero terms of the Taylor expansion are $\boldsymbol{m_{X,Y}(s,t)}$

$$\boldsymbol{m_{X,Y}(s,t) = \left(1 + \frac{5}{4}(t+s) + \frac{7}{8}(t+s)^2 + ...\right) \cdot \left(1 + \frac{1}{24}(t-s)^2 + ...\right)}$$

$$\boldsymbol{= 1 + \frac{5}{4}(t+s) + \frac{7}{8}(t+s)^2 + \frac{1}{24}(t-s)^2 + ...}$$

$$\boldsymbol{= 1 + \frac{5}{4}(t+s) + \frac{5}{3}ts + \frac{11}{12}(t^2+s^2) + ...}$$

- $$\mathbb{E}\boldsymbol{XY = \frac{\partial^2}{\partial t \partial s} m_{X,Y}(0,0) = \frac{5}{3}}$$

- $$\boldsymbol{m_X(t) = 1 + \frac{5}{4}t + \frac{11}{12}t^2 + \frac{95}{192}t^3 + ...}$$

- $$\mathbb{E}\boldsymbol{X = \frac{5}{4}} \qquad \mathbb{E}\boldsymbol{X^2 = 2! \cdot \frac{11}{12} = \frac{11}{6}} \qquad \mathbb{E}\boldsymbol{X^3 = 3! \cdot \frac{95}{192} = \frac{95}{32}}$$

- using the marginal density from Test 2 we can also compute

$$\boldsymbol{\int_0^1 (2x/3)dx + \int_1^{1.5} (2/3)dx + \int_{1.5}^{2.5} ((5-2x)/3)dx = 1}$$

$$\boldsymbol{\int_0^1 x(2x/3)dx + \int_1^{1.5} x(2/3)dx + \int_{1.5}^{2.5} x((5-2x)/3)dx = \frac{5}{4}}$$

$$\boldsymbol{\int_0^1 x^2(2x/3)dx + \int_1^{1.5} x^2(2/3)dx + \int_{1.5}^{2.5} x^2((5-2x)/3)dx = \frac{11}{6}}$$

$$\boldsymbol{\int_0^1 x^3(2x/3)dx + \int_1^{1.5} x^3(2/3)dx + \int_{1.5}^{2.5} x^3((5-2x)/3)dx = \frac{95}{32}}$$

End of the quiz