Show all work. A correct answer with no solution will give only a partial credit.

Write each problem on a separate page. Each answer should be clearly written in the end of the page.

Preferably, make a single pdf file and submit in HuskyCT.

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[**(1a)**] Let $\boldsymbol{X}$ be a Geometric random variable with $\boldsymbol{p = 1/3}$ and $\boldsymbol{Y}$ be a Poisson random variable with $\boldsymbol{\lambda = 2}$. Assuming that $\boldsymbol{X}$ and $\boldsymbol{Y}$ are independent, find $\mathbb{P}(\boldsymbol{X+Y} \geqslant 4)$.

Solution 1:

$$\mathbb{P}(X+Y \geqslant 4) = \frac{1}{3}e^{-2}\sum_{m=1}^{\infty}\left(\frac{2}{3}\right)^{m-1}\left(\sum_{n=\max(0,4-m)}^{\infty}\frac{2^n}{n!}\right) = 1 - \frac{67}{27e^2}$$

Solution 2:

$$\begin{aligned}
\mathbb{P}(X+Y \geqslant 4) &= 1 - \mathbb{P}(X+Y \leqslant 3) = \\
&= 1 - \frac{1}{3}e^{-2}\sum_{m=1}^{3}\left(\frac{2}{3}\right)^{m-1}\left(\sum_{n=0}^{m-3}\frac{2^n}{n!}\right) = \\
&= 1 - \mathbb{P}(X{=}1,2,3,Y{=}0) - \mathbb{P}(X{=}1,2,Y{=}1) - \mathbb{P}(X{=}1,Y{=}2) = \\
&= 1 - e^{-2}\left(\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+2\left(\frac{1}{3}+\frac{2}{9}\right)+2\cdot\frac{1}{3}\right) = 1 - \frac{67}{27e^2} \\
&= 1 - \frac{e^{-2}}{3}\left(1+\frac{2}{3}+\frac{4}{9}+2\left(1+\frac{2}{3}\right)+2\right) = 1 - \frac{67}{27e^2}
\end{aligned}$$

[**(1b)**] In the same situation find $\mathbf{Cov}(\boldsymbol{X+Y, XY})$.

Solution 1:

$\mathrm{Cov}(X+Y,XY) = \mathbb{E}(X+Y)XY - \mathbb{E}(X+Y)\mathbb{E}XY =$

$= \mathbb{E}X^2\mathbb{E}Y + \mathbb{E}X\mathbb{E}Y^2 - (\mathbb{E}X)^2\mathbb{E}Y - \mathbb{E}X(\mathbb{E}Y)^2 = 30 + 18 - 18 - 12 = 18$

because from the table of distributions we have

$\mathbb{E}X = 3$, $\mathbb{E}X^2 = (\mathbb{E}X)^2 + \mathrm{Var}\,(X) = 9 + 6 = 15$

$\mathbb{E}Y = 2$, $\mathbb{E}Y^2 = (\mathbb{E}Y)^2 + \mathrm{Var}\,(Y) = 4 + 2 = 6$

Solution 2:

$\mathrm{Cov}(X+Y,XY) = \mathbb{E}(X+Y)XY - \mathbb{E}(X+Y)\mathbb{E}XY =$

$= \mathbb{E}X^2\mathbb{E}Y + \mathbb{E}X\mathbb{E}Y^2 - (\mathbb{E}X)^2\mathbb{E}Y - \mathbb{E}X(\mathbb{E}Y)^2 =$

$= (\mathbb{E}X^2 - (\mathbb{E}X)^2)\mathbb{E}Y + \mathbb{E}X(\mathbb{E}Y^2 - (\mathbb{E}Y)^2) = \mathrm{Var}\,(X)\cdot\mathbb{E}Y + \mathbb{E}X\cdot\mathrm{Var}\,(Y) = 6\cdot 2 + 2\cdot 3 = 18$

[(2a)] Let X, Y be uniformly distributed in the rectangle defined by $-1 < x-y < 1, \quad 1 < x+y < 4$. Find the marginal density $f_X(x)$.

Solution 1 and Solution 2 will be explained in a video lecture. Answer:

$$f_X(x) = \begin{cases} 0 & \text{if } x < 0 \text{ or } x > 2.5 \\ 2x/3 & \text{if } 0 < x < 1 \\ 2/3 & \text{if } 1 < x < 1.5 \\ (5-2x)/3 & \text{if } 1.5 < x < 2.5 \end{cases}$$

For ***Solution 1***, we compute that the area of the rectangle is $A = 3$, and so $f(x, y) = 1/3$. Then we can use

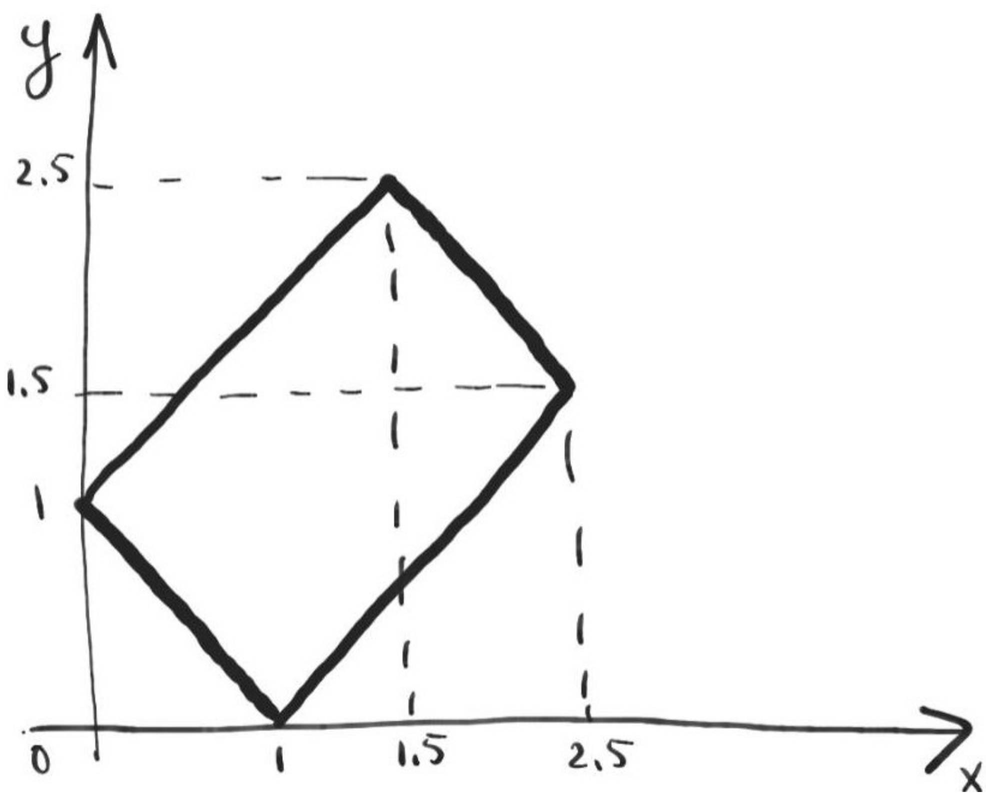
$$f_X(x) = \int_A f(x, y)dy$$

$f_X(x) = \int_{1-x}^{1+x} \frac{1}{3} dy = 2x/3$ if $0 < x < 1$.

$f_X(x) = \int_{-1+x}^{1+x} \frac{1}{3} dy = 2/3$ if $1 < x < 1.5$.

$f_X(x) = \int_{-1+x}^{4-x} \frac{1}{3} dy = (5-2x)/3$ if $1.5 < x < 2.5$.

For ***Solution 2***, use not calculus but geometry: we note that $f_X(x) = L(x)/A$ where $L(x)$ is the length of the vertical line segment above x and inside the area.



[(**2b**)] In the same situation find $\mathbb{E}(\boldsymbol{Y}|\boldsymbol{X})$.

Solution 1 and Solution 2 will be explained in a video lecture. For ***Solution 1***, $\boldsymbol{f(x,y) = 1/3}$ and

$$\mathbb{E}(\boldsymbol{Y}|\boldsymbol{X}) = \frac{1}{f_X(x)} \int_A y f(x,y) dy = \frac{1}{f_X(x)} \int_A y f(x,y) dy = \int_A y f_{Y|X=x}(y) dy$$

$$f_{Y|X=x} = \frac{f(x,y)}{f_X(x)}$$

$\mathbb{E}(\boldsymbol{Y}|\boldsymbol{X}) = \int_{1-x}^{1+x} y \frac{1}{2x} dy = \frac{(1+x)^2-(1-x)^2}{4x} = 1$ if $0 < x < 1$.

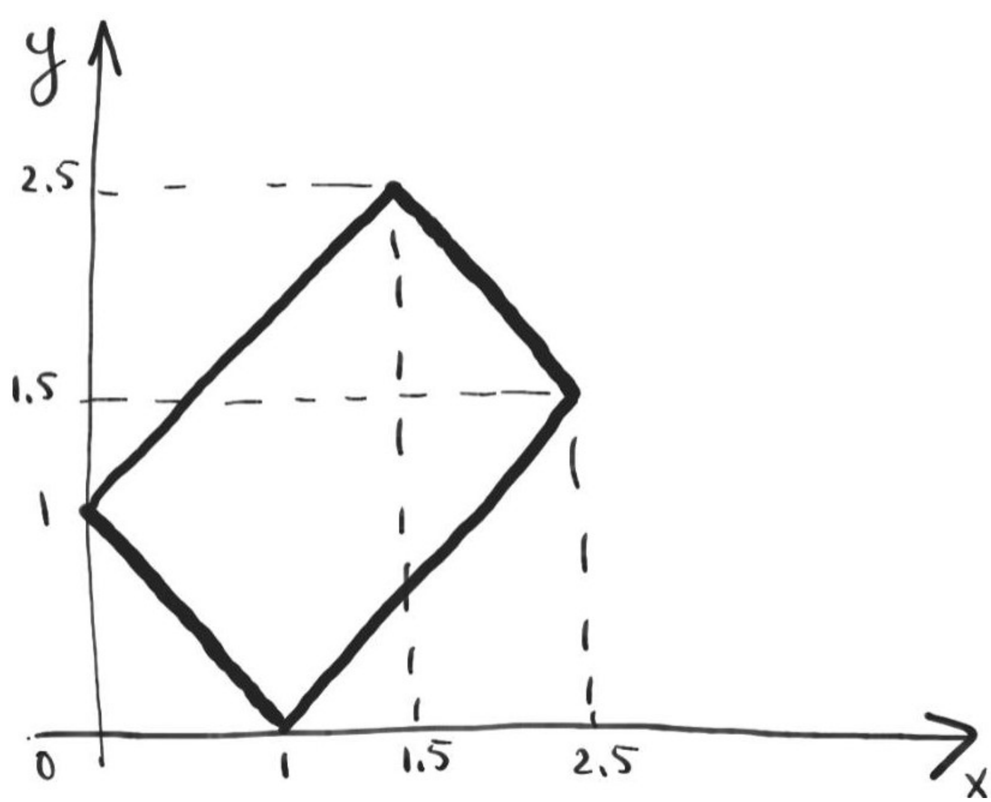
$\mathbb{E}(\boldsymbol{Y}|\boldsymbol{X}) = \int_{-1+x}^{1+x} \frac{y}{2} dy = \frac{(1+x)^2-(-1+x)^2}{4} = x$ if $1 < x < 1.5$.

$\mathbb{E}(\boldsymbol{Y}|\boldsymbol{X}) = \int_{-1+x}^{4-x} \frac{y}{5-2x} dy = \frac{(4-x)^2-(-1+x)^2}{5-2x} = \frac{15-6x}{5-2x} = \frac{2}{3}$ if $1.5 < x < 2.5$.

Answer:

$$\mathbb{E}(\boldsymbol{Y}|\boldsymbol{X}) = \begin{cases} \text{DNE} & \text{if } x < 0 \text{ or } x > 2.5 \\ 1 & \text{if } 0 < x < 1 \\ x & \text{if } 1 < x < 1.5 \\ 2/3 & \text{if } 1.5 < x < 2.5 \end{cases}$$

For ***Solution 2***, use not calculus but geometry: we use that $f_X(x) = L(x)/A$ where $L(x)$ is the length of the vertical line segment above x and inside the area. Hence $\mathbb{E}(\boldsymbol{Y}|\boldsymbol{X}) = M(x)$ where $M(x)$ is the middle point of the vertical line segment above x and inside the area.



[(**2c**)] In the same situation find $\mathbf{Cov}(\boldsymbol{X}, \boldsymbol{Y})$.

Solution 1 and Solution 2 will be explained in a video lecture. Answer:

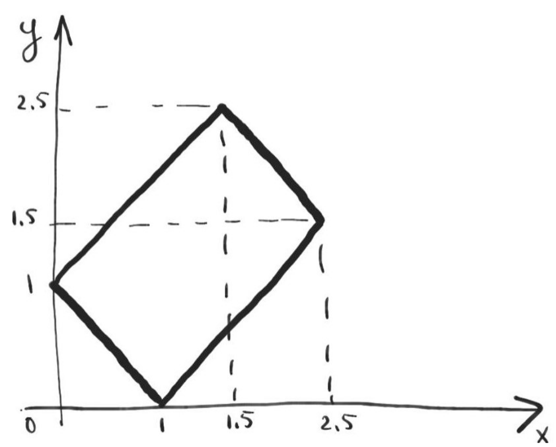
$$\mathbf{Cov}(\boldsymbol{X}, \boldsymbol{Y}) = \mathbb{E}\boldsymbol{XY} - \mathbb{E}\boldsymbol{X} \cdot \mathbb{E}\boldsymbol{Y} = \frac{\mathbf{5}}{\mathbf{3}} - \left(\frac{\mathbf{5}}{\mathbf{4}}\right)^{\mathbf{2}} = \frac{\mathbf{5}}{\mathbf{48}}$$

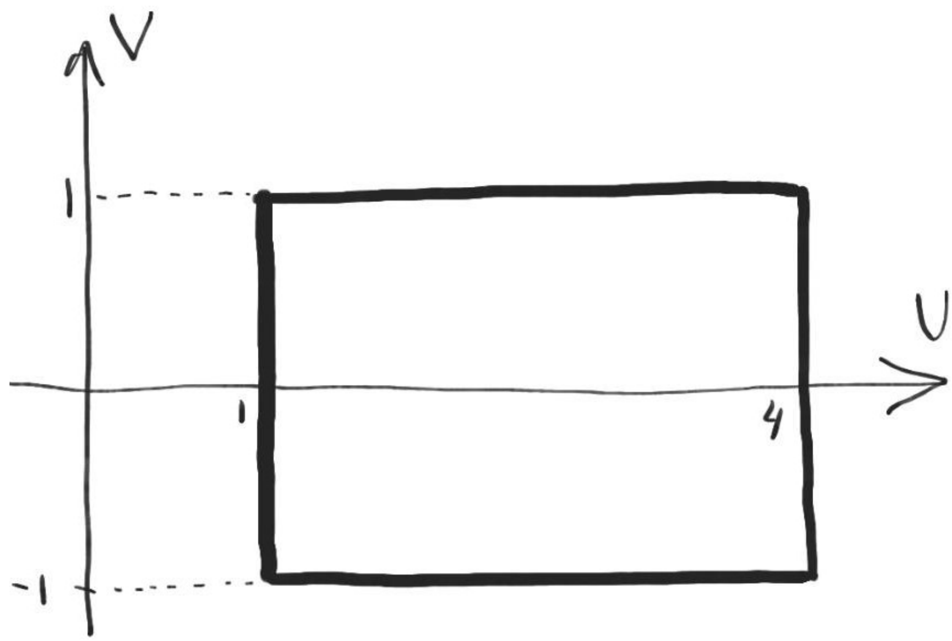
Solution 1:

$$\mathbb{E}\boldsymbol{XY} = \int_{\boldsymbol{A}} \boldsymbol{xyf(x,y)dxdy} =$$

$$\frac{1}{3}\left(\int_0^1 \int_{1-x}^{1+x} xy\, dydx + \int_1^{1.5} \int_{-1+x}^{1+x} xy\, dydx + \int_{1.5}^{2.5} \int_{-1+x}^{4-x} xy\, dydx\right) = \ldots =$$

$$= \frac{1}{3}\left(\frac{2}{3} + \frac{19}{12} + \frac{11}{4}\right) = \frac{1}{3}\left(\frac{8}{12} + \frac{19}{12} + \frac{33}{12}\right) = \frac{5}{3}$$





Solution 2: $\boldsymbol{U} = \boldsymbol{X} + \boldsymbol{Y}, \quad \boldsymbol{V} = \boldsymbol{X} - \boldsymbol{Y}$

$$\mathbb{E}\boldsymbol{XY} = \frac{1}{4}\mathbb{E}(\boldsymbol{U}+\boldsymbol{V})(\boldsymbol{U}-\boldsymbol{V}) = \frac{1}{4}\left(\mathbb{E}\boldsymbol{U}^2 - \mathbb{E}\boldsymbol{V}^2\right) = \frac{1}{4}\left(\frac{1}{3}\int_1^4 u^2\, du - \frac{1}{2}\int_{-1}^1 v^2\, dv\right) =$$

$$= \frac{1}{4}\left(\frac{64-1}{9} - \frac{2}{6}\right) = \frac{5}{3}$$