Phason Engineering in Aperiodic Resonant Structures

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Workshop: Quasi-periodic spectral and topological analysis
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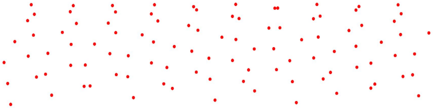
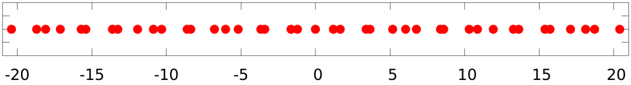
What is the phason of a generic pattern?



A Closer Look at Patterns



$$x_n(\phi) = n + 0.4 \sin\big(2\pi(n\theta + \phi)\big), \quad [\theta = \text{fixed}]$$

Observations:

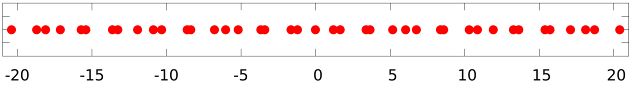
- Clearly we have a natural degree of freedom, ϕ, which lives on a circle
- Changes of ϕ produce global changes of the pattern
- If we shift the pattern rigidly such that point a sits at the origin

$$\{x_n\}_{n\in\mathbb{Z}} \mapsto \{x'_n\}_{n\in\mathbb{Z}}, \quad x'_n(\phi) = x_{n+a}(\phi)$$

this is equivalent to $\phi \mapsto \phi + a\theta$.

Let's call ϕ the phason of the pattern.

What if the algorithm is unknown?



For this particular pattern,

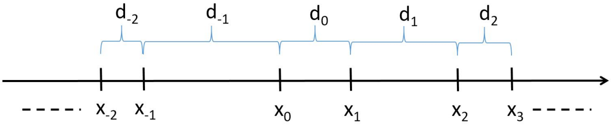
there is a relation between the dynamical system ($\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ is the circle)

$$\sigma_n : \mathbb{S}^1 \to \mathbb{S}^1, \quad \sigma_n(\phi) = (\phi + n\theta) \bmod 1$$

and the rigid shifts of the pattern which gets every point to sit at the origin once.

Question: Can we see this circle from the pattern, without knowing the algorithm?

Astonishingly (at least to me): YES!



Obvious facts about pattern $\mathcal{P} = \{x_n\}_{n\in\mathbb{Z}}$:

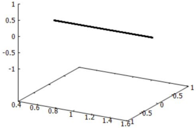
- If $x_0 = 0$, $\mathcal{P}$ can be reproduced from the sequence $\{d_n\}_{n\in\mathbb{Z}}$.
- Hence, $\mathcal{P}$ is a point of $[0,1]^\infty$.
- If $\mathcal{P}$ is rigidly shifted
$$\mathcal{P} \to \tau_a(\mathcal{P}) = \{d_{n-a}\}_{n\in\mathbb{Z}}, \quad a \in \mathbb{Z},$$
such that each point sits once at the origin, then $\mathcal{P}$ traces a shape in $[0,1]^\infty$.
- This shape Ξ together with the action τ of $\mathbb{Z}$ form a dynamical system (Ξ, τ).

Statement:

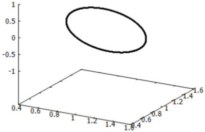
(Ξ, τ) and $(\mathbb{S}^1, \sigma)$ are identical as dynamical systems.

Seeing is Believing

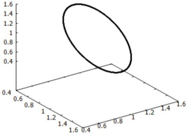
$$x_n(\phi) = n + 0.4\sin\left(2\pi(n\theta + \phi)\right), \quad \theta \neq \mathbb{Q}$$



Projection on d_0

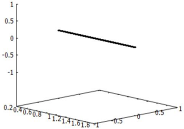


Projection on (d_0, d_1)

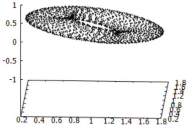


Projection on (d_0, d_1, d_2)

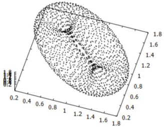
$$x_n(\phi_1, \phi_2) = n + 0.2\sin\left(2\pi(n\theta_1 + \phi_1)\right) + 0.2\sin\left(2\pi(n\theta_2 + \phi_2)\right), \quad \theta_1,\ \theta_2,\ \theta_1/\theta_2 \neq \mathbb{Q}$$



Projection on d_0



Projection on (d_0, d_1)



Projection on (d_0, d_1, d_2)

Phason Space and Phason Defined

Facts for patterns in $\mathbb{R}^d$:

- The pattern can be re-constructed from the vectors connecting near-neighbors
- Hence, the pattern is again a point in $[0,1]^\infty$
- If $\mathcal{P}$ is rigidly shifted
$$\mathcal{P} \to \mathcal{P} - p, \quad p \in \mathcal{P},$$
such that each point sits once at the origin, then $\mathcal{P}$ traces a shape in $[0,1]^\infty$ (in $\mathbb{R}^d$, $[0,1]$ becomes the space of Voronoi cells)
- Then

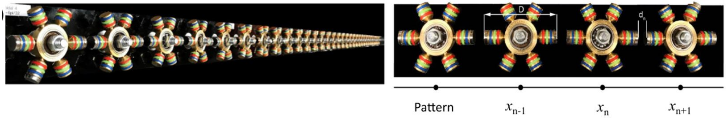
  The phason is the shape Ξ traced by this process (its closure, more precisely)
- When the points labeled by $\mathbb{Z}^d$, there is more to it, namely,

  a whole dynamical system (Ξ, τ).

Computing Ξ is notoriously hard:

- For quasi-crystals, Ξ's are Canorized tori.
- For quasi-periodic patterns, Ξ's are tori.

Why is this interesting for a meta-material scientist?



Facts and Assumptions:

- The resonators are identical.
- The (linearized) dynamics takes place in the Hilbert space spanned by $|n\rangle$
- The generic dynamical matrices look like:

$$D = \sum_{m,n} h_{m,n}(\mathcal{P})\, |m\rangle\langle n|$$

- Once the resonators are chosen, the coupling matrices $h_{n,m}$ are entirely determined by the pattern (i.e. they are functions of $\mathcal{P} \in [0,1]^\infty$).

Can we sort things out at such general setting?

YES, because the algebra of these D's is Small

Magical Facts:

- Galilean invariance implies:

$$h_{m,n}(\mathcal{P}) = h_{m-a,n-a}(\tau_a \mathcal{P}), \quad a \in \mathbb{Z}.$$

- Taking $a = n$, one index can be dropped and:

$$D = \sum_q S^q \sum_n h_q(\tau_n \mathcal{P}) \, |n\rangle\langle n|, \quad S|n\rangle = |n+1\rangle.$$

Conclusion:

All Galilean invariant dynamical matrices belong to the algebra generated by S and commuting diagonal operators:

$$T_f = \sum_n f(\tau_n \mathcal{P})|n\rangle\langle n|,$$

with f defined over the phason space Ξ. They obey the commutation relations:

$$T_f \, S = S \, T_{f \circ \tau_1}$$

Let's Compute one this Algebra for 1D Quasi-Periodic Patterns

For ALL cases where

$$\Xi = \mathbb{R}/\mathbb{Z}, \quad \tau_a(x) = (x + a\theta) \bmod 1$$

the following apply:

- Since f can Fourier decomposed, all T_f's are generated from:

$$T = \sum_n e^{\imath 2n\pi\theta} |n\rangle\langle n|$$

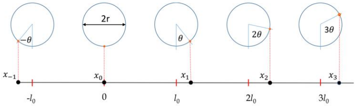
- The commutation relation is:

$$T\, S = e^{\imath 2\pi\theta} S\, T.$$

Conclusion:

No matter how wild the pattern is, if Ξ is a circle, the dynamical matrix belongs to the algebra of magnetic translations in 2D which generates IQHE

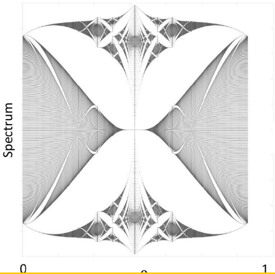
Not Convinced? Let's Compute the Spectrum of a Generic Hamiltonian



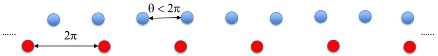
$$x_n = n + 0.4[\sin(n\theta + \phi) - \sin(\phi)]$$

Take:

$$D = \sum_{p,p' \in \mathcal{P}} W(|p - p'|)\,|p\rangle\langle p'| \quad [\text{numerically } W(x) = e^{-3|x|}]$$



How about this?



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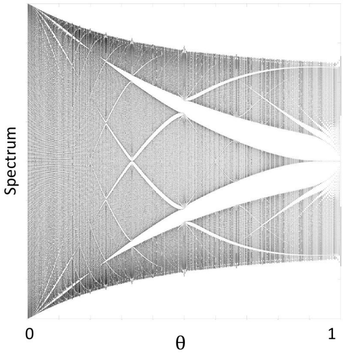
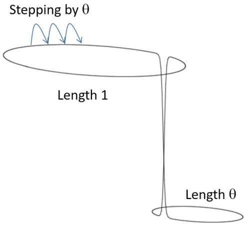




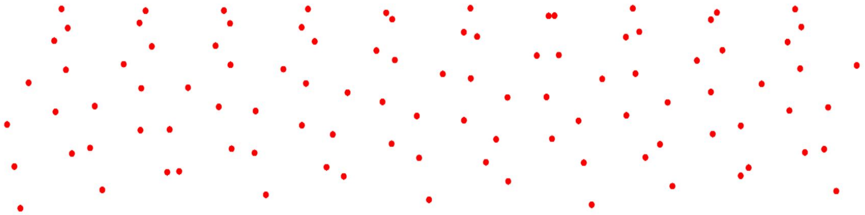
As a result:

- $\Xi = \mathbb{R}/(1+\theta)\mathbb{Z}$ and τ is the shift by θ.
- The algebra of the D's is generated by:

$$T\,S = e^{\imath 2\pi\theta'}\,S\,T, \quad \theta' = \theta/(1+\theta)$$

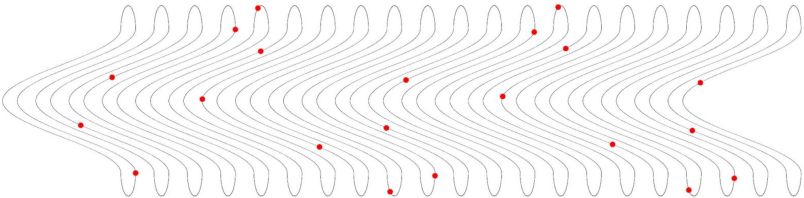
Quiz Time!!

Can we understand the spectrum of $D = \sum_{p,p' \in \mathcal{P}} W(|p-p'|)\,|p\rangle\langle p'|$ over this pattern??



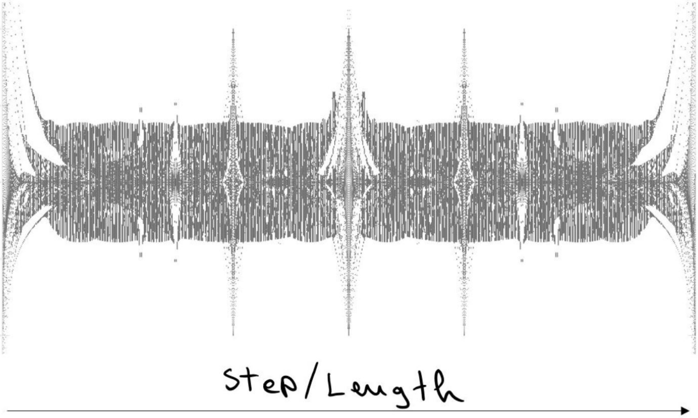
A Hint ...

The pattern was generated with the circle algorithm (but using a strong deforming lens!):

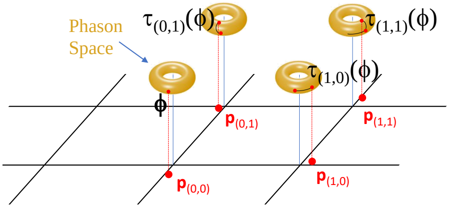


Hence, the phason space is again the circle.

The Answer is Clear



Generic Algorithm for Quasi-Periodic Structures

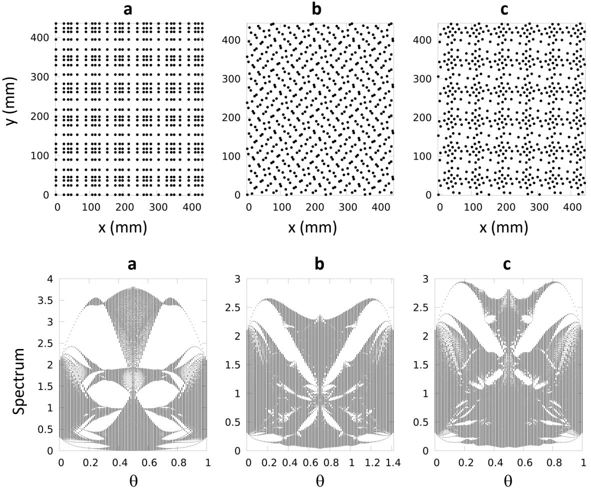


$$\mathbb{T}^{d'} = \mathbb{R}^{d'}/\mathcal{L}', \quad \tau_{\boldsymbol{n}}(\phi) = \Big(\phi + \sum_{i=1}^{d} n_i \boldsymbol{a}_i\Big) \bmod \mathcal{L}', \quad \phi \in \mathbb{T}^{d'}, \quad \boldsymbol{a}_i \in \mathbb{R}^d \subseteq \mathbb{R}^{d'}$$

If $F : \mathbb{T}^{d'} \to \mathbb{R}^d$ is a continuous map and

$$\mathcal{P}_\phi = \{\boldsymbol{p}_{\boldsymbol{n}}(\phi)\}_{n \in \mathbb{Z}^d}, \quad \boldsymbol{p}_{\boldsymbol{n}}(\phi) = \boldsymbol{p}_0 + \sum_{i=1}^{d} n_i \boldsymbol{a}_i + F\big(\tau_{\boldsymbol{n}}(\phi)\big), \quad \phi \in \mathbb{T}^{d'}$$

2D Examples (Spectra computed with $D = \sum_{x,x' \in \mathcal{P}} e^{-|x-x'|} |x\rangle\langle x'|$)



In this case $\Xi = \mathbb{T}^{d'}$, so we have

- d' number of T_f's, $T_1, \ldots, T_{d'}$ (Fourier transform over d'-torus)
- d number of space shifts, $S_1 = T_{d'+1}, \ldots S_d = T_{d'+d}$.

Fact:

$$T_i\, T_j = e^{\imath 2\pi\theta_{ij}}\, T_j\, T_i \tag{1}$$

with the matrix $\Theta = \{\theta_{ij}\}$ fully determined by the two lattices $\mathcal{L}$ and $\mathcal{L}'$. Specifically, if A is the transformation matrix, $\boldsymbol{a}_j = \sum_{j=1}^{d'} A_{ji}\boldsymbol{a}'_i$, $j = \overline{1, d}$, then

$$\theta_{ij} = -\theta_{ji} = A_{ji}, \quad i = \overline{1, d'}, \quad j = \overline{1, d}, \tag{2}$$

and zero for the rest of the indices.

Conclusion: We are generating QHE in higher dimensions!