Phason Engineering in Aperiodic Resonant Structures

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What is the phason of a generic pattern?
A Closer Look at Patterns

\[ x_n(\phi) = n + 0.4 \sin (2\pi (n\theta + \phi)), \quad [\theta = \text{fixed}] \]

Observations:

- Clearly we have a natural degree of freedom, \( \phi \), which lives on a circle.
- Changes of \( \phi \) produce global changes of the pattern.
- If we shift the pattern rigidly such that point \( a \) sits at the origin,

\[
\{x_n\}_{n \in \mathbb{Z}} \mapsto \{x'_n\}_{n \in \mathbb{Z}}, \quad x'_n(\phi) = x_{n+a}(\phi)
\]

this is equivalent to \( \phi \mapsto \phi + a\theta \).

Let’s call \( \phi \) the phason of the pattern.
What if the algorithm is unknown?

For this particular pattern, there is a relation between the dynamical system \((S^1 = \mathbb{R}/\mathbb{Z} \text{ is the circle})\)

\[
\sigma_n : S^1 \rightarrow S^1, \quad \sigma_n(\phi) = (\phi + n\theta) \mod 1
\]

and the rigid shifts of the pattern which gets every point to sit at the origin once.

Question: Can we see this circle from the pattern, without knowing the algorithm?

Astonishingly (at least to me): YES!
Detecting the Phason Space from the Pattern

**Obvious facts about pattern** $\mathcal{P} = \{x_n\}_{n \in \mathbb{Z}}$:

- If $x_0 = 0$, $\mathcal{P}$ can be reproduced from the sequence $\{d_n\}_{n \in \mathbb{Z}}$.
- Hence, $\mathcal{P}$ is a point of $[0, 1]^\infty$.
- If $\mathcal{P}$ is rigidly shifted
  \[ \mathcal{P} \rightarrow \tau_a(\mathcal{P}) = \{d_n - a\}_{n \in \mathbb{Z}}, \quad a \in \mathbb{Z}, \]
  such that each point sits once at the origin, then $\mathcal{P}$ traces a shape in $[0, 1]^\infty$.
- This shape $\Xi$ together with the action $\tau$ of $\mathbb{Z}$ form a dynamical system $(\Xi, \tau)$.

**Statement:**

$(\Xi, \tau)$ and $(S^1, \sigma)$ are identical as dynamical systems.
\[
x_n(\phi) = n + 0.4 \sin \left(2\pi (n\theta + \phi)\right), \quad \theta \neq \mathbb{Q}
\]

\[
x_n(\phi_1, \phi_2) = n + 0.2 \sin \left(2\pi (n\theta_1 + \phi_1)\right) + 0.2 \sin \left(2\pi (n\theta_2 + \phi_2)\right), \quad \theta_1, \theta_2, \theta_1/\theta_2 \neq \mathbb{Q}
\]
Phason Space and Phason Defined

Facts for patterns in $\mathbb{R}^d$:

- The pattern can be re-constructed from the vectors connecting near-neighbors.
- Hence, the pattern is again a point in $[0, 1]^\infty$.
- If $\mathcal{P}$ is rigidly shifted
  
  \[ \mathcal{P} \rightarrow \mathcal{P} - p, \quad p \in \mathcal{P}, \]

  such that each point sits once at the origin, then $\mathcal{P}$ traces a shape in $[0, 1]^\infty$ (in $\mathbb{R}^d$, $[0, 1]$ becomes the space of Voronoi cells).

- Then
  
  The phason is the shape $\Xi$ traced by this process (its closure, more precisely).

- When the points labeled by $\mathbb{Z}^d$, there is more to it, namely,
  
  a whole dynamical system $(\Xi, \tau)$.

Computing $\Xi$ is notoriously hard:

- For quasi-crystals, $\Xi$'s are Canorized tori.
- For quasi-periodic patterns, $\Xi$'s are tori.
Why is this interesting for a meta-material scientist?
Facts and Assumptions:

- The resonators are identical.
- The (linearized) dynamics takes place in the Hilbert space spanned by $|n\rangle$.
- The generic dynamical matrices look like:

$$D = \sum_{m,n} h_{m,n}(\mathcal{P}) |m\rangle \langle n|$$

- Once the resonators are chosen, the coupling matrices $h_{n,m}$ are entirely determined by the pattern (i.e. they are functions of $\mathcal{P} \in [0,1]^\infty$).

Can we sort things out at such general setting?
YES, because the algebra of these D’s is Small

Magical Facts:

- Galilean invariance implies:
  \[ h_{m,n}(\mathcal{P}) = h_{m-a,n-a}(\tau_a \mathcal{P}), \quad a \in \mathbb{Z}. \]

- Taking \( a = n \), one index can be dropped and:
  \[ D = \sum_q S^q \sum_n h_q(\tau_n \mathcal{P}) |n\rangle \langle n|, \quad S |n\rangle = |n + 1\rangle. \]

Conclusion:

All Galilean invariant dynamical matrices belong to the algebra generated by \( S \) and commuting diagonal operators:

\[ T_f = \sum_n f(\tau_n \mathcal{P}) |n\rangle \langle n|, \]

with \( f \) defined over the phason space \( \Xi \). They obey the commutation relations:

\[ T_f S = S T_{f \circ \tau_1} \]
Let's Compute one this Algebra for 1D Quasi-Periodic Patterns

For ALL cases where

$$\Xi = \mathbb{R}/\mathbb{Z}, \quad \tau_a(x) = (x + a\theta) \mod 1$$

the following apply:

- Since $f$ can Fourier decomposed, all $T_f$’s are generated from:

  $$T = \sum_n e^{i2n\pi\theta} |n\rangle \langle n|$$

- The commutation relation is:

  $$T S = e^{i2\pi\theta} S T.$$

Conclusion:

No matter how wild the pattern is, if $\Xi$ is a circle, the dynamical matrix belongs to the algebra of magnetic translations in 2D which generates IQHE.
Not Convinced? Let’s Compute the Spectrum of a Generic Hamiltonian

\[ x_n = n + 0.4[\sin(n\theta + \phi) - \sin(\phi)] \]

Take:

\[ D = \sum_{p, p' \in P} W(|p - p'|) |p\rangle \langle p'| \quad \text{[numerically } W(x) = e^{-3|x|} \text{]} \]
How about this?

Take:

\[ D = \sum_{p, p' \in \mathcal{P}} W(|p - p'|) |p\rangle\langle p'| \quad \text{[numerically } W(x) = e^{-3|x|}\text{]} \]
As a result:

- $\Xi = \mathbb{R}/(1 + \theta)\mathbb{Z}$ and $\tau$ is the shift by $\theta$.

- The algebra of the $D$’s is generated by:

$$T S = e^{i2\pi \theta'} S T, \quad \theta' = \theta/(1 + \theta)$$
Can we understand the spectrum of $D = \sum_{p,p' \in \mathcal{P}} W(|p - p'|) |p\rangle\langle p'|$ over this pattern??
A Hint ...

The pattern was generated with the circle algorithm (but using a strong deforming lens!):

Hence, the phason space is again the circle.
The Answer is Clear

[Image of a diagram with 'step/Length']
Generic Algorithm for Quasi-Periodic Structures

\[ T^{d'} = \mathbb{R}^{d'} / \mathcal{L}', \quad \tau_n(\phi) = \left( \phi + \sum_{i=1}^{d} n_i a_i \right) \mod \mathcal{L}', \quad \phi \in T^{d'}, \quad a_i \in \mathbb{R}^d \subseteq \mathbb{R}^{d'} \]

If \( F : T^{d'} \rightarrow \mathbb{R}^d \) is a continuous map and

\[ \mathcal{P}_\phi = \{ p_n(\phi) \}_{n \in \mathbb{Z}^d}, \quad p_n(\phi) = p_0 + \sum_{i=1}^{d} n_i a_i + F(\tau_n(\phi)), \quad \phi \in T^{d'} \]
2D Examples (Spectra computed with $D = \sum_{x,x' \in \mathcal{P}} e^{-|x-x'|} |x\rangle\langle x'|$)
In this case $\Xi = \mathbb{T}^{d'}$, so we have

* $d'$ number of $T_f$'s, $T_1, \ldots, T_{d'}$ (Fourier transform over $d'$-torus)
* $d$ number of space shifts, $S_1 = T_{d'+1}, \ldots S_d = T_{d'+d}$.

**Fact:**

$$T_i T_j = e^{i 2 \pi \theta_{ij}} T_j T_i$$  \hspace{1cm} (1)

with the matrix $\Theta = \{\theta_{ij}\}$ fully determined by the two lattices $\mathcal{L}$ and $\mathcal{L}'$. Specifically, if $A$ is the transformation matrix, $\mathbf{a}_j = \sum_{i=1}^{d'} A_{ji} \mathbf{a}'_i$, $j = 1, d'$, then

$$\theta_{ij} = -\theta_{ji} = A_{ji}, \quad i = 1, d', \quad j = 1, d,$$  \hspace{1cm} (2)

and zero for the rest of the indices.

**Conclusion:** We are generating QHE in higher dimensions!