Topological properties of tilings
From structure to spectrum

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PAZY
EXCELLING IN SCIENCE

Technion
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ISRAEL SCIENCE FOUNDATION

Virtual Workshop on new approaches to quasi-periodic spectral and topological analysis, May 2021

Is there a relation between structure and spectrum in aperiodic tilings ?

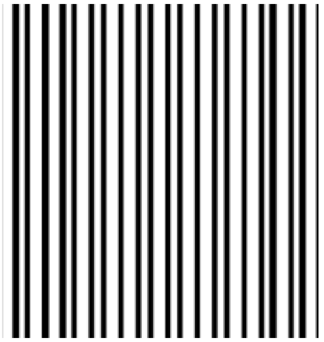
A equivalent of Bloch theorem for tilings

1. E. Akkermans, Y. Don, J. Rosenberg and C. L. Schochet, *Relating Diffraction and Spectral Data of Aperiodic Tilings: Towards a Bloch theorem*, J. Geom. Phys. **165**, 104217 (2021).

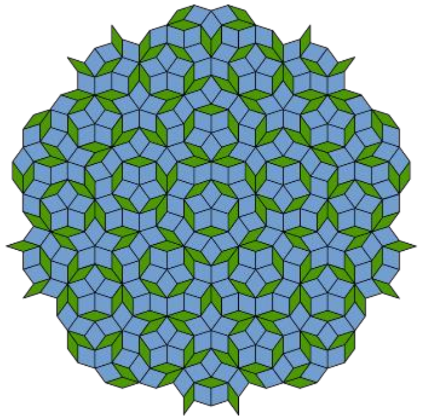
Outline

Tilings

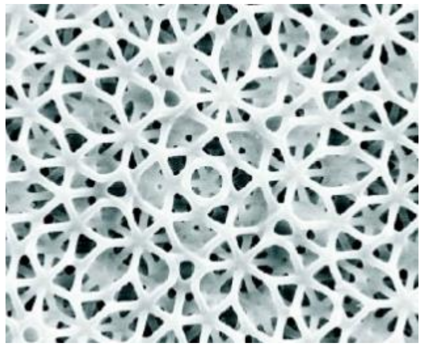
1D



2D



3D



Aperiodic Tilings

Problem: too big; ill defined

Quasicrystals a b a a b a b a a b a a b a b a a b a b a a b a a b a b a a b a a b

Thue-Morse a b b a b a a b b a a b a b b a b a a b a b b a a b b a b a a b

...

Wang Tiles

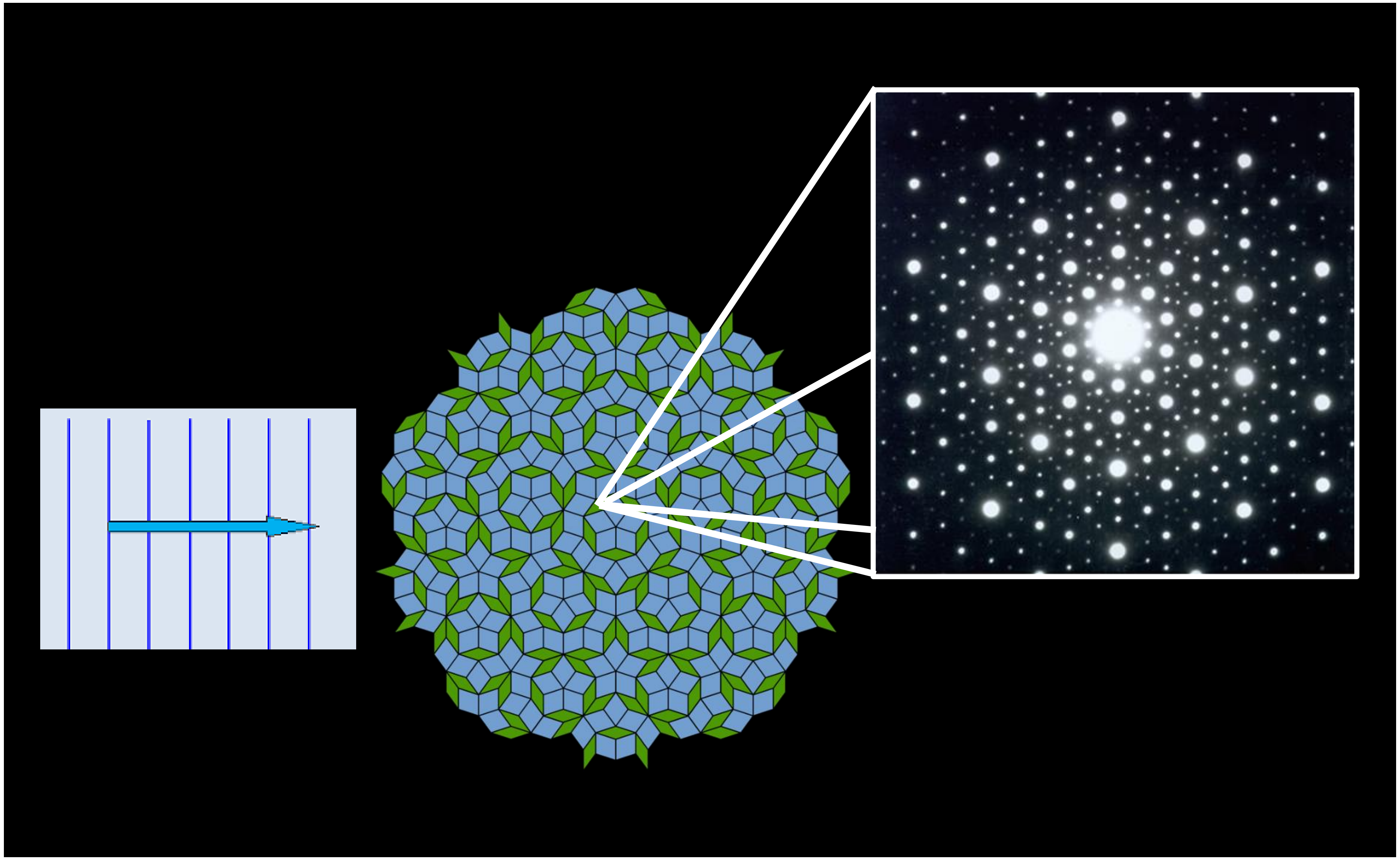
L-systems

...

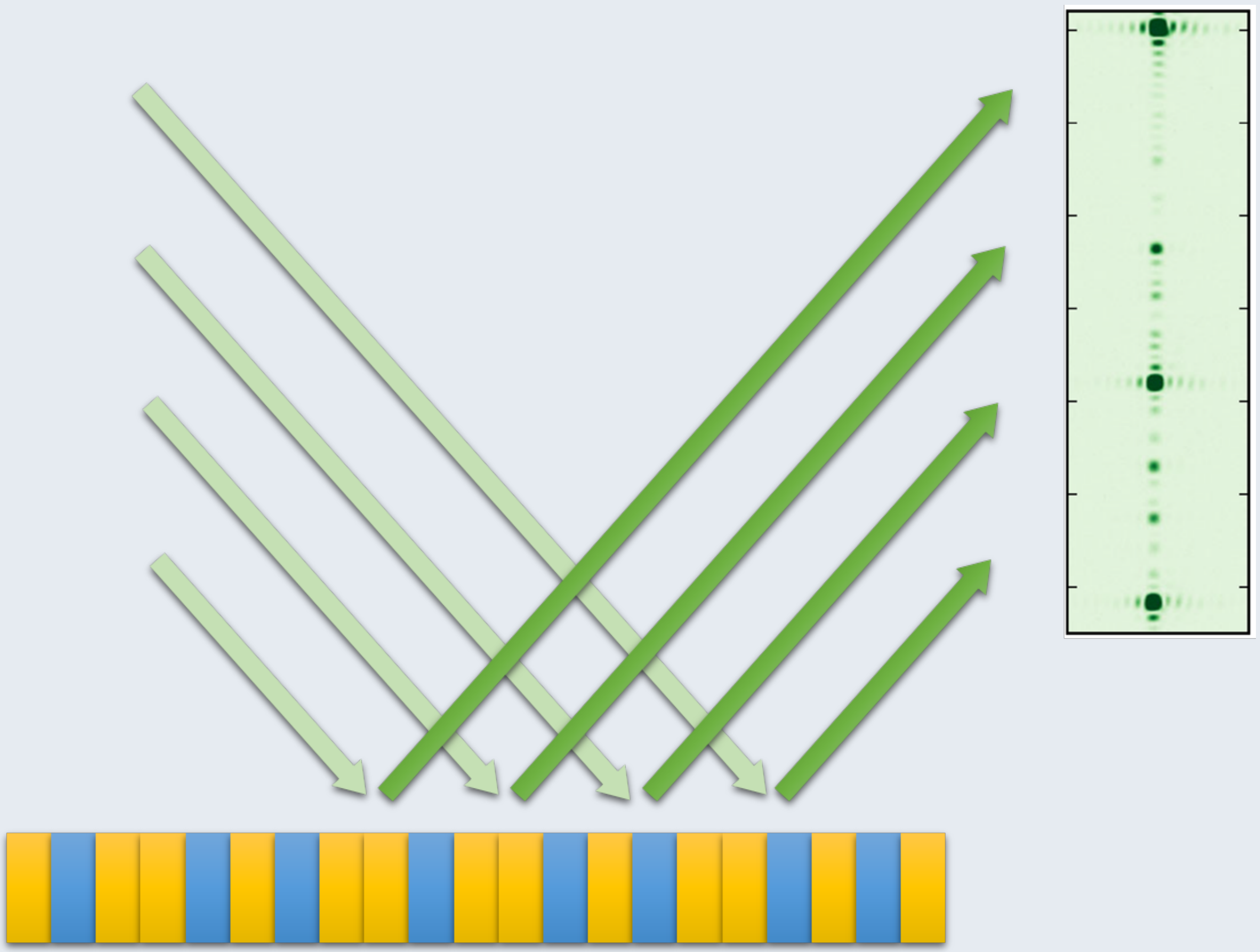
Periodic a a a b a a a b a a a b a a a b a a a b a a a b a a a b a a a b

How to Characterize Tilings – Structure?

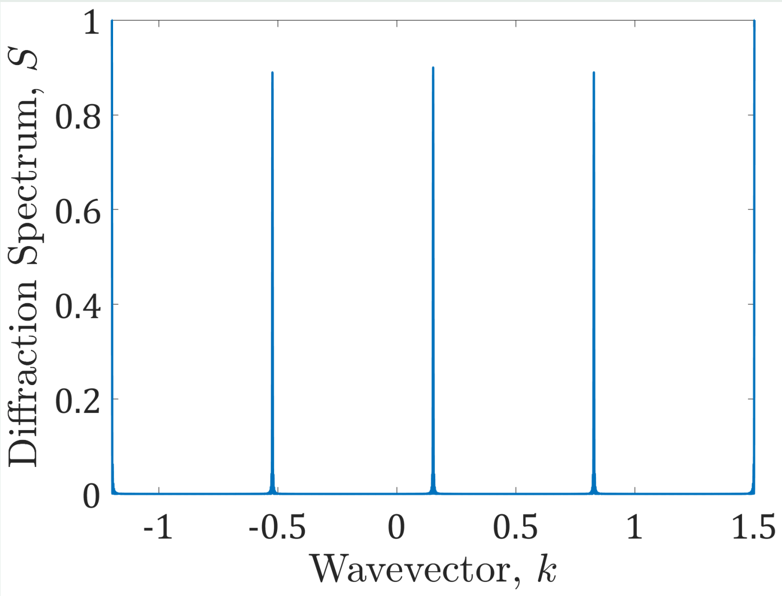
Diffraction pattern - Structure

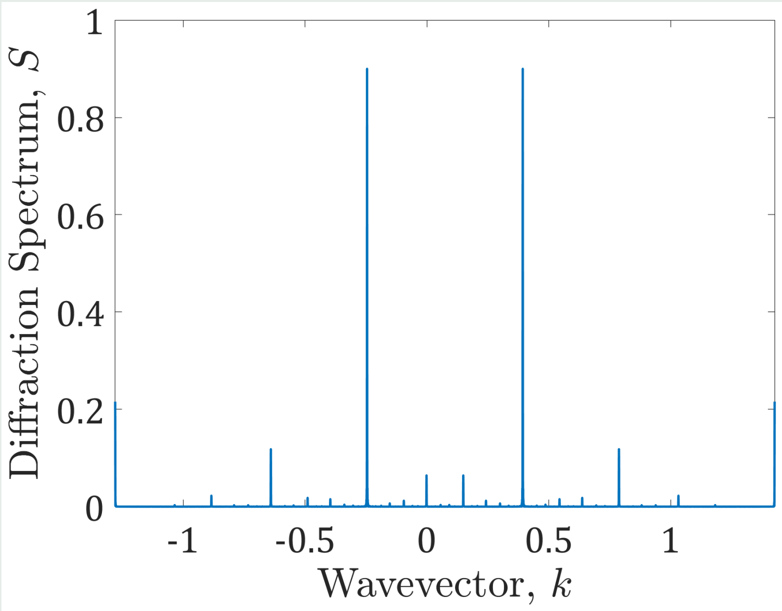


Diffraction (X-ray) pattern

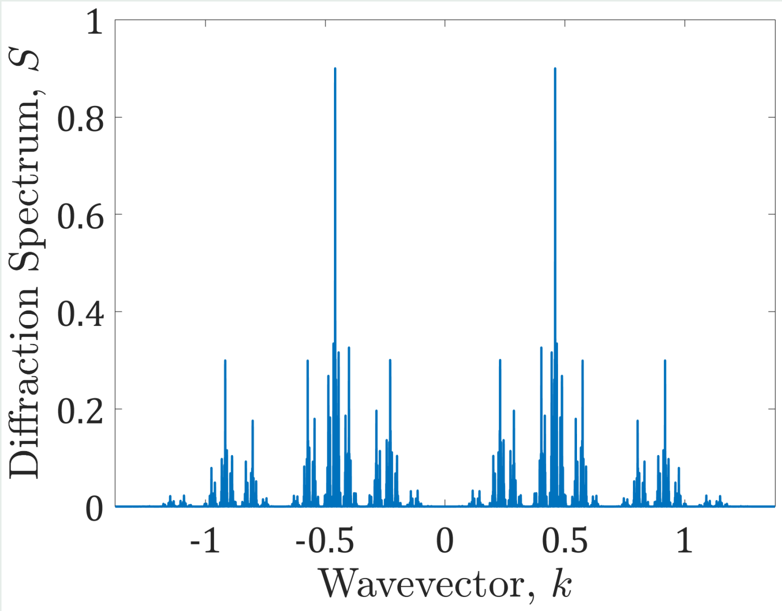


Periodic | **Quasiperiodic** | **Aperiodic**



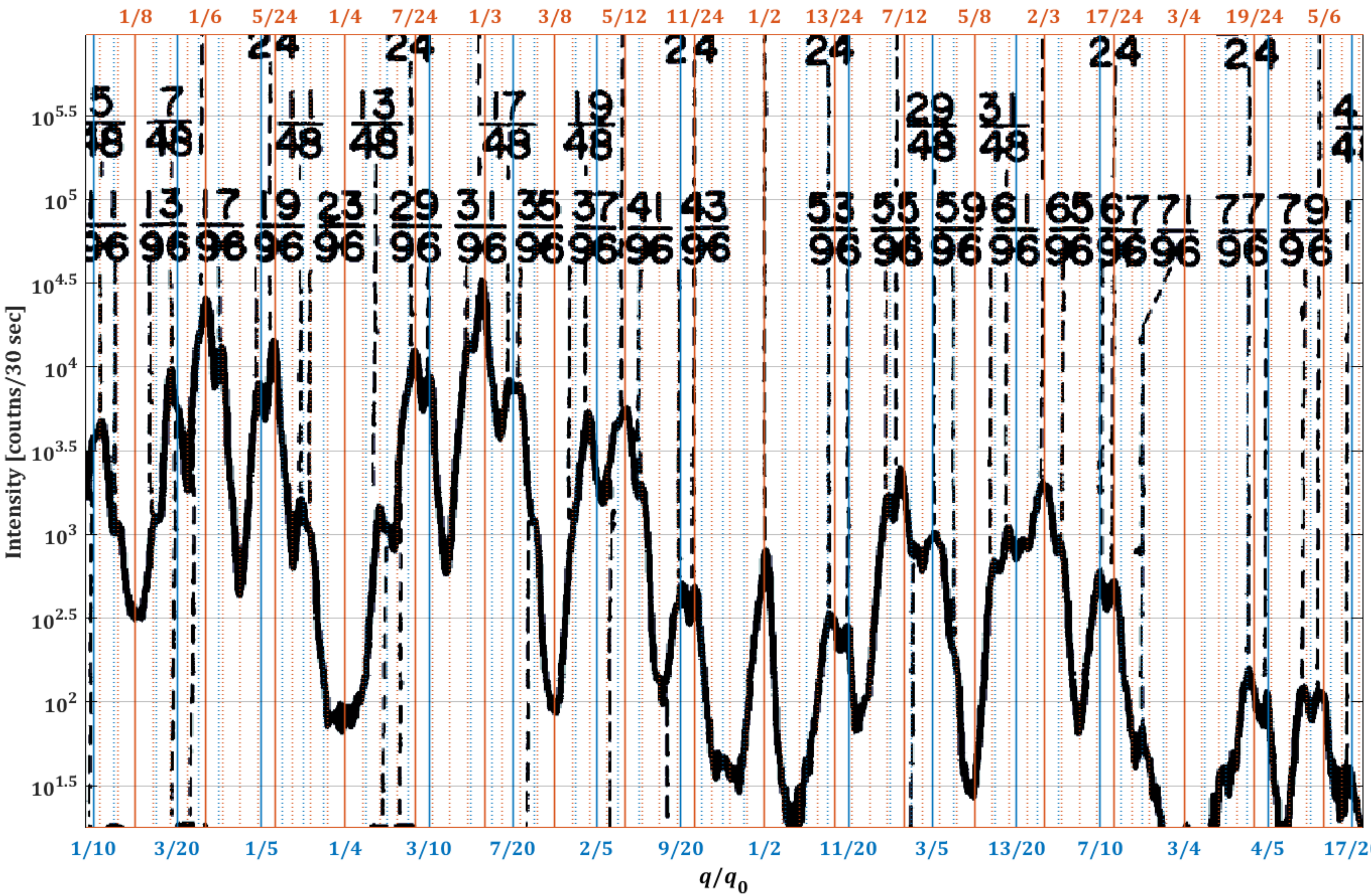


Fibonacci



Thue-Morse

Existence of a Bragg peaks (PP) diffraction pattern is often unclear (e.g. Thue-Morse)



F. Axel and H. Terauchi, Phys. Rev. Lett. **66**, 2223–2226 (1991)

How to Characterize Tilings – Spectrum?

How to Characterize Tilings – Spectrum?

Spectrum & Integrated Density of States

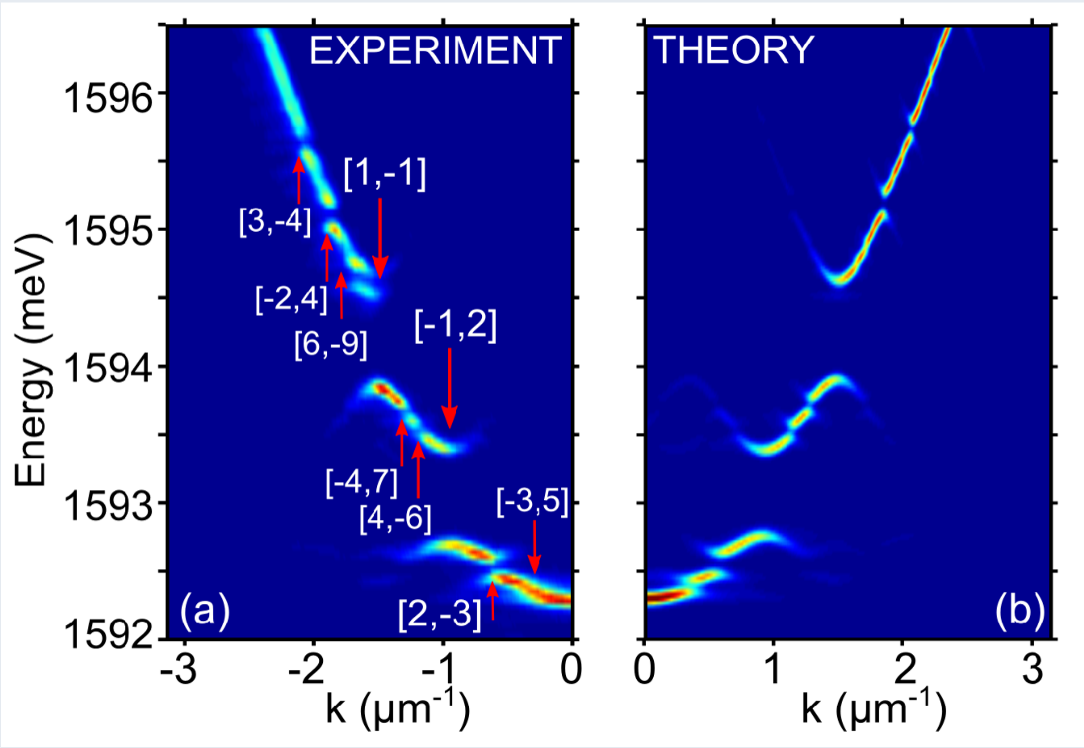
- Solve a Hamiltonian $H(E)$ (Fibonacci)

$$H\psi(x) = E\,\psi(x)$$

a b a a b a b a a b a a b a b a a b a b a a b a a b a b a a b a a b

- Find the spectrum:
  - dispersion $E(k)$
  - integrated density of states

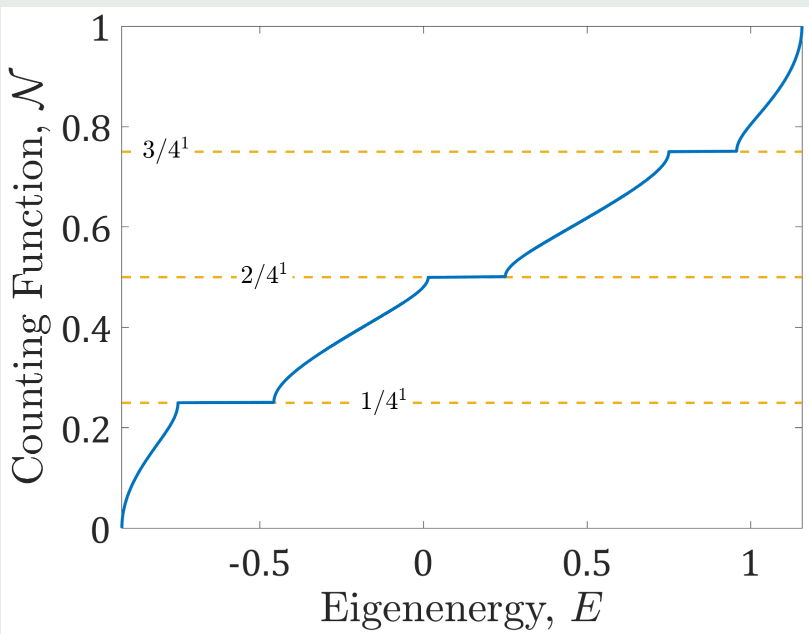
$$H(E) \to \begin{cases} \varrho(E) & \text{DOS} \\ \mathcal{N}(E) & \text{IDOS} \end{cases}$$



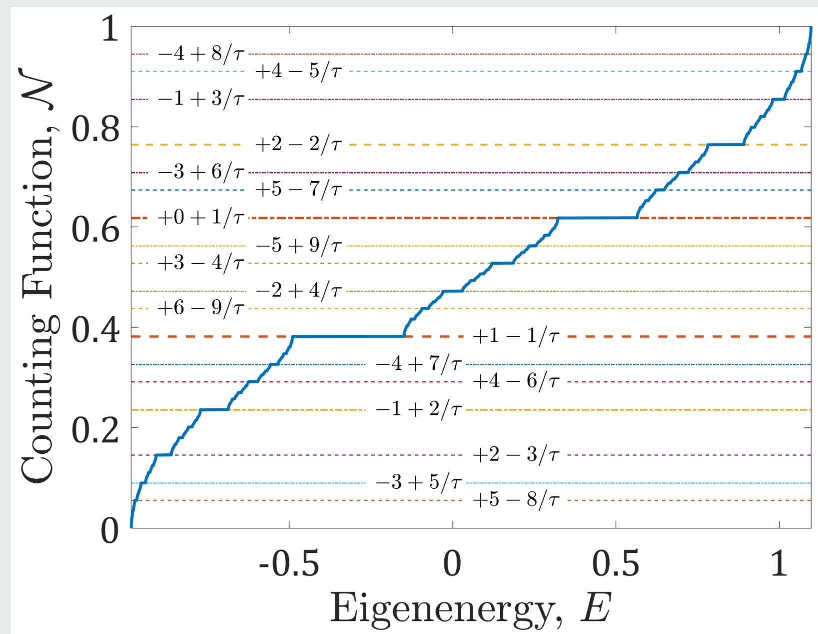
❑ D. Tanese et al., Phys. Rev. Lett. **112**, 146404 (2013)

How to Characterize Tilings – Spectrum?

Periodic

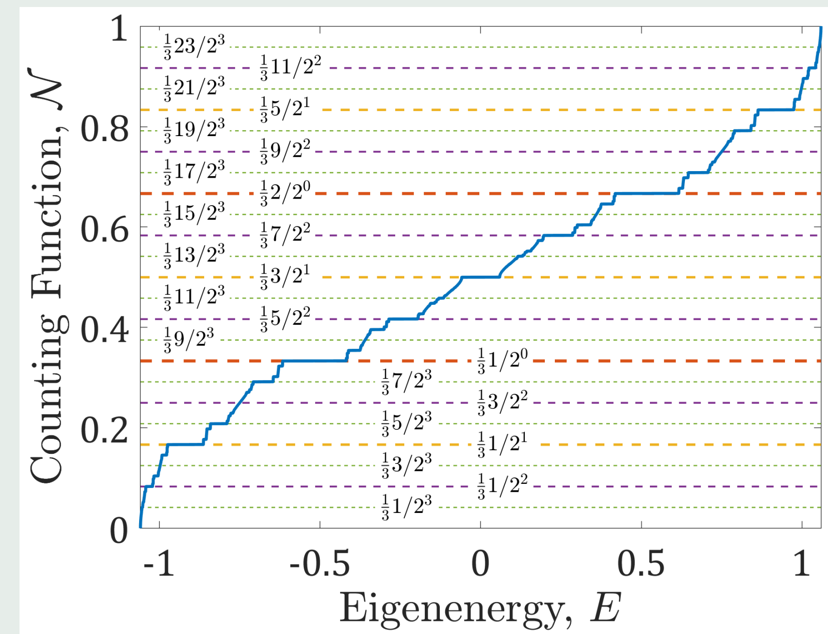


Quasiperiodic



Fibonacci

Aperiodic



Thue-Morse

Correspondence between Structure and Spectrum?

Bloch theorem

Periodic case: we know the connection

$$\text{finite \# of peaks} \xleftrightarrow[\text{correspondence}]{\text{1 to 1}} \text{finite \# of gaps}$$

Aperiodic case: this is not necessarily true

- We show that—at least for one family—there is a connection

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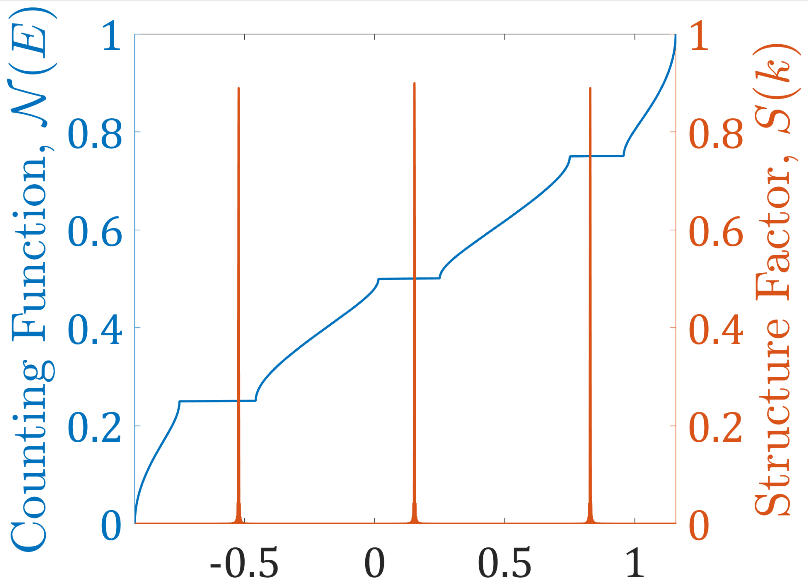
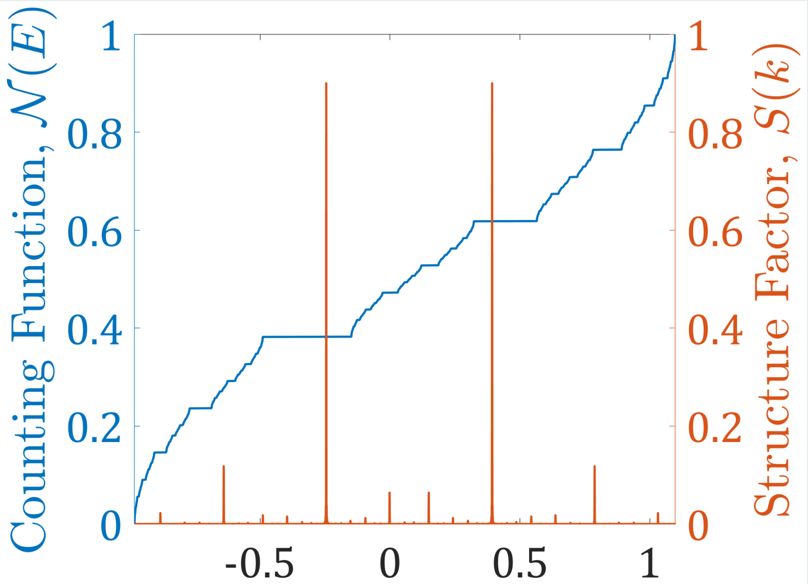
Correspondence between Structure and Spectrum?

Periodic

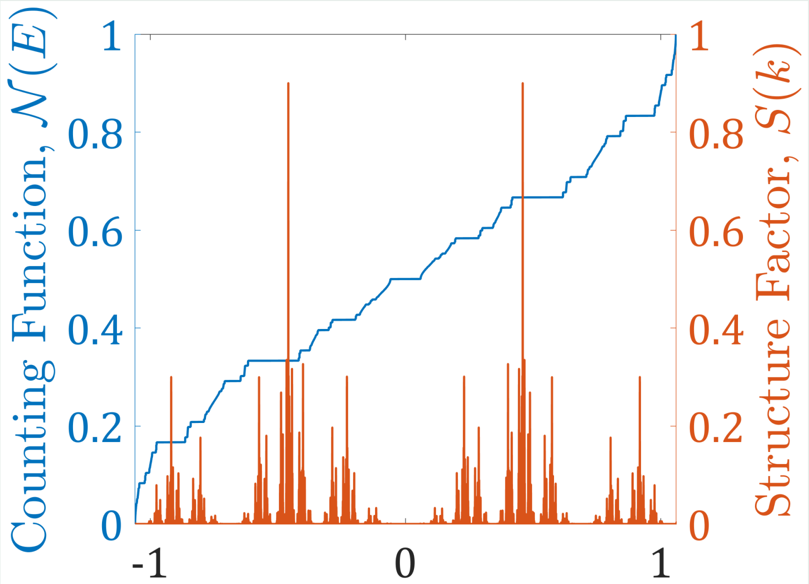


Quasiperiodic



Fibonacci

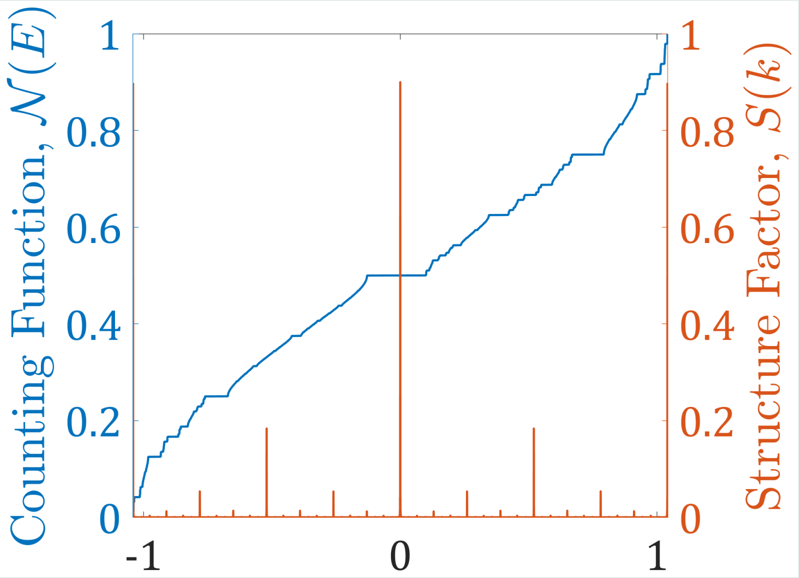
Aperiodic



Thue-Morse

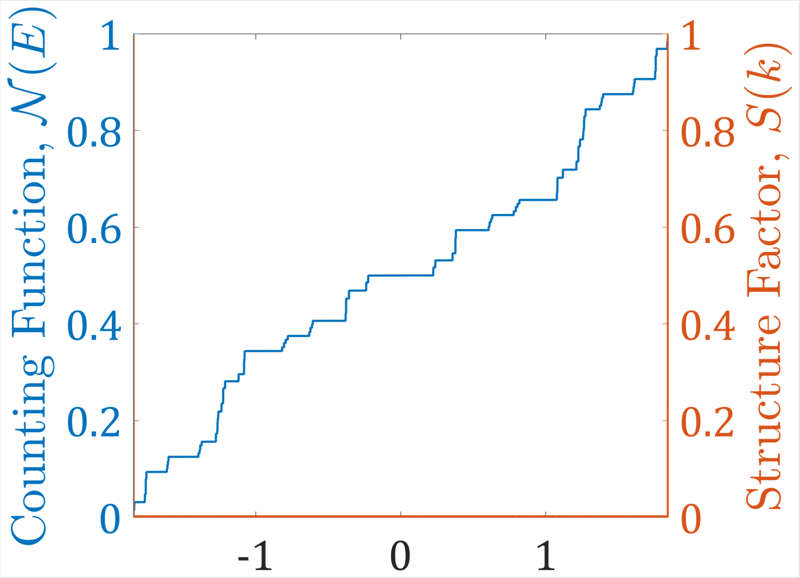
Correspondence between Structure and Spectrum?

Aperiodic



Period Doubling

Aperiodic



Rudin-Shapiro

Showing the connection - Finding the tools to discriminate between tilings

The tool: topological invariants

We use the Čech cohomology $\check{H}^1$

- to calculate Bragg peaks
- to compute topological numbers
- to show correspondence

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Outline

How to Construct Aperiodic Tilings?

Problem

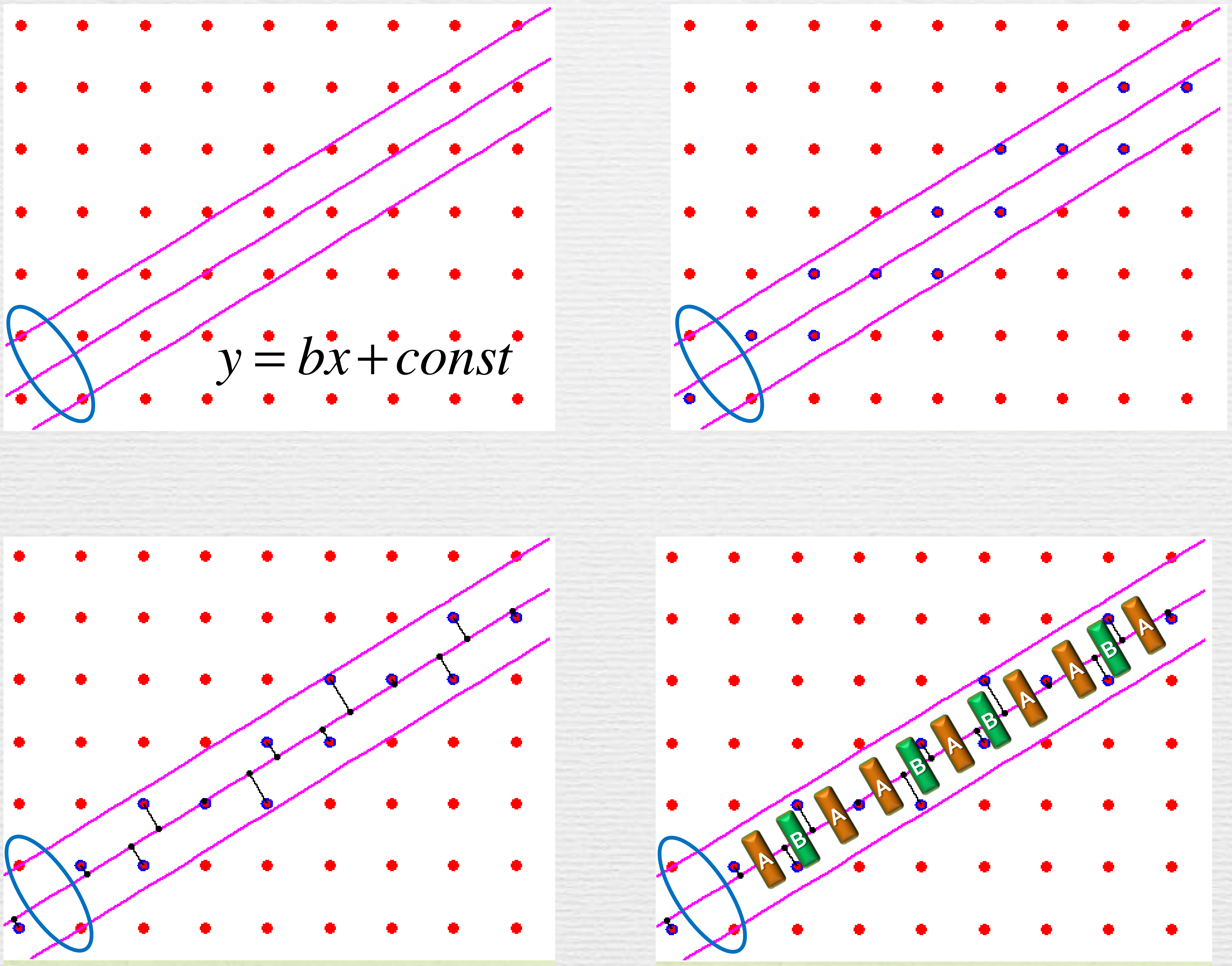
How to tile a space without repeating a pattern indefinitely?

Solution

There are several methods

- We start with the Cut and Project (C&P) scheme

Start from a 2D lattice $L = \mathbb{Z}^2$



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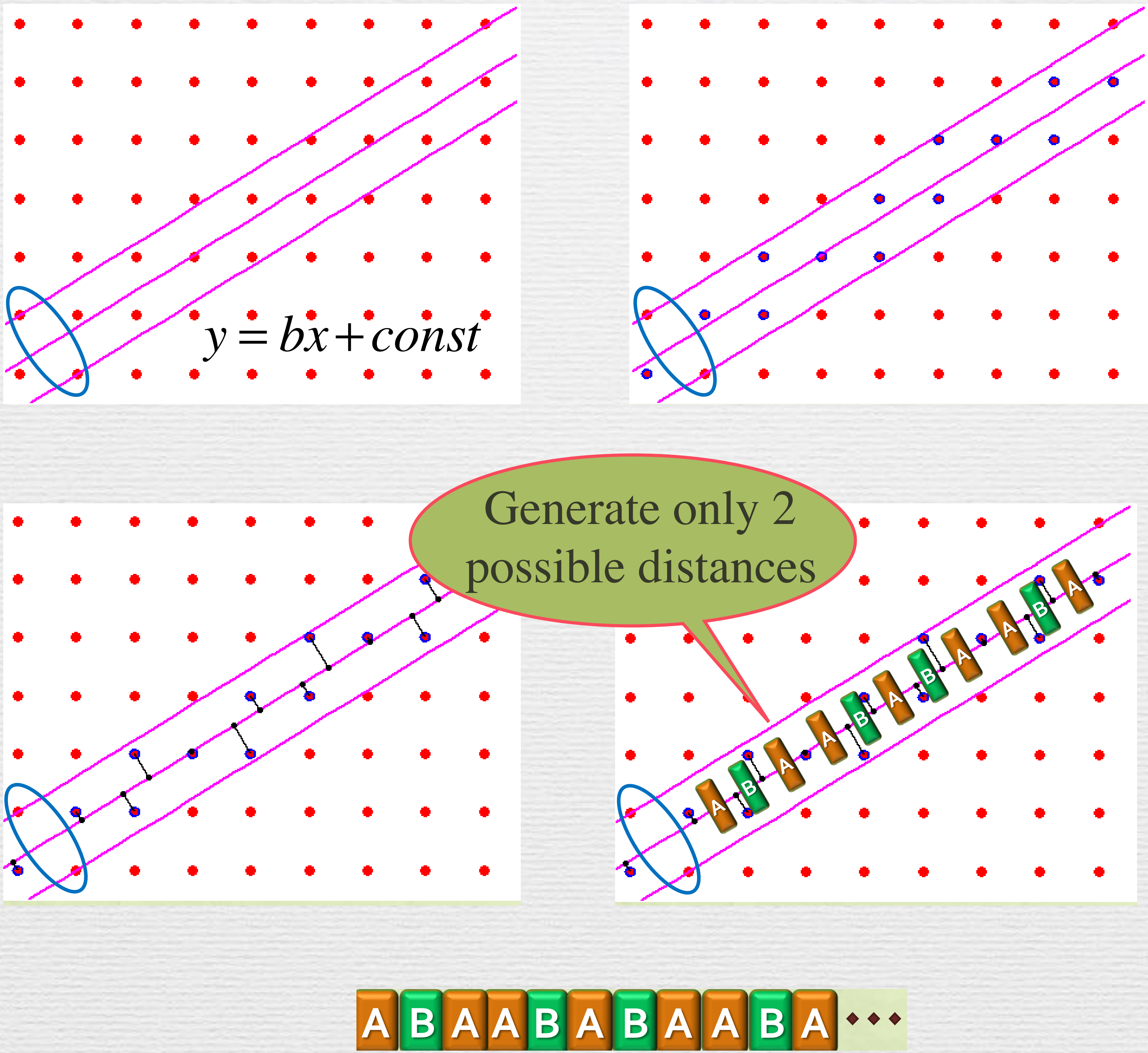
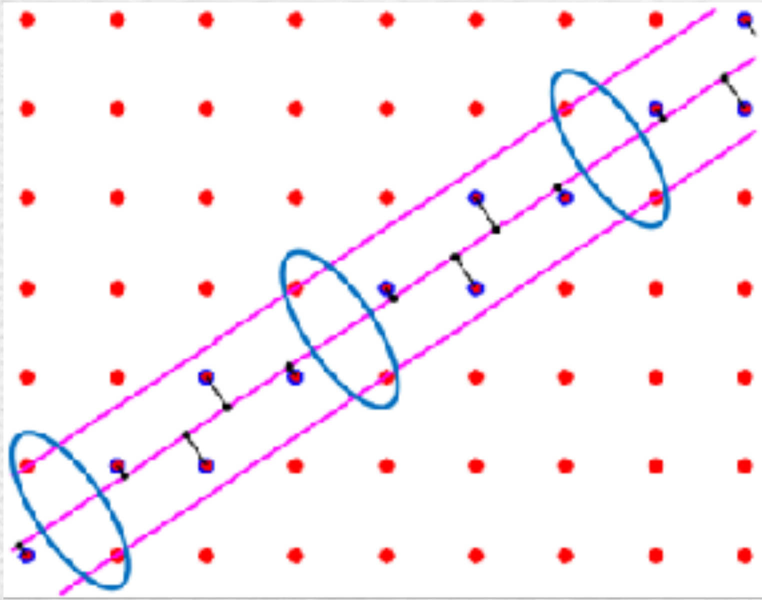
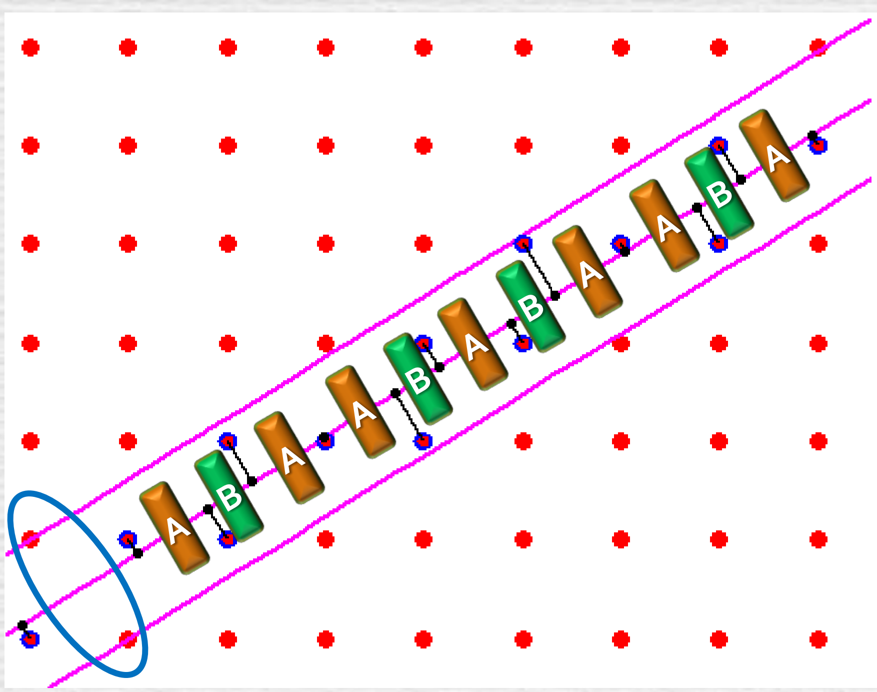


For a rational slope : periodic superlattice



$$y=\tfrac{2}{3}x+const$$

For an irrational slope : quasi-periodic structure



$$y=\tau^{-1}x+const$$

$$\tau = {(1+\sqrt{5})}/{2}$$

golden mean

Different ways to build tiling chains

- Characteristic function
- Cut & Project

Characteristic function

$$\chi_n = sign\left[\cos\left(2\pi n \tau^{-1} + \phi\right) - \cos\left(\pi \tau^{-1}\right)\right]$$

$-1 =$ B

$+1 =$ A

Characteristic function

$$\chi_n = sign\left[\cos\left(2\pi n\tau^{-1}+\phi\right)-\cos\left(\pi\tau^{-1}\right)\right]$$

$-1 =$ B

$+1 =$ A

$$F_N(\phi)=\left[\chi_1\,\chi_2\cdots\chi_n\cdots\chi_N\right] \Leftrightarrow$$ A B A A B A B A A B A A B ⋯

Characteristic function

$$\chi_n = sign\left[\cos\left(2\pi n \tau^{-1} + \phi\right) - \cos\left(\pi \tau^{-1}\right)\right]$$

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$$F_N(\phi) = \left[\chi_1\, \chi_2 \cdots \chi_n \cdots \chi_N\right] \Leftrightarrow$$ A B A A B A B A A B A A B • • •

The angle ϕ is a (legitimate) degree of freedom.

ϕ is known as a phason

$$\tau = \sqrt{5}+1 \big/ 2 \approx 1.62$$

Characteristic function

$$\chi_n = sign\left[\cos\left(2\pi n\tau^{-1} + \phi\right) - \cos\left(\pi\tau^{-1}\right)\right]$$

ϕ is an innocuous and thus ignored modulation phase.

For an infinite Fibonacci chain :

$$\phi_\infty = 3\pi\sigma = 3\pi\tau^{-1}$$

Define instead

$$\chi_n = sign\left[\cos\left(2\pi n\tau^{-1} + \phi_\infty + \Delta\phi\right) - \cos\left(\pi\tau^{-1}\right)\right]$$

$$\tau = \left(1+\sqrt{5}\right)/2$$

golden mean

Characteristic function

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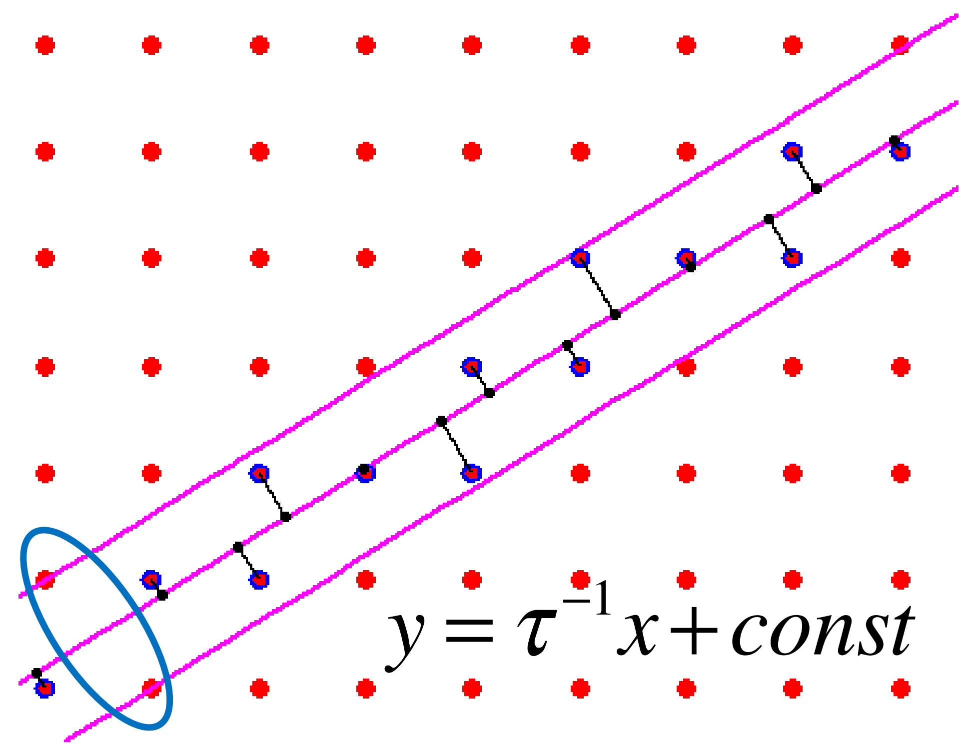
golden mean

C&P method

Is it possible to give a meaning to $\Delta\phi$ using the C&P method ?



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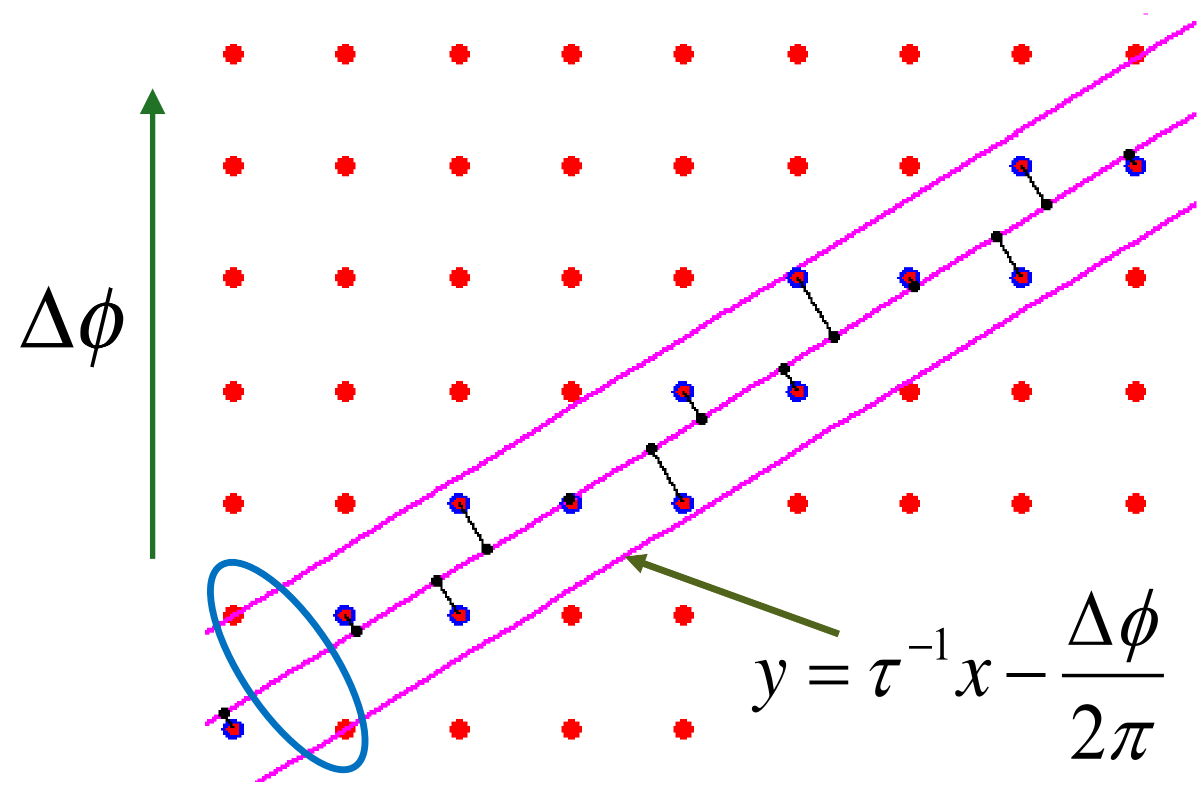

We understand the meaning of $\Delta\phi$

Meaning of the phason ϕ : a gauge field

A gauge degree of freedom

- Take a characteristic function

$$\chi(n,\phi) = \mathrm{sign}\left[\cos\left(2\pi n\,\lambda_1^{-1} + \phi\right) - \cos\left(\pi\lambda_1^{-1}\right)\right]$$

?

A gauge degree of freedom

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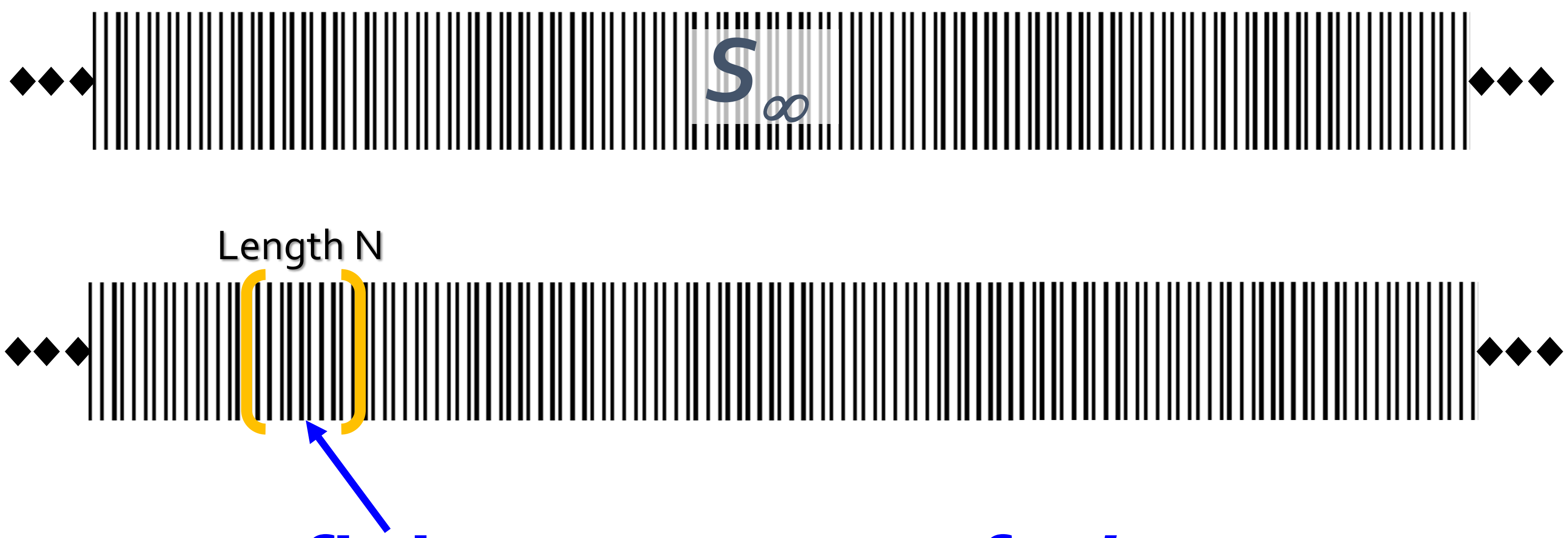
$$\chi(n, \phi) = \text{sign}\left[\cos\left(2\pi n \lambda_1^{-1} + \phi\right) - \cos\left(\pi \lambda_1^{-1}\right)\right]$$



A finite segment of *size N*

A gauge degree of freedom

- Take a characteristic function

$$\chi(n,\phi) = \text{sign}\left[\cos\left(2\pi n\lambda_1^{-1} + \phi\right) - \cos\left(\pi\lambda_1^{-1}\right)\right]$$

Discrete gauge: Choice of origin

A gauge degree of freedom

- Take a characteristic function

$$\chi(n,\phi) = \mathrm{sign}\left[\cos\left(2\pi n\lambda_1^{-1} + \phi\right) - \cos\left(\pi\lambda_1^{-1}\right)\right]$$

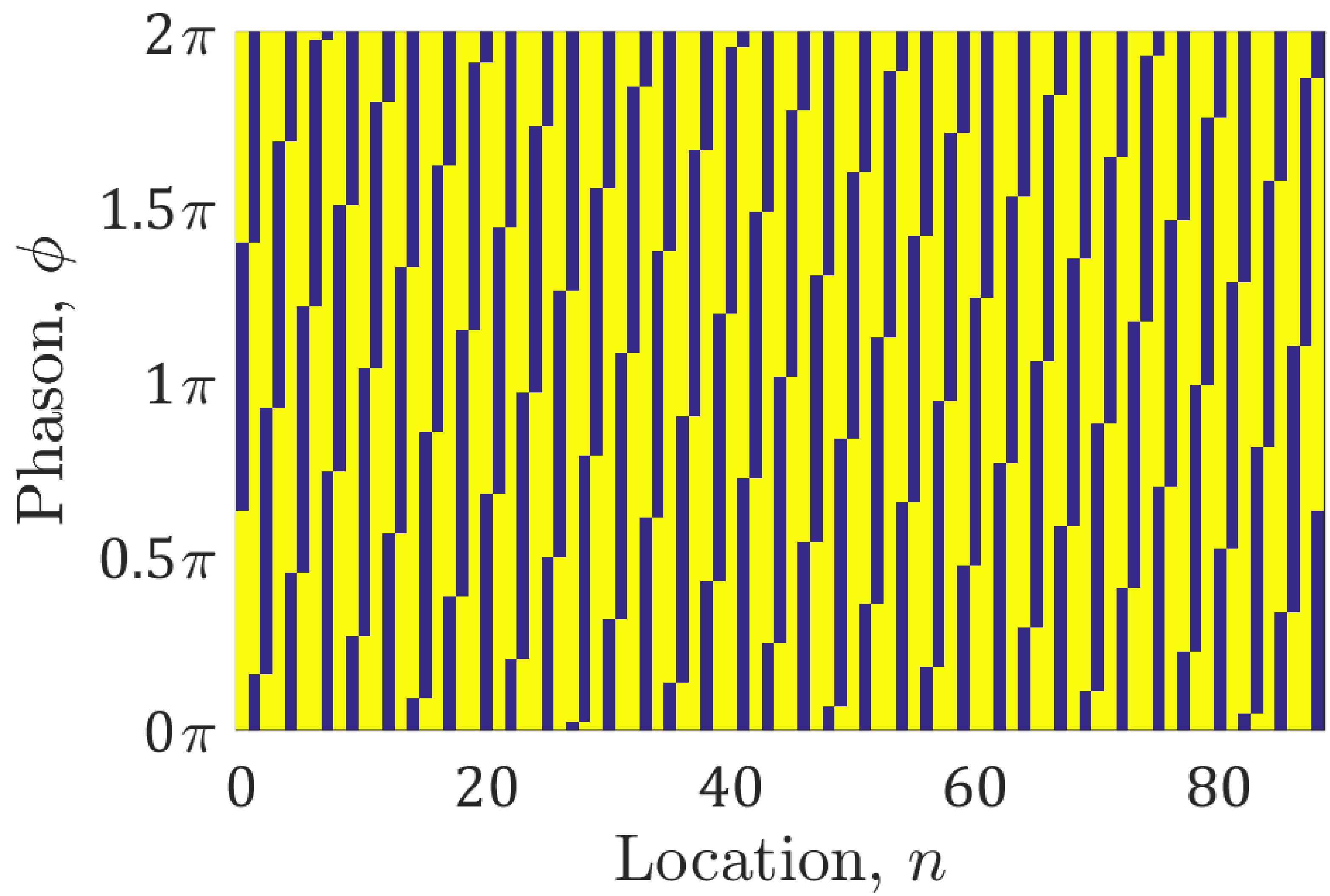
$$F_N(\phi) = \left[\chi_1\,\chi_2\cdots\chi_n\cdots\chi_N\right] \Leftrightarrow$$

A torus

- Take a characteristic function

$$\chi(n, \phi) = \text{sign}\left[\cos\left(2\pi n \lambda_1^{-1} + \phi\right) - \cos\left(\pi \lambda_1^{-1}\right)\right]$$

with $n = 0 \ldots F_N - 1$ and $[0, 2\pi] \ni \phi$



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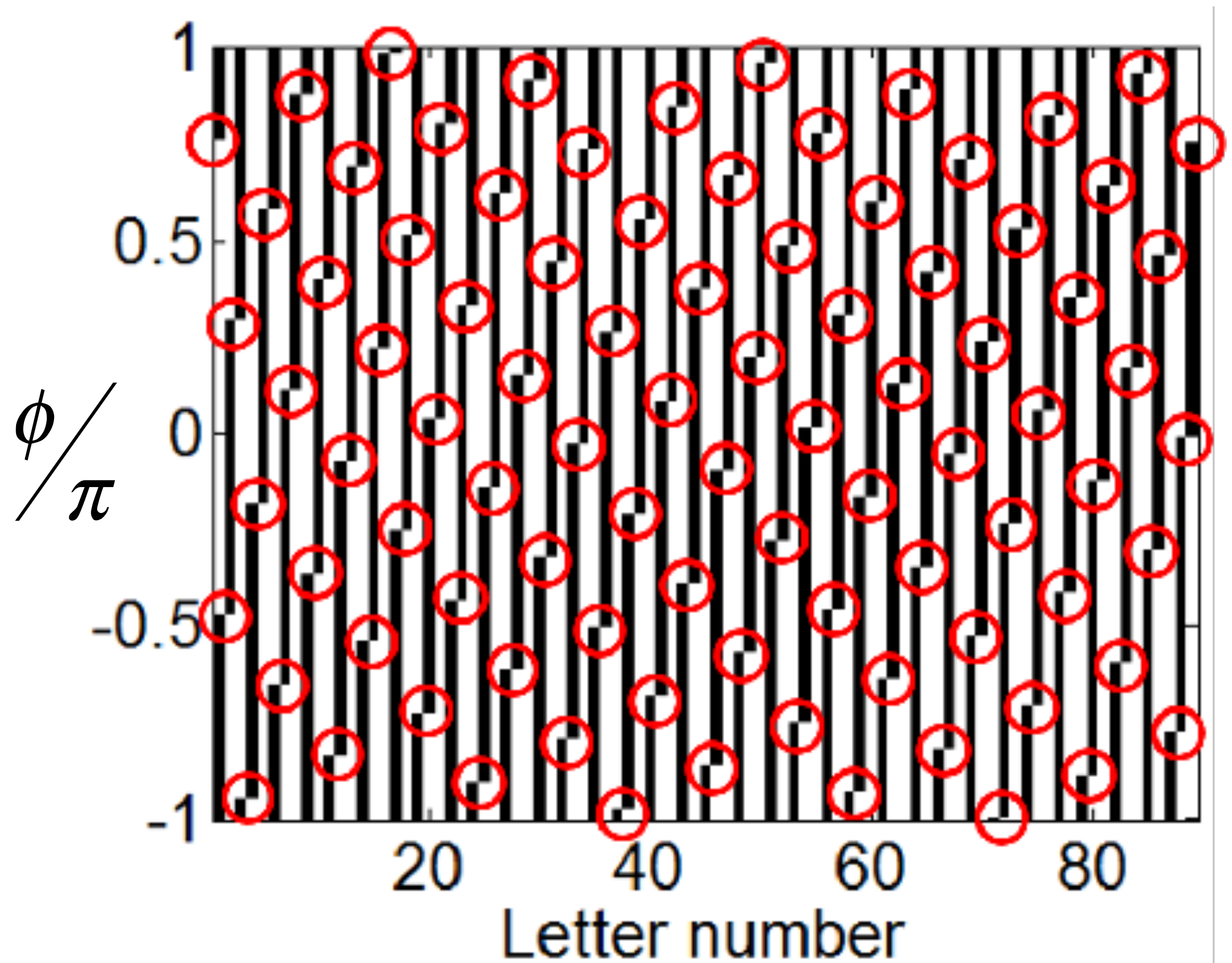
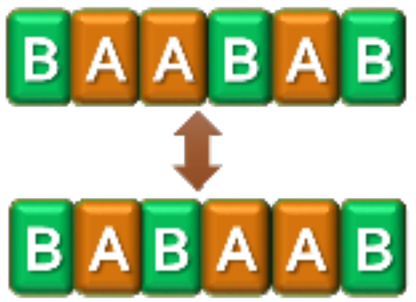
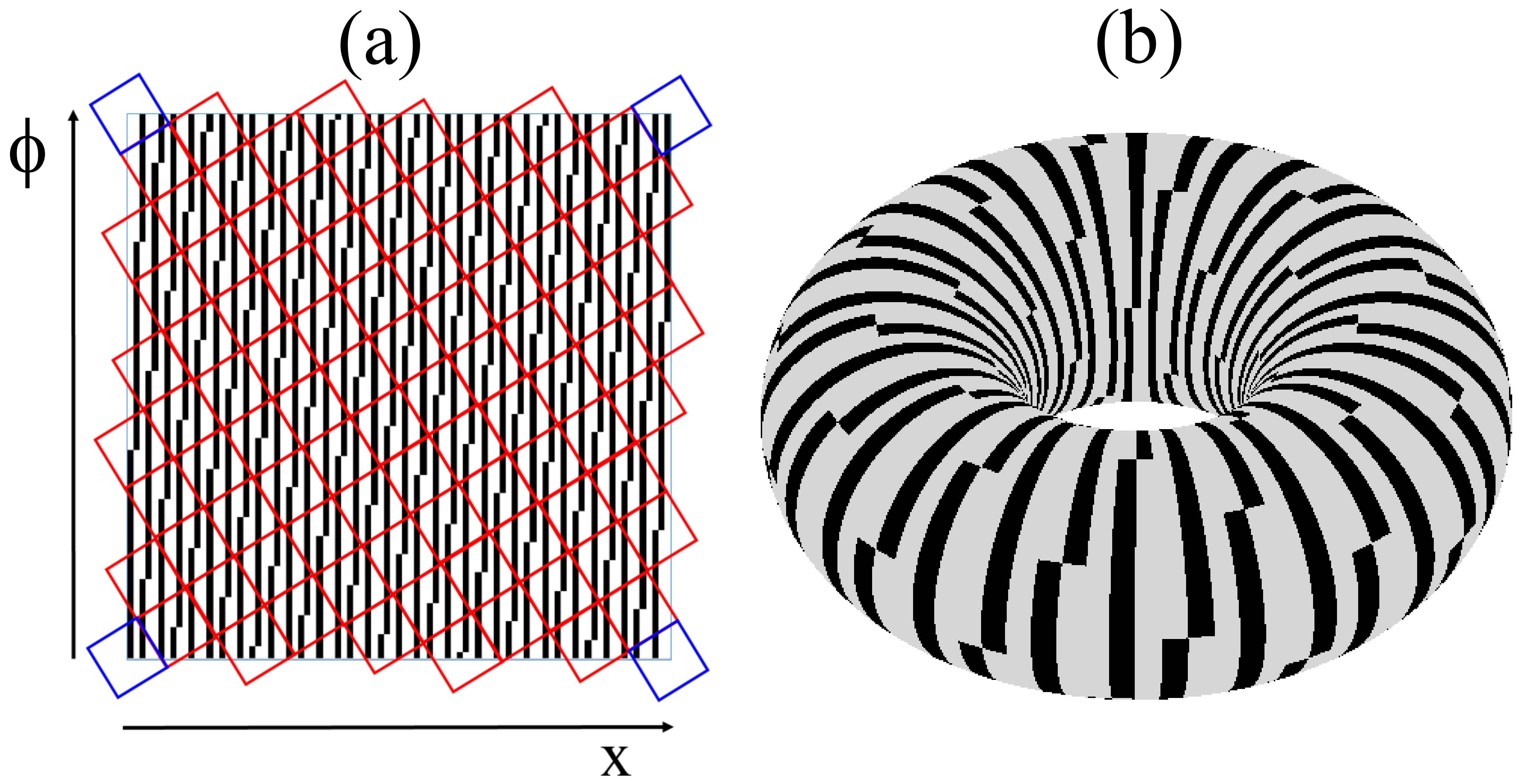


Scanning ϕ generates **local** structural changes.



A torus



A gauge degree of freedom

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$$\chi(n,\phi) = \text{sign}\left[\cos\left(2\pi n\,\lambda_1^{-1} + \phi\right) - \cos\left(\pi\lambda_1^{-1}\right)\right]$$

Are there spectral consequences
of these structural properties ?

Almost No…

Atomic distributions - Structure factor

- Distributions of identical atoms in $1D$
- Use language of tilings: two tiles (letters) a and b
- Distribution of letters underlies distribution of atoms

- Define atomic density

$$\rho(x) = \sum_k \delta(x - x_k)$$

The diffraction pattern of the infinite chain $\rho(x) = \sum_k \delta(x - x_k)$ is given by

$$g(\xi) = \int_{-\infty}^{+\infty} \mathrm{d}x\, \rho(x)\, \mathrm{e}^{-i\xi x} = \sum_k \mathrm{e}^{-\mathrm{i}\xi x_k}$$

with structure factor

$$S(\xi) = |g(\xi)|^2$$

Definition

Diffraction spectrum has a Bragg peak (atomic part) at ξ_B iff

$$\xi_B x_c \xrightarrow{c\to\infty} 0 \pmod{2\pi}$$

for $\{x_c\}_{c=1}^{\infty} \subset \{x_k\}_{k=1}^{\infty}$, so that $g(\xi_B) \to \delta(\xi - \xi_B)$

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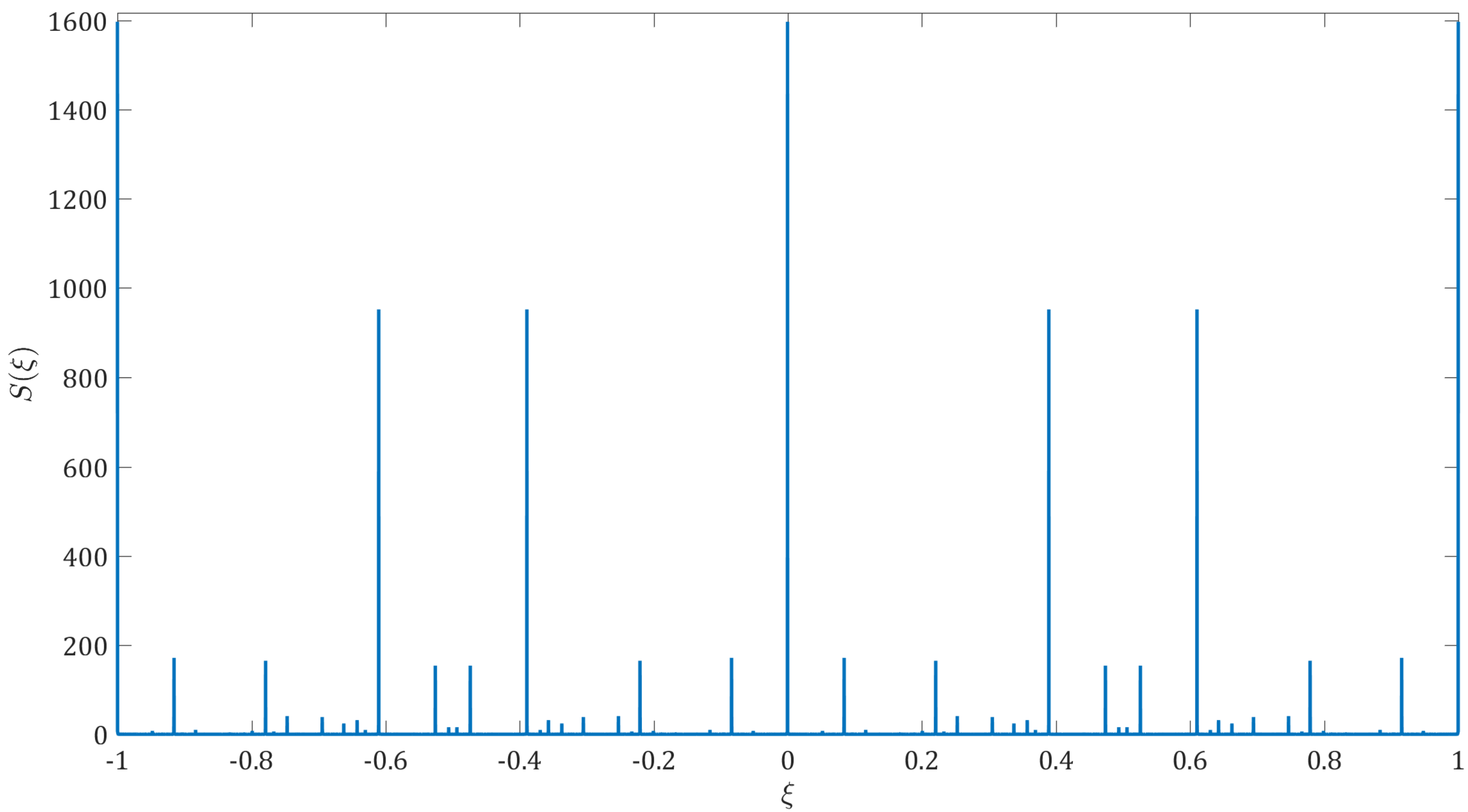


The extra phase - Winding numbers

- Take a characteristic function

$$\chi(n,\phi) = \text{sign}\left[\cos\left(2\pi n\lambda_1^{-1} + \phi\right) - \cos\left(\pi\lambda_1^{-1}\right)\right]$$

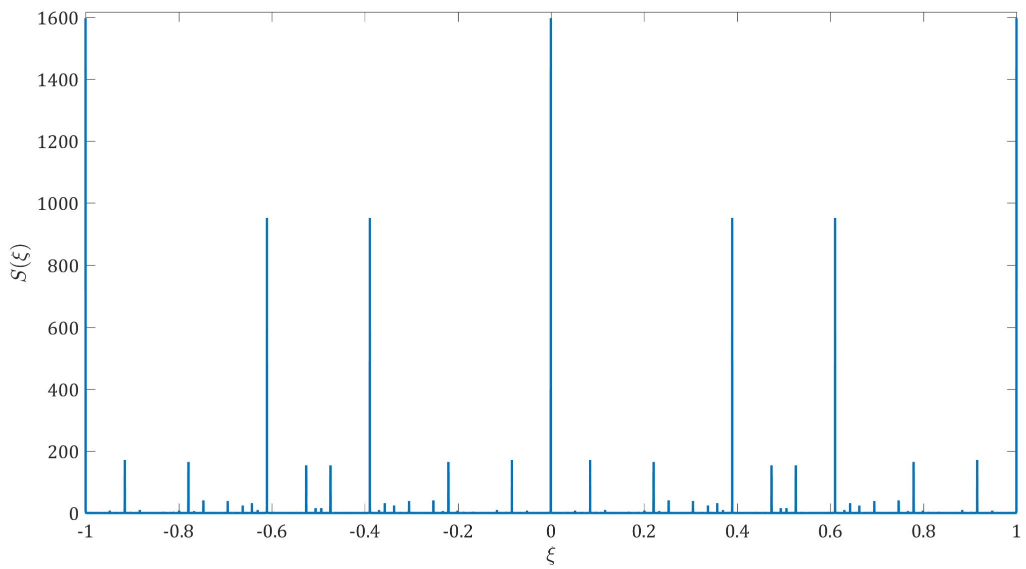
with $n = 0 \ldots F_N - 1$ and $[0, 2\pi] \ni \phi \to \phi_\ell = \frac{2\pi}{F_N}\ell$

- Discrete Fourier transform w.r.t. n

$$g(\xi,\phi) = \sum_{n=0}^{F_N - 1} \omega^{-\xi n}\chi(n,\phi), \quad \omega = e^{\frac{2\pi i}{F_N}}$$

- Structure factor S and phase θ

$$S(\xi,\phi) = |g(\xi,\phi)|^2$$



Structure factor is ϕ - independent

- Take a characteristic function

$$\chi(n,\phi) = \text{sign}\left[\cos\left(2\pi n \lambda_1^{-1} + \phi\right) - \cos\left(\pi\lambda_1^{-1}\right)\right]$$

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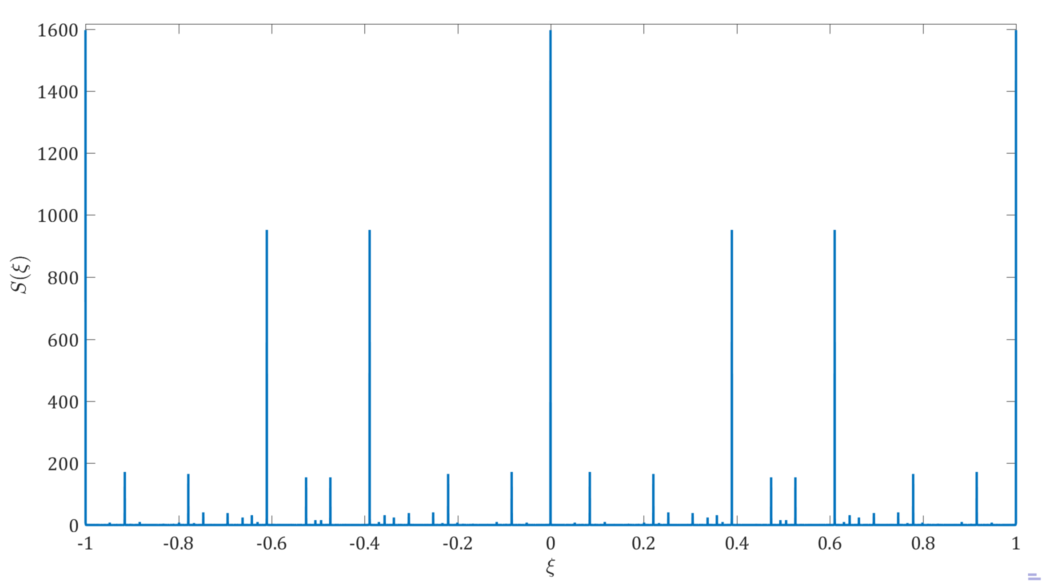
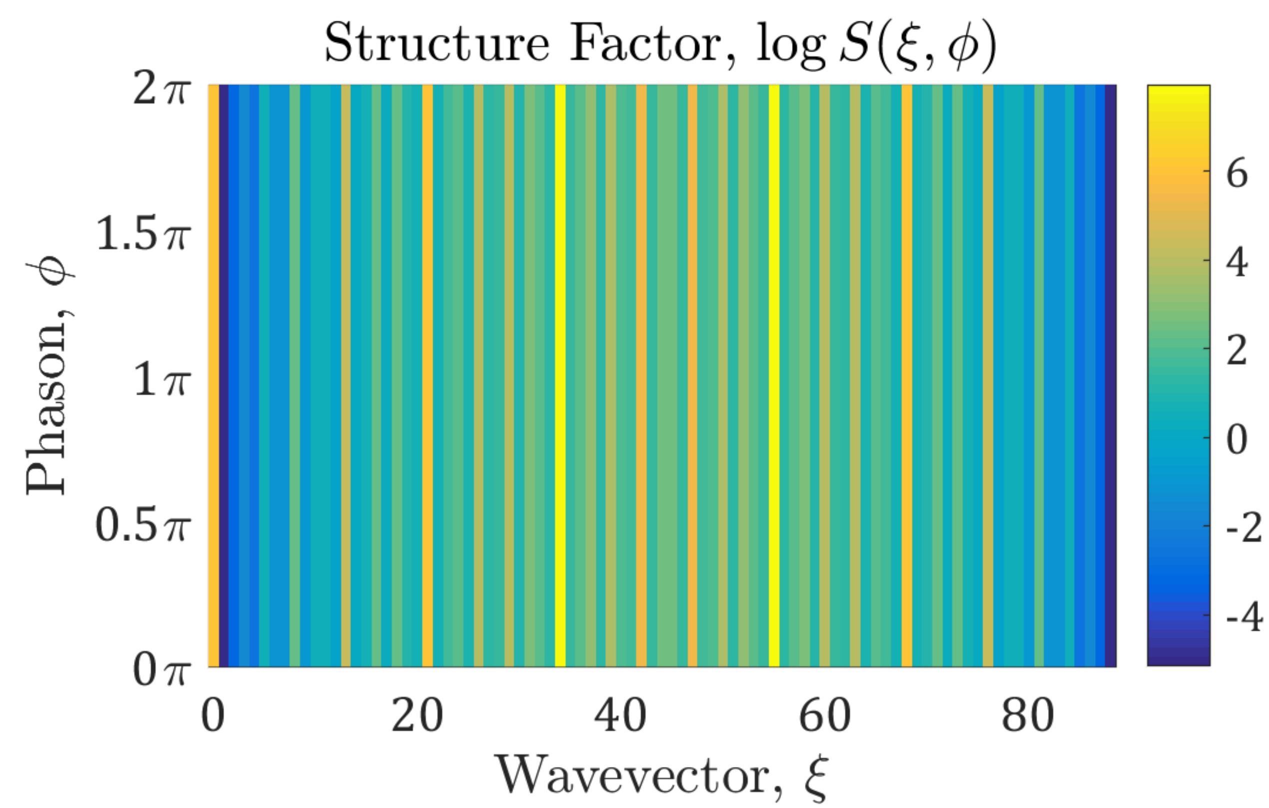
- Structure factor S

$$S(\xi,\phi) = |g(\xi,\phi)|^2$$





The extra phase - Winding numbers

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- Structure factor S and phase θ

$$S\left(\xi,\phi\right)=\left|g\left(\xi,\phi\right)\right|^2,\quad \theta\left(\xi,\phi\right)=\arg g\left(\xi,\phi\right)$$

usually disregarded

The extra phase - Winding numbers

- Take a characteristic function

$$\chi(n,\phi) = \text{sign}\left[\cos\left(2\pi n\lambda_1^{-1} + \phi\right) - \cos\left(\pi\lambda_1^{-1}\right)\right]$$

with $n = 0 \ldots F_N - 1$ and $[0, 2\pi] \ni \phi \to \phi_\ell = \frac{2\pi}{F_N}\ell$

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- Structure factor S and phase θ

$$S(\xi,\phi) = |g(\xi,\phi)|^2, \quad \theta(\xi,\phi) = \arg g(\xi,\phi)$$

- Winding number at ξ_0

$$W_{\xi_0} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial\theta(\xi=\xi_0,\phi)}{\partial\phi}\,d\phi$$

The extra phase - Winding numbers

$$g(\xi,\phi) = \sum_{n=0}^{F_N-1} \omega^{-\xi n}\chi(n,\phi)\,, \quad \omega = e^{\frac{2\pi i}{F_N}}$$

Structure factor S and phase θ

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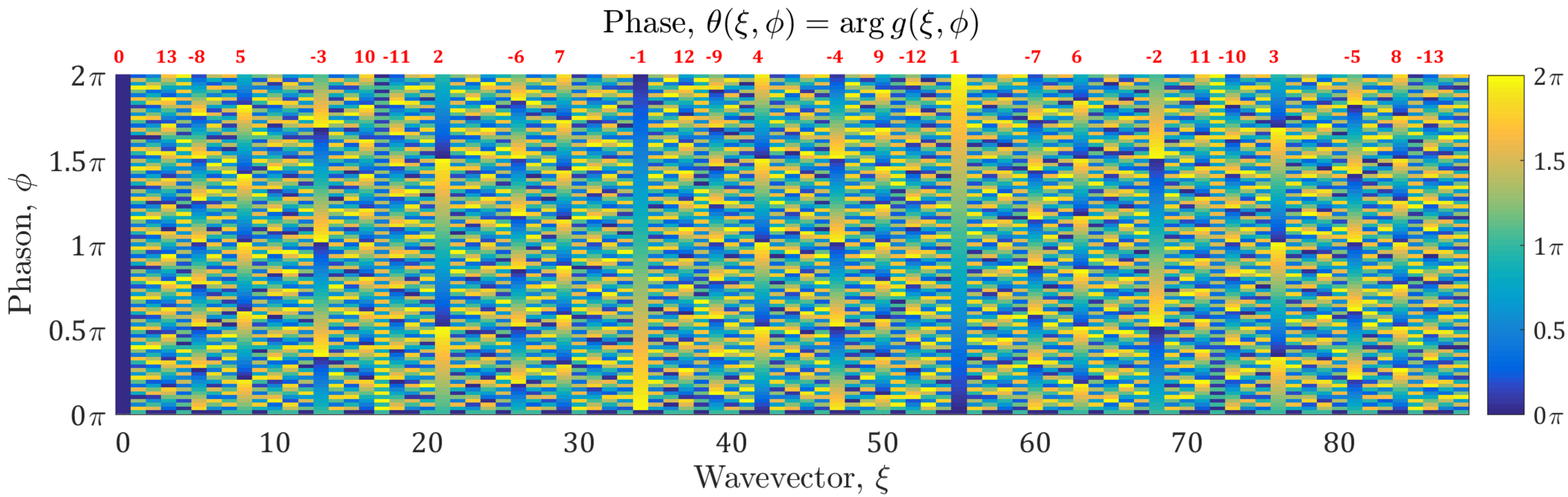
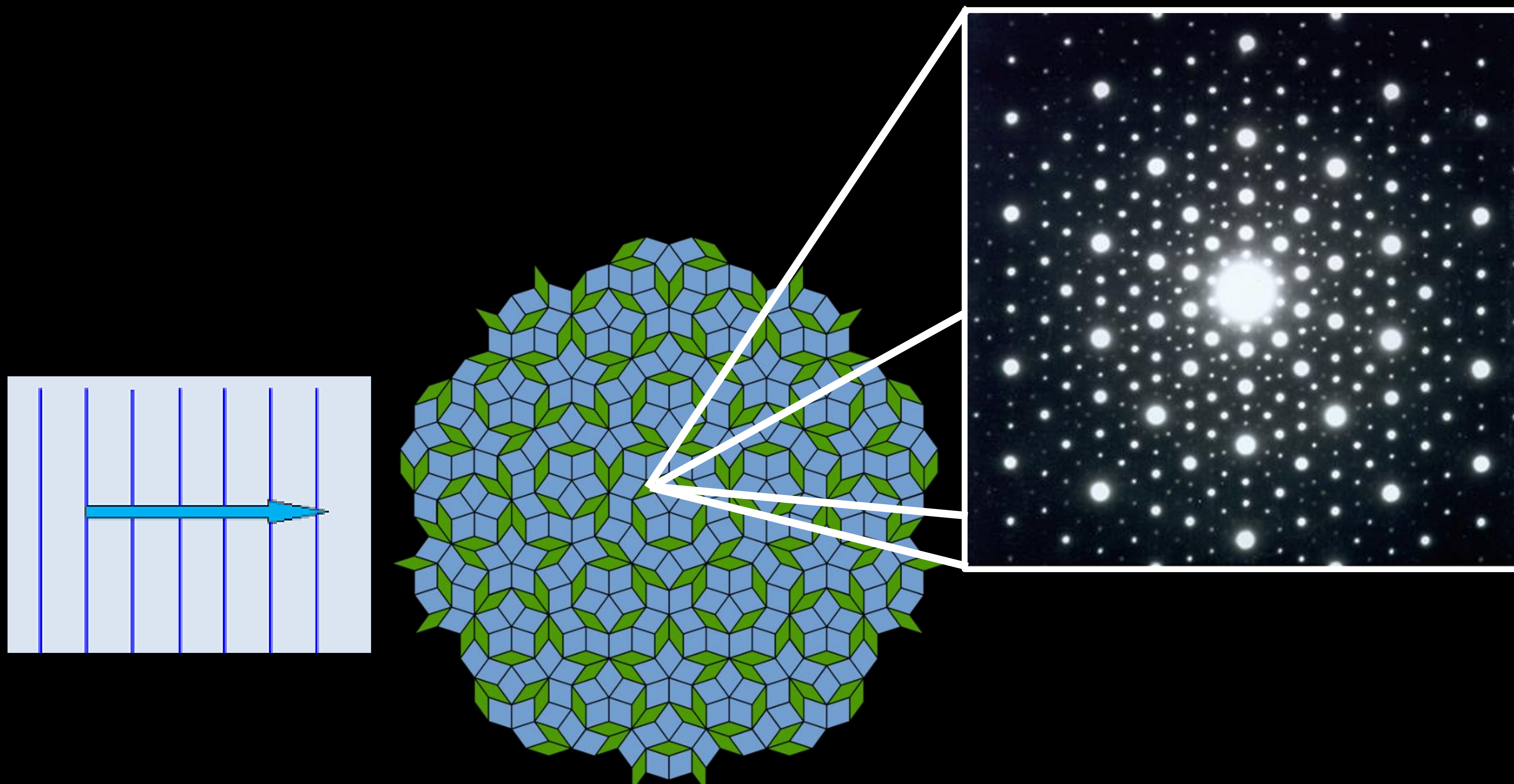


These are topological numbers!

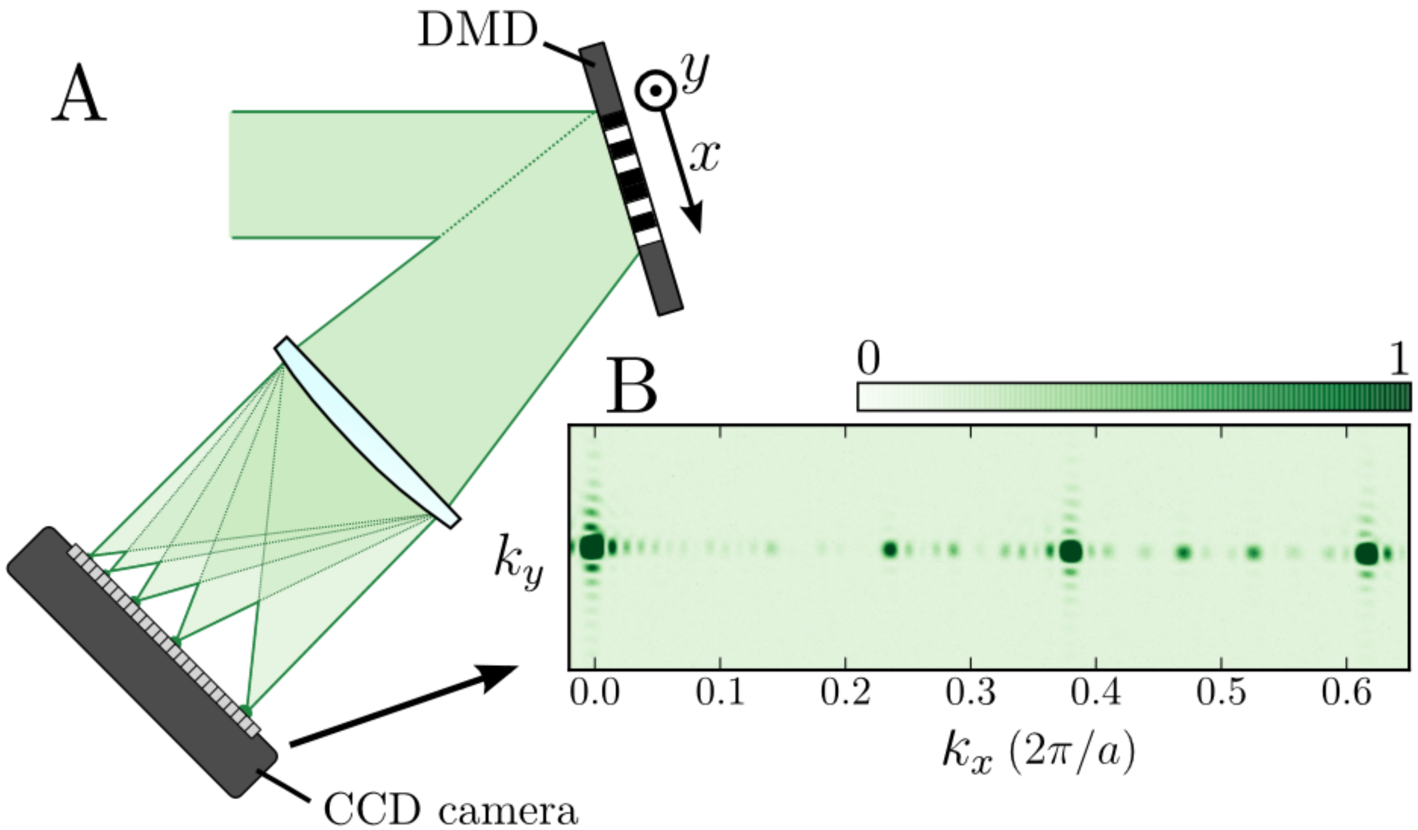
Measuring the structural winding numbers

A. Dareau, E. Levy, F. Gerbier and J. Beugnon and E.A 2017



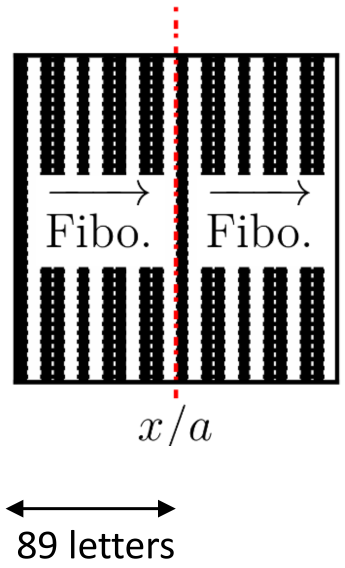
A diffraction measurement of winding numbers

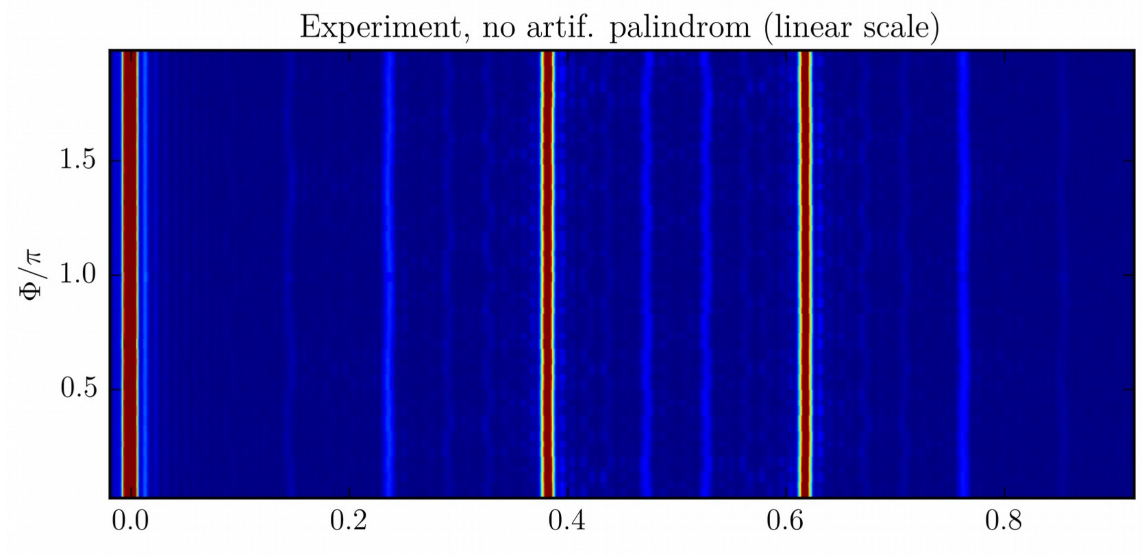
Fibonacci finite string



A. Dareau, E. Levy, F. Gerbier and J. Beugnon and E.A 2017

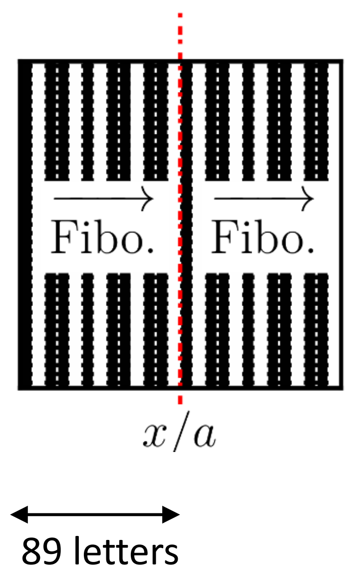
DMD Pattern

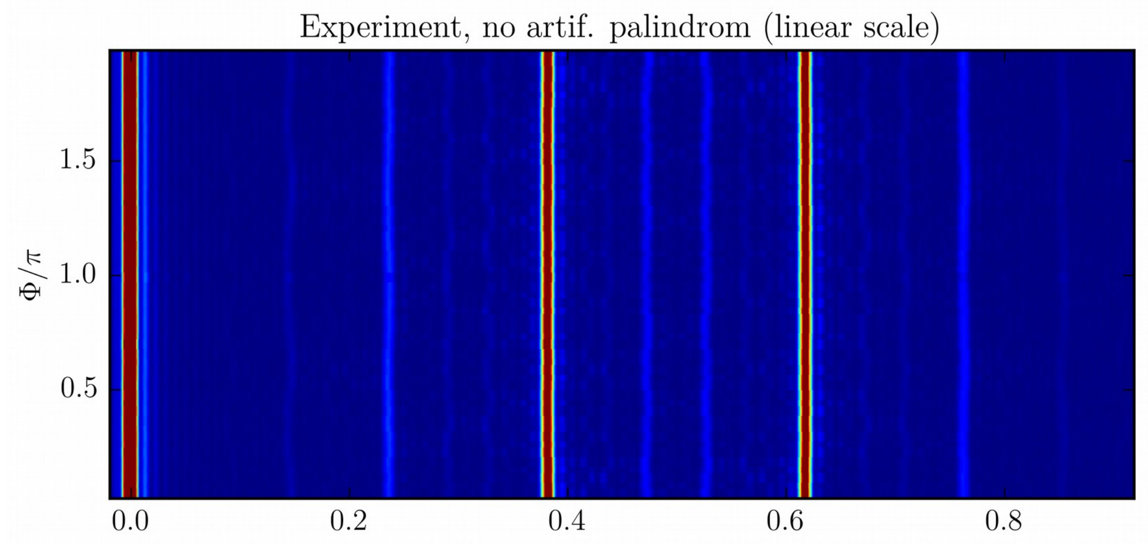




No effect of ϕ

DMD Pattern





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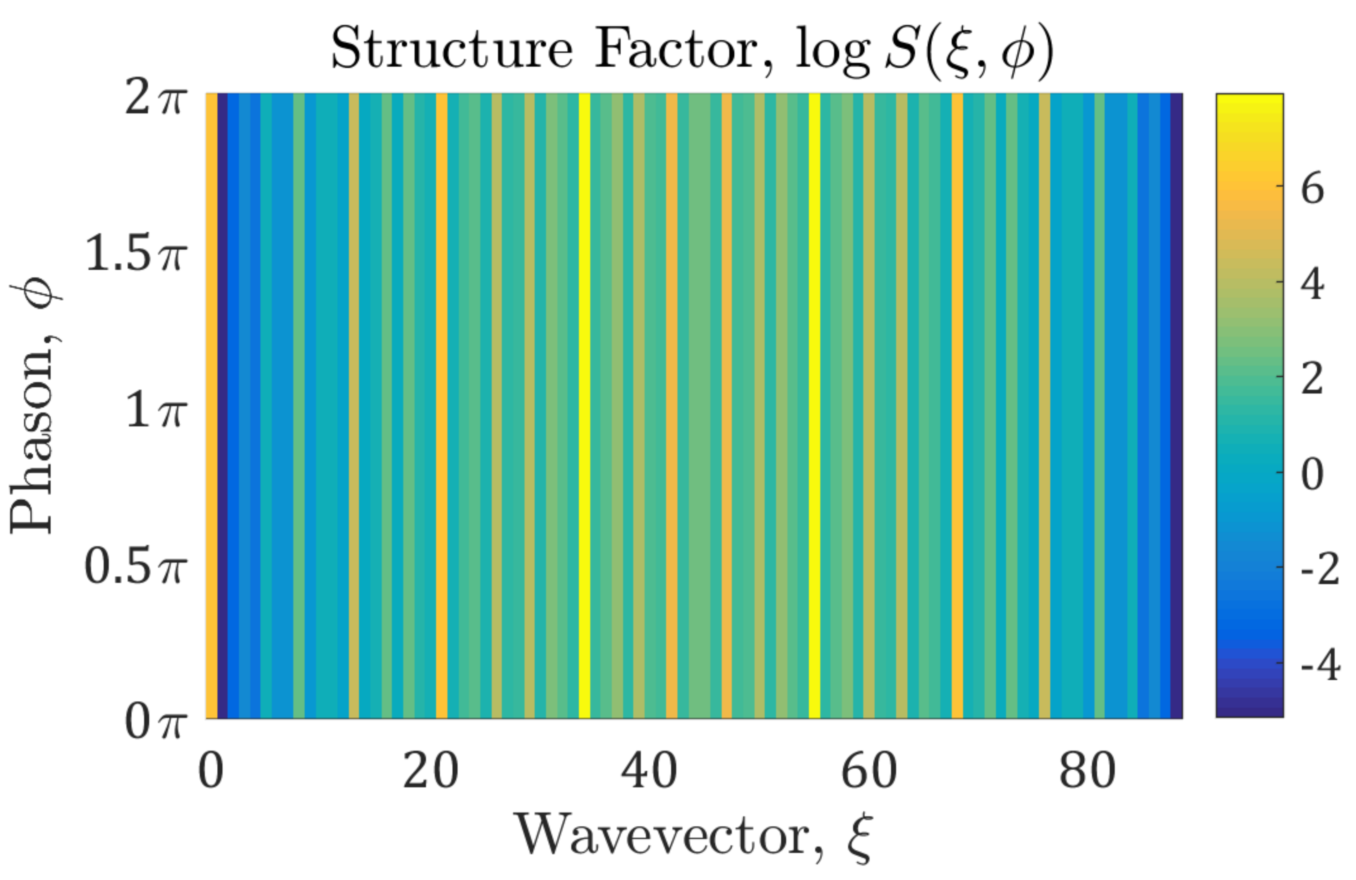
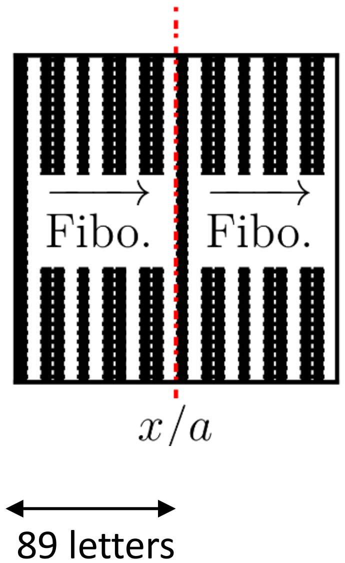
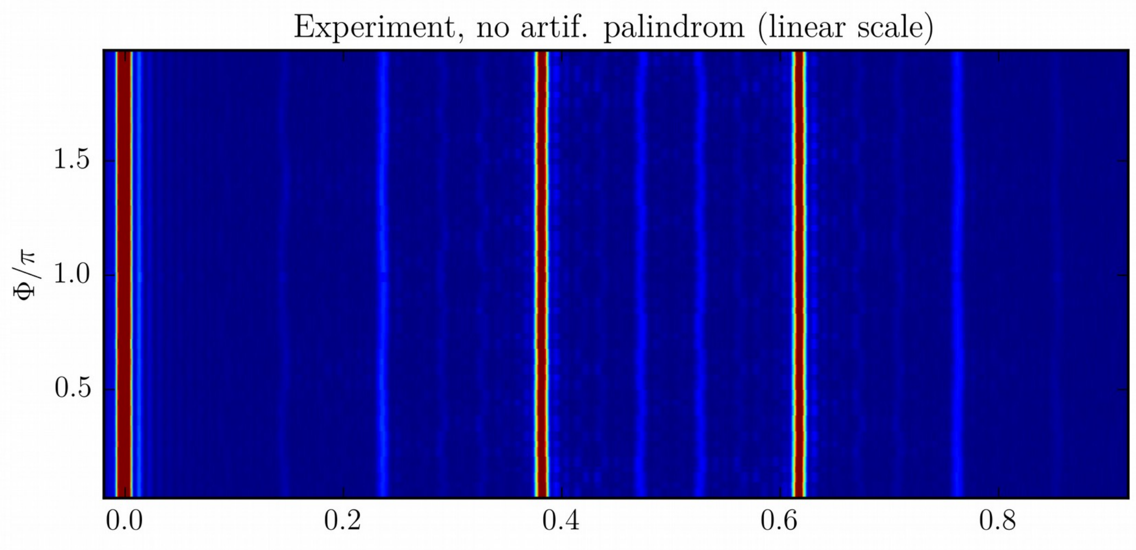


DMD Pattern

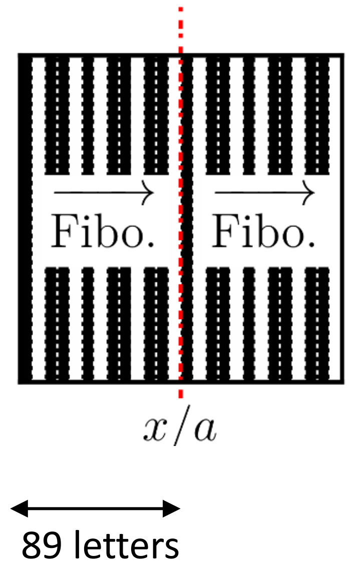


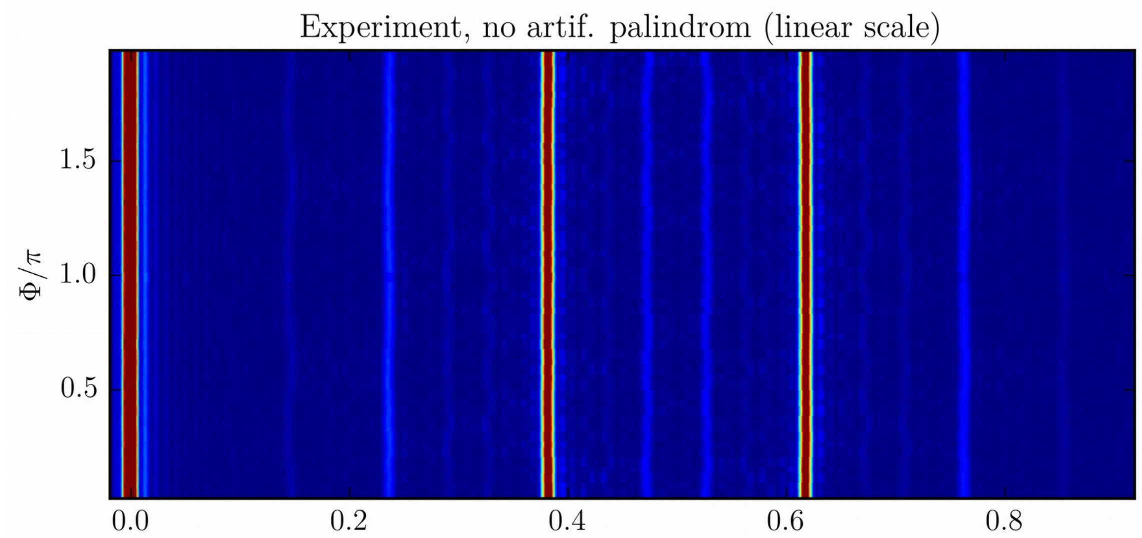


No effect of ϕ

Creating edge states

DMD Pattern





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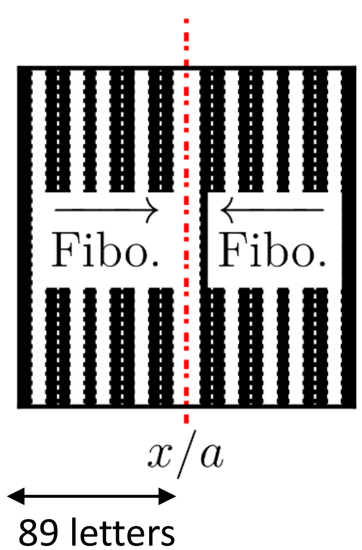
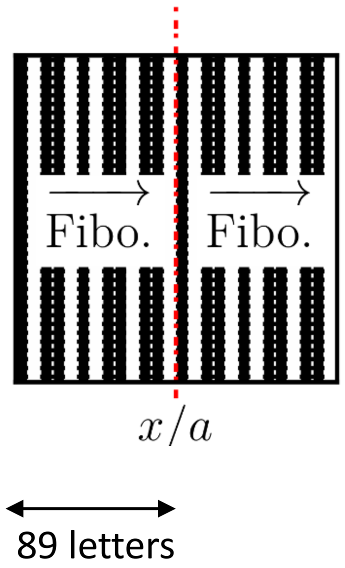
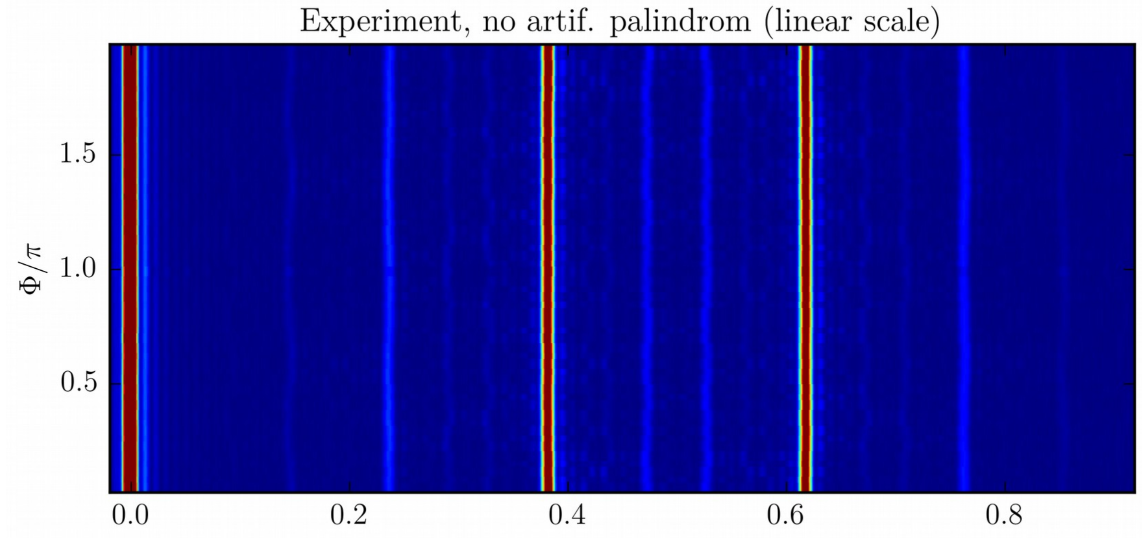


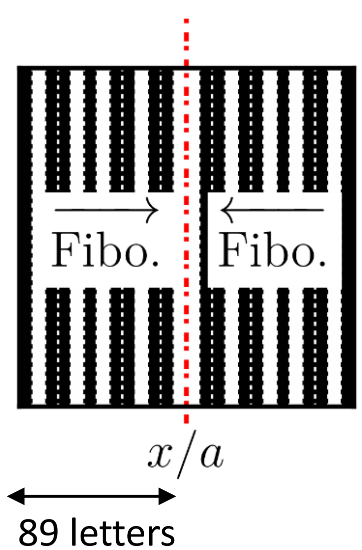
DMD Pattern

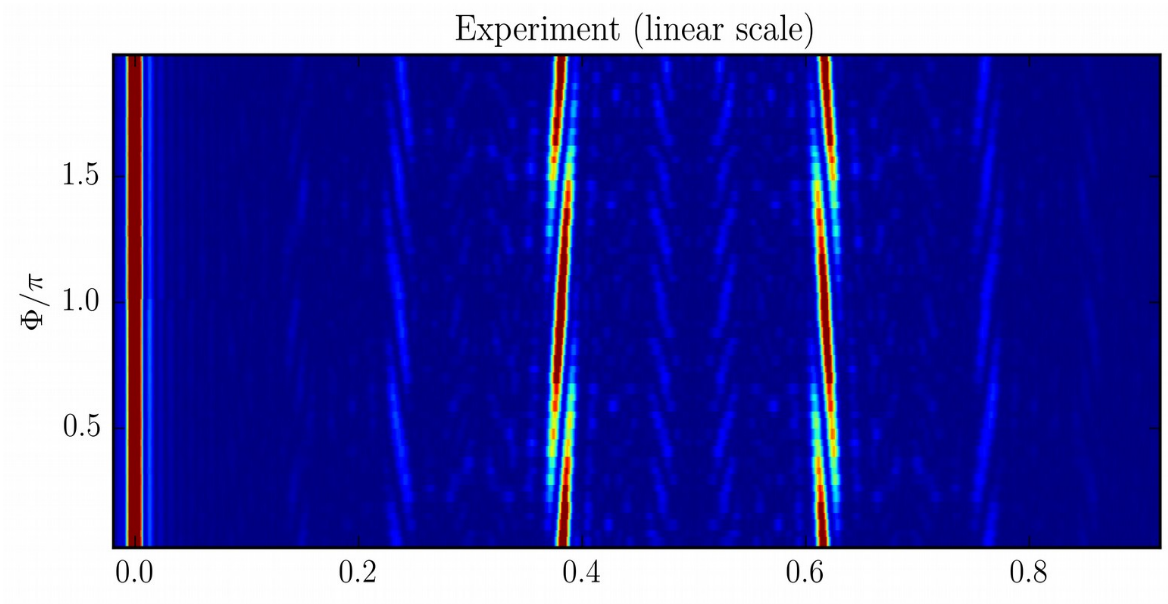




No effect of ϕ

DMD Pattern

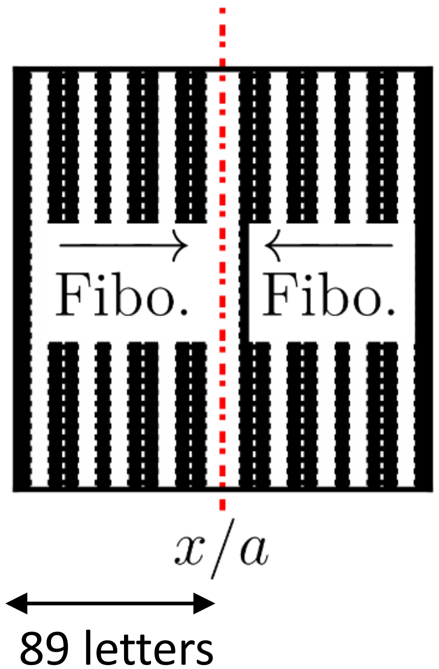




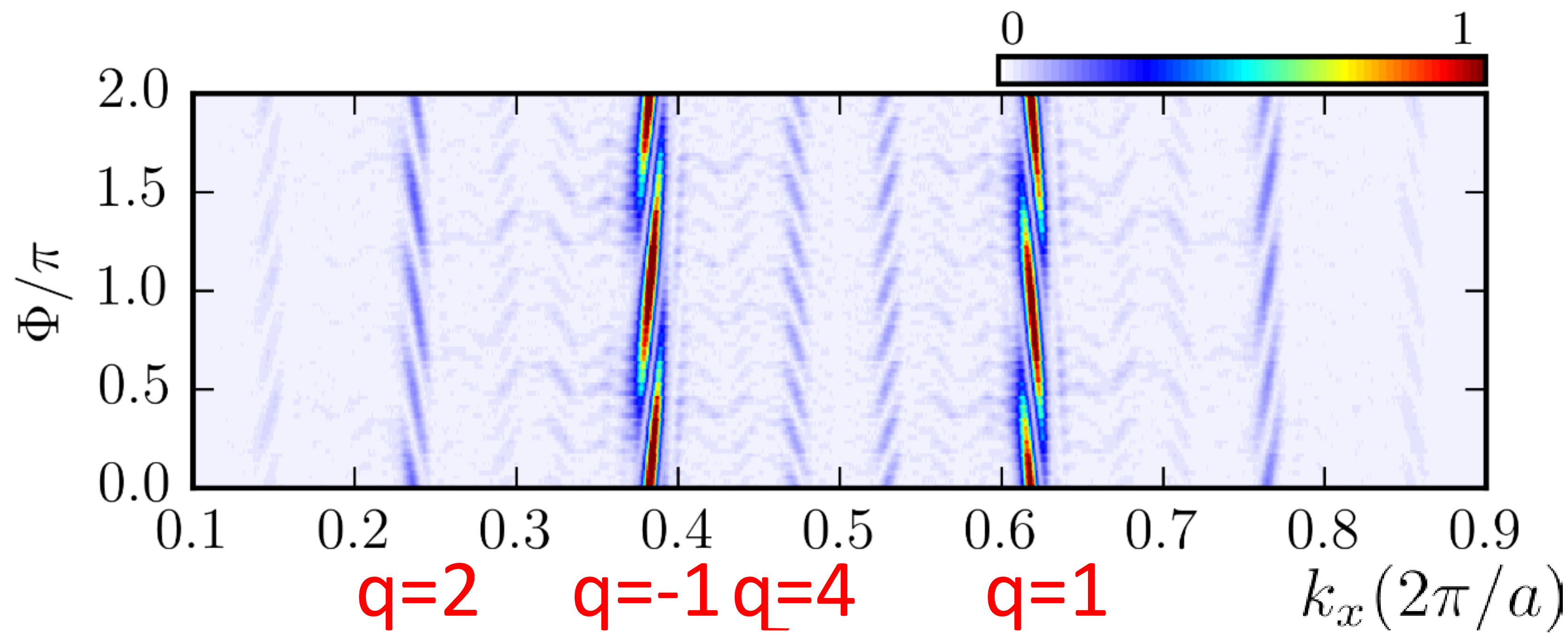
There is an effect of the phason ϕ

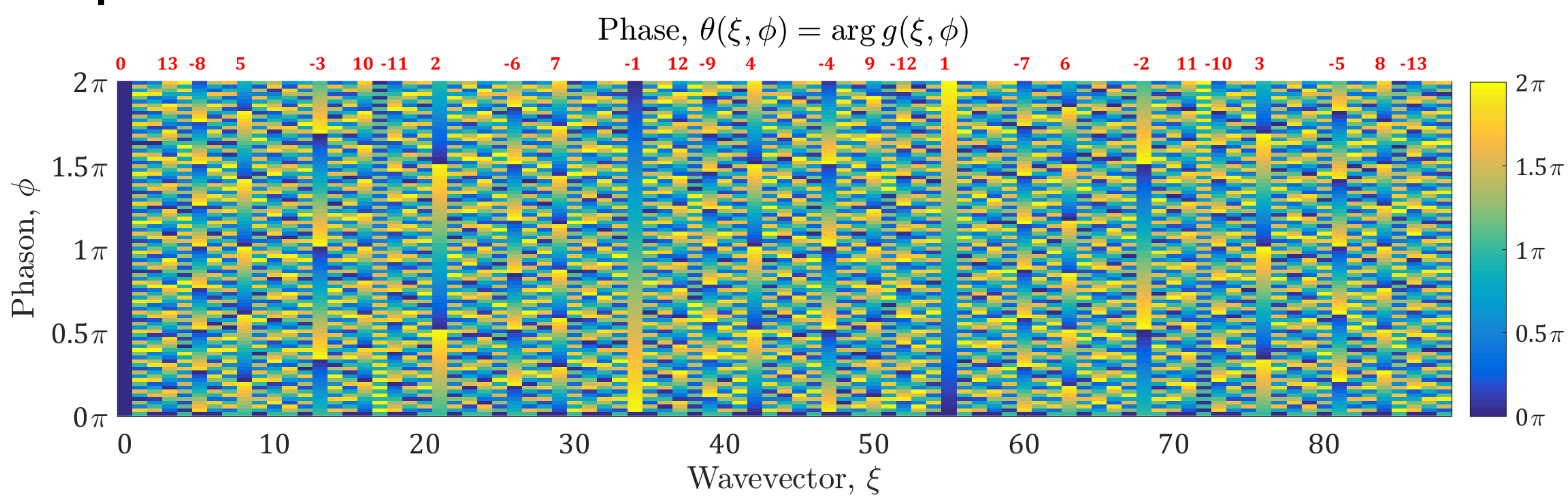
A diffraction measurement of winding numbers

DMD Pattern



Diffraction pattern



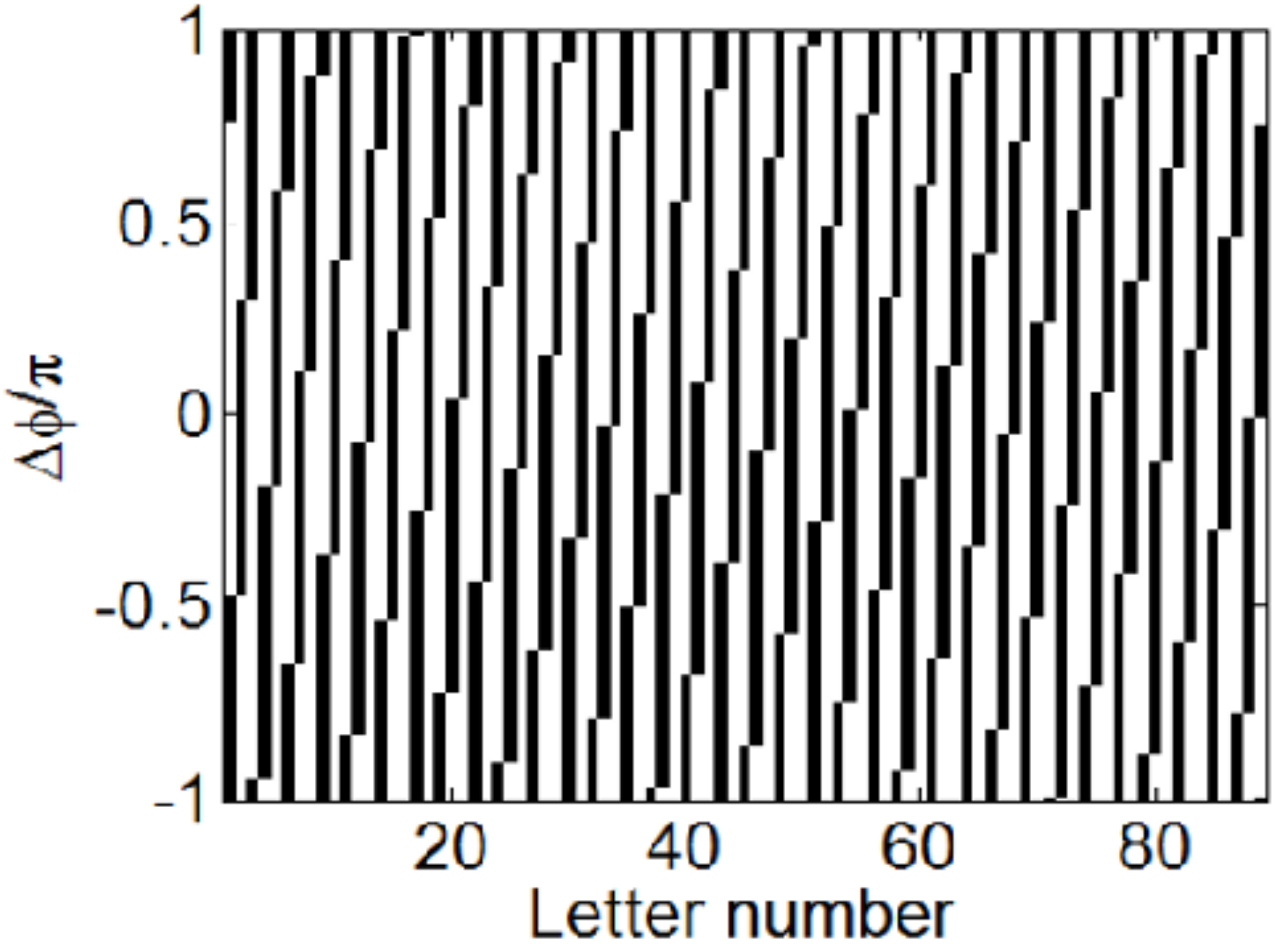


$$W_{\xi_0} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial\theta\left(\xi=\xi_0,\phi\right)}{\partial\phi}\,\mathrm{d}\phi$$

2D diffraction experiment

Instead of 

consider all realisations



A. Dareau, E. Levy, F. Gerbier and J. Beugnon and E.A 2017

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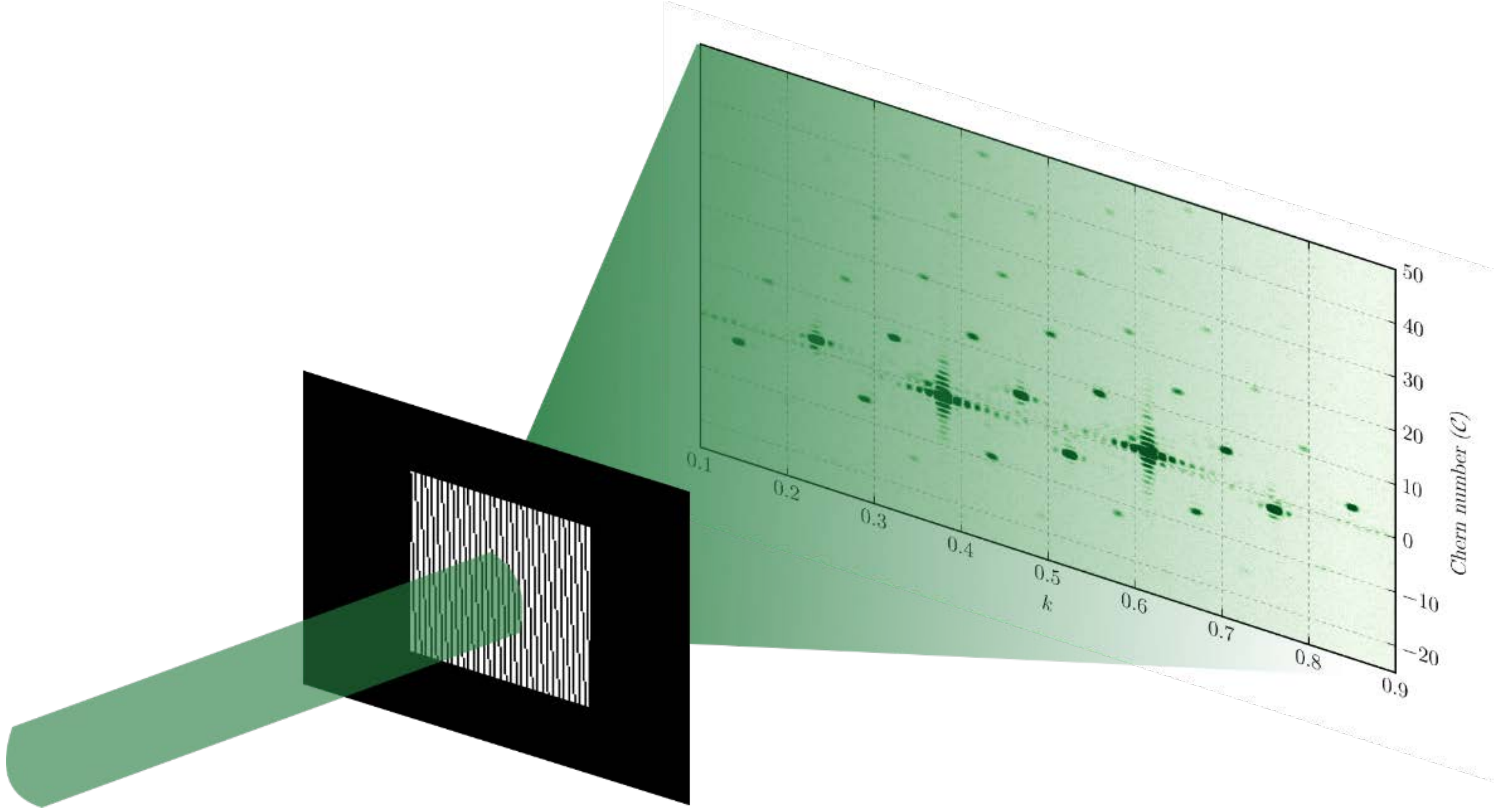
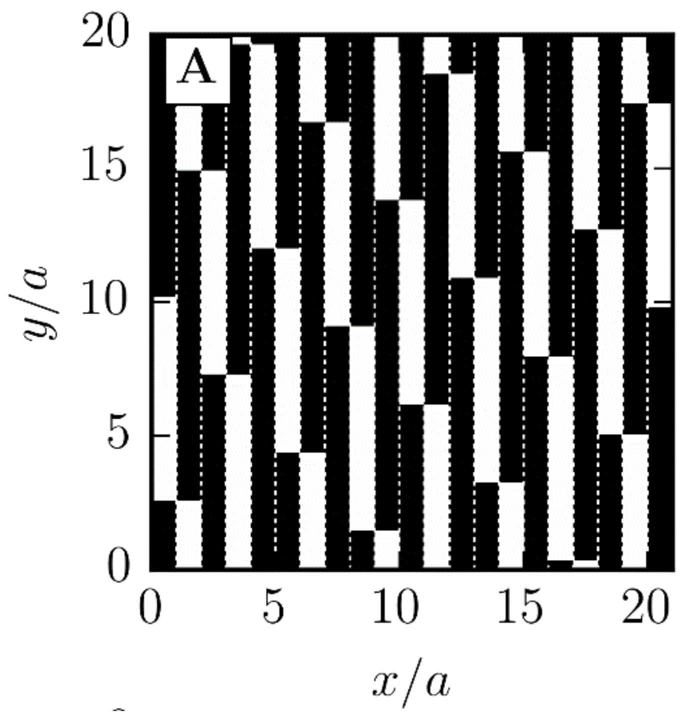
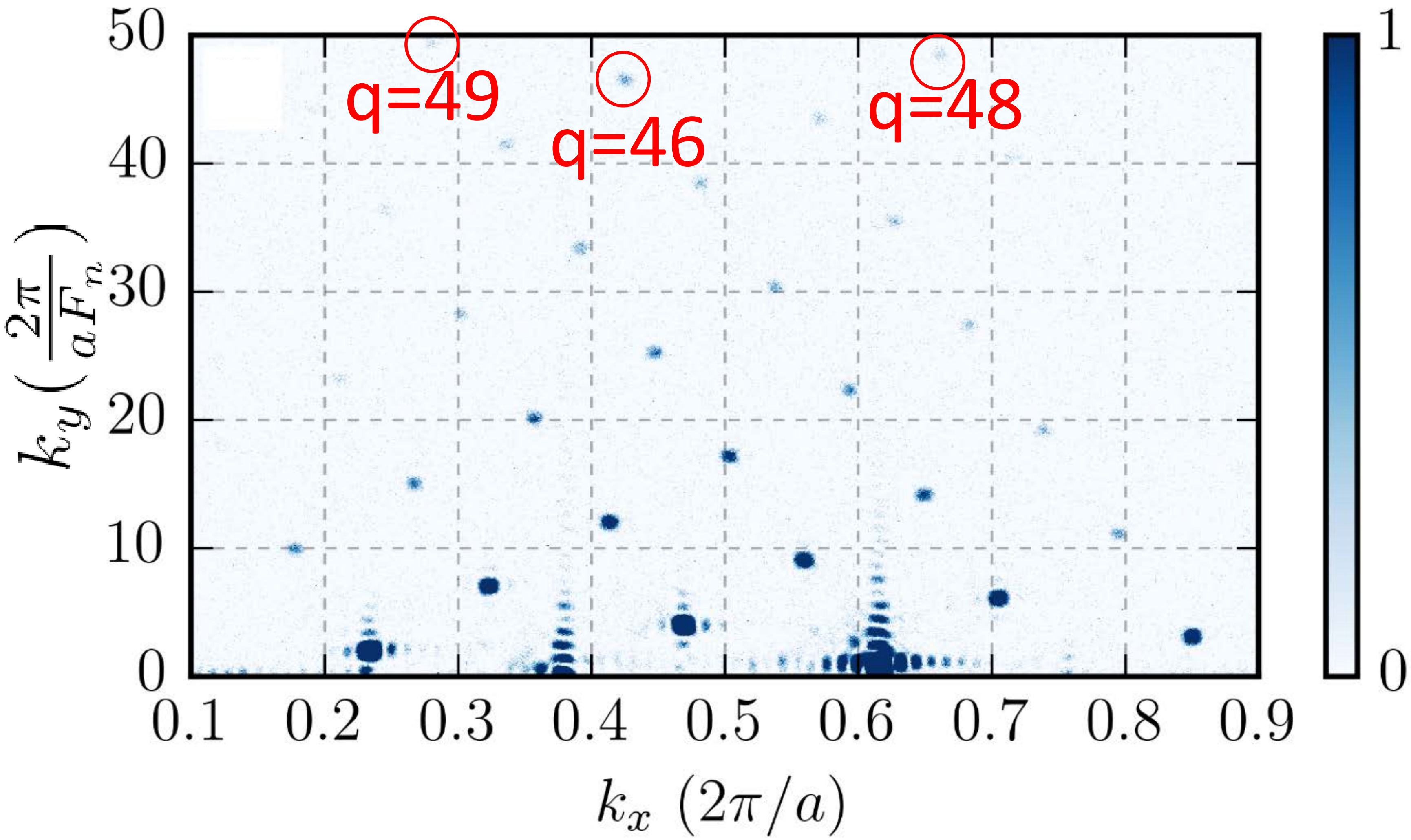


DMD Pattern



y axis is associated with Φ

Diffraction pattern



Spectral Features

So far, we presented structural features

culminating in topological winding numbers

What about spectral ones?

How to Characterize Tilings – Spectrum?

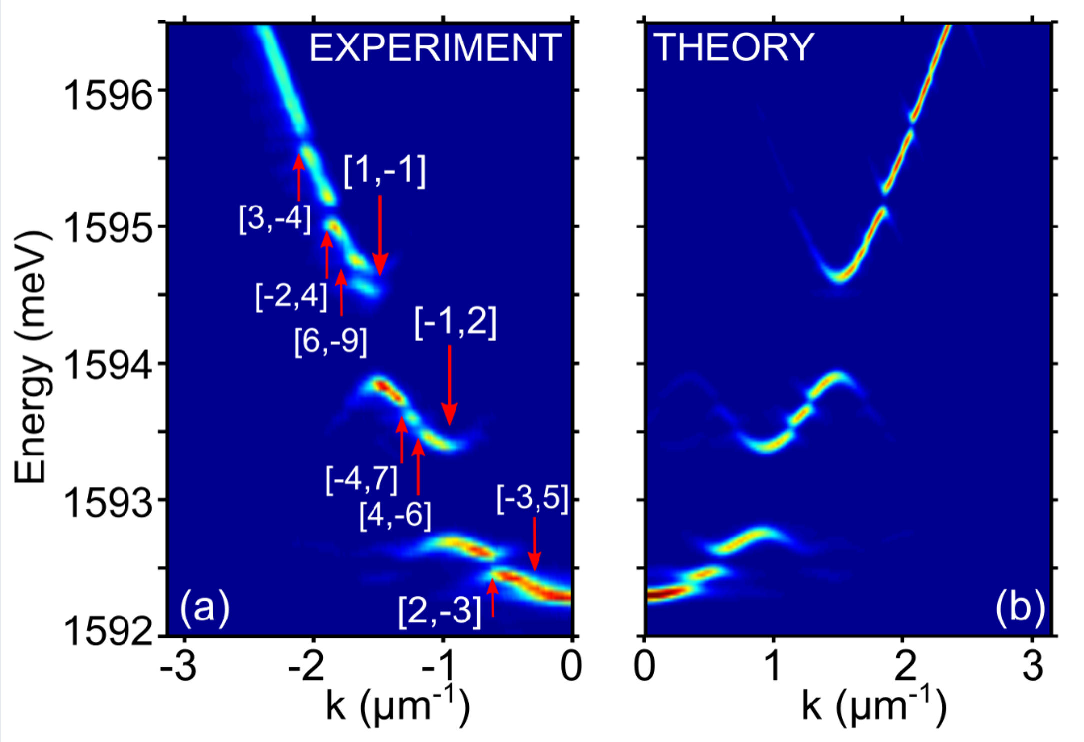
Spectrum & Integrated Density of States

- Solve a Hamiltonian $H(E)$ (Fibonacci)

$$H\psi(x) = E\,\psi(x)$$

- Find the spectrum:
  - dispersion $E(k)$
  - integrated density of states

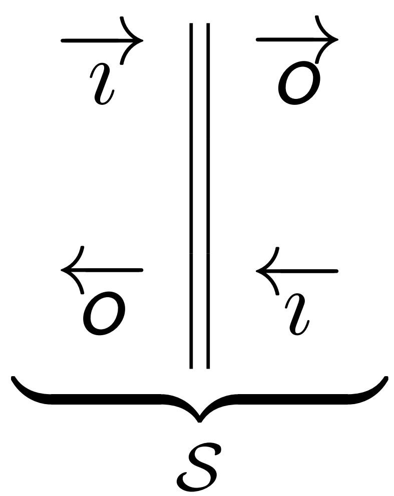
$$H(E) \to \begin{cases} \varrho(E) & \text{DOS} \\ \mathcal{N}(E) & \text{IDOS} \end{cases}$$



❑ D. Tanese et al., Phys. Rev. Lett. **112**, 146404 (2013)

Scattering formalism

- Take $1D$ wave system of size L bounded by two semi-infinite free systems



$$\Rightarrow \quad \begin{pmatrix}\overrightarrow{o}\\ \overleftarrow{o}\end{pmatrix} = \begin{pmatrix}\overrightarrow{r}(k) & t(k)\\ t(k) & \overleftarrow{r}(k)\end{pmatrix}\begin{pmatrix}\overrightarrow{\imath}\\ \overleftarrow{\imath}\end{pmatrix} \equiv \mathcal{S}\begin{pmatrix}\overrightarrow{\imath}\\ \overleftarrow{\imath}\end{pmatrix}$$

Scattering formalism

- The $\mathcal{S}$-matrix is diagonalized to

$$\mathcal{S} \mapsto \begin{pmatrix} e^{i\phi_1} & 0 \\ 0 & e^{i\phi_2} \end{pmatrix} \quad \Rightarrow \quad \det \mathcal{S} = e^{2i\delta(k)}$$

with $\delta(k) = \frac{1}{2}(\phi_1(k) + \phi_2(k))$

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with $\delta(k) = \frac{1}{2}(\phi_1(k) + \phi_2(k))$

- Find density of modes with Krein-Schwinger formula ,

$$\varrho(k) - \varrho_0(k) = \frac{1}{2\pi} \operatorname{Im} \frac{d}{dk} \ln \det \mathcal{S}(k)$$

G. Dunne, E. Levy, E.A.,"Optics of Aperiodic Structures: Fundamentals and Device Applications", L. dal Negro ed., Pan Stanford Publishing, (2013)

- The normalized IDOS is given by

$$\mathcal{N}(\nu) - \mathcal{N}_0(\nu) = \frac{1}{2\pi} \operatorname{Im} \log \det \mathscr{S}(\nu, \phi)$$

independent of ϕ

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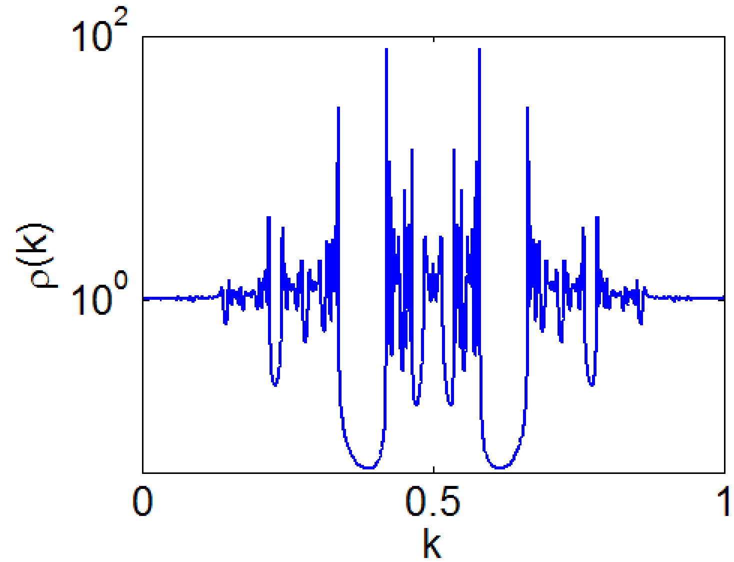
independent of ϕ

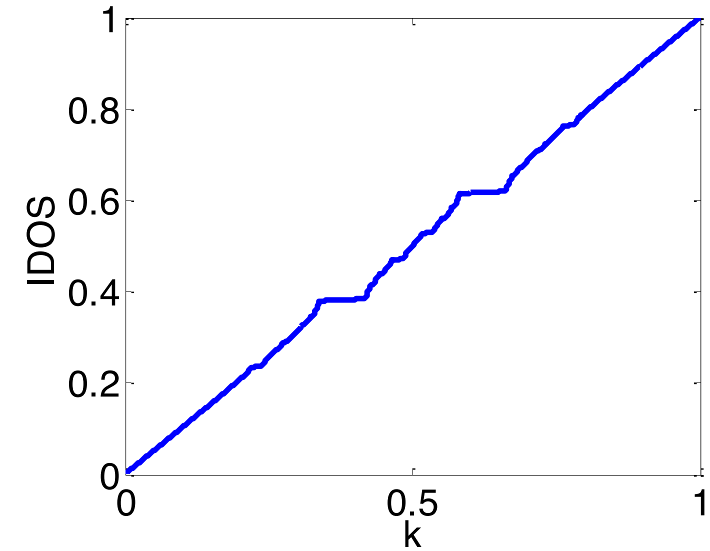
- For C&P, gaps appear at

$$\mathcal{N}_{\text{gap}} = p + q\, s \pmod 1$$

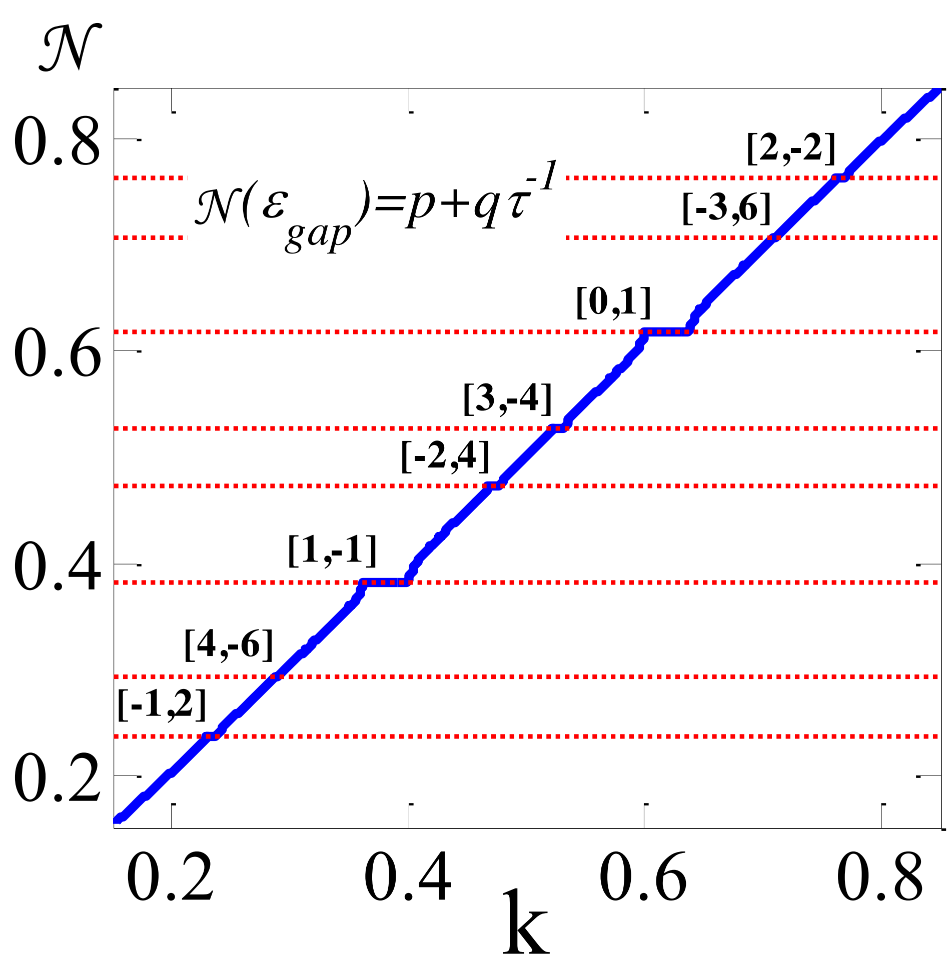
this is the GLT



Density of modes



IDOS- counting function



Gap Labeling Theorem (GLT)

- The $\mathscr{S}$-matrix is a 2×2 unitary matrix (in $1d$) : $\mathscr{S} \sim \begin{pmatrix} e^{i\gamma_1} & 0 \\ 0 & e^{i\gamma_2} \end{pmatrix}$
    - Uniquely identified by 2 phases
    - That can be written **universally**

1. A ϕ-independent spectral total phase shift

$$\delta(\nu) = \frac{1}{2}(\gamma_1 + \gamma_2) = \frac{1}{2} \operatorname{Im} \log \det \mathscr{S}(\nu, \phi)$$

with $\mathcal{N}(\nu) - \mathcal{N}_0(\nu) = \frac{1}{\pi}\delta(\nu)$

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1. A ϕ-independent spectral total phase shift

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2. A ϕ-dependent spectral chiral phase by

$$\alpha(\nu_{gap}, \phi) = \gamma_1 - \gamma_2 = \operatorname{Im} \operatorname{Tr}[\sigma_z \log \mathscr{S}(\nu_{gap}, \phi)]$$

Where there is a ϕ-dependent phase – there is a **winding**!

Spectral Winding

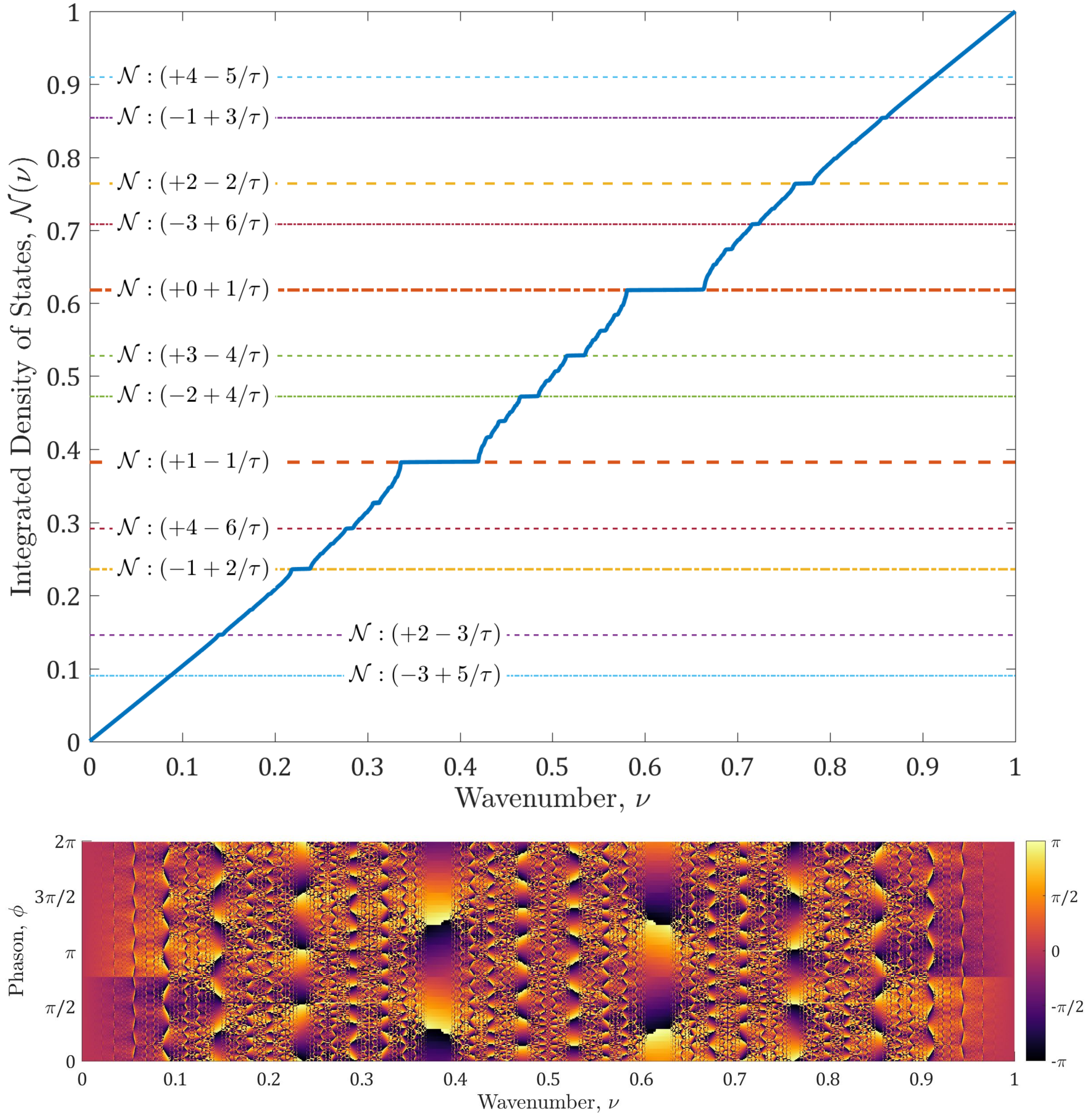
- To each gap $\mathcal{N}_{\text{gap}} = q\, s \pmod 1$, count the winding

$$\mathcal{W}_{\phi}\left[\alpha\right] = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{\partial \alpha\left(\nu_{\text{gap}}, \phi\right)}{\partial \phi} \, \mathrm{d}\phi$$

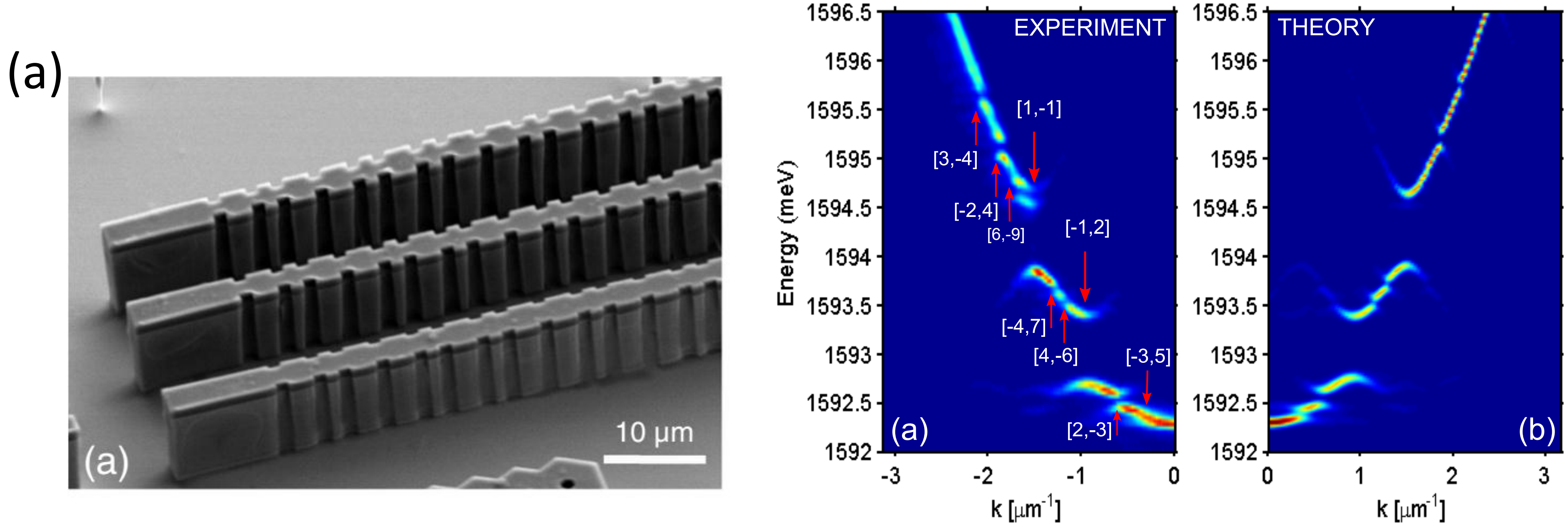
- Numerical calculation yields

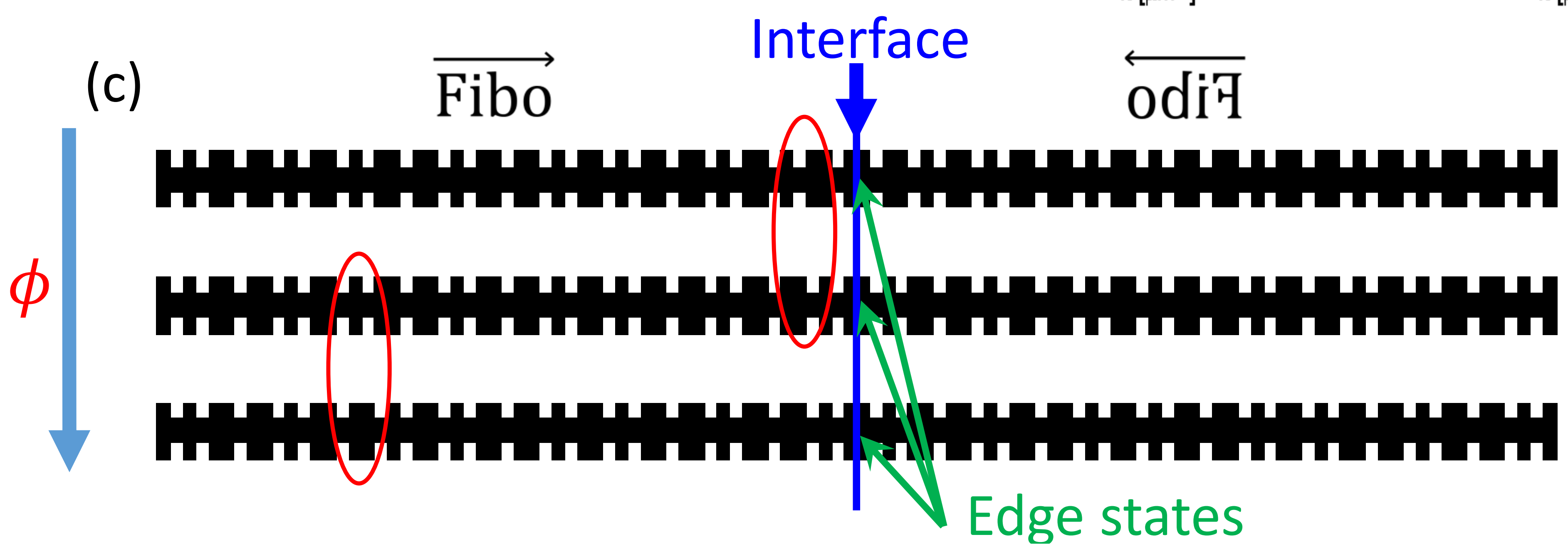
$$\mathcal{W}_{\phi}\left[\alpha\right] = 2q$$

IDOS and Chiral Phase (Fibonacci)



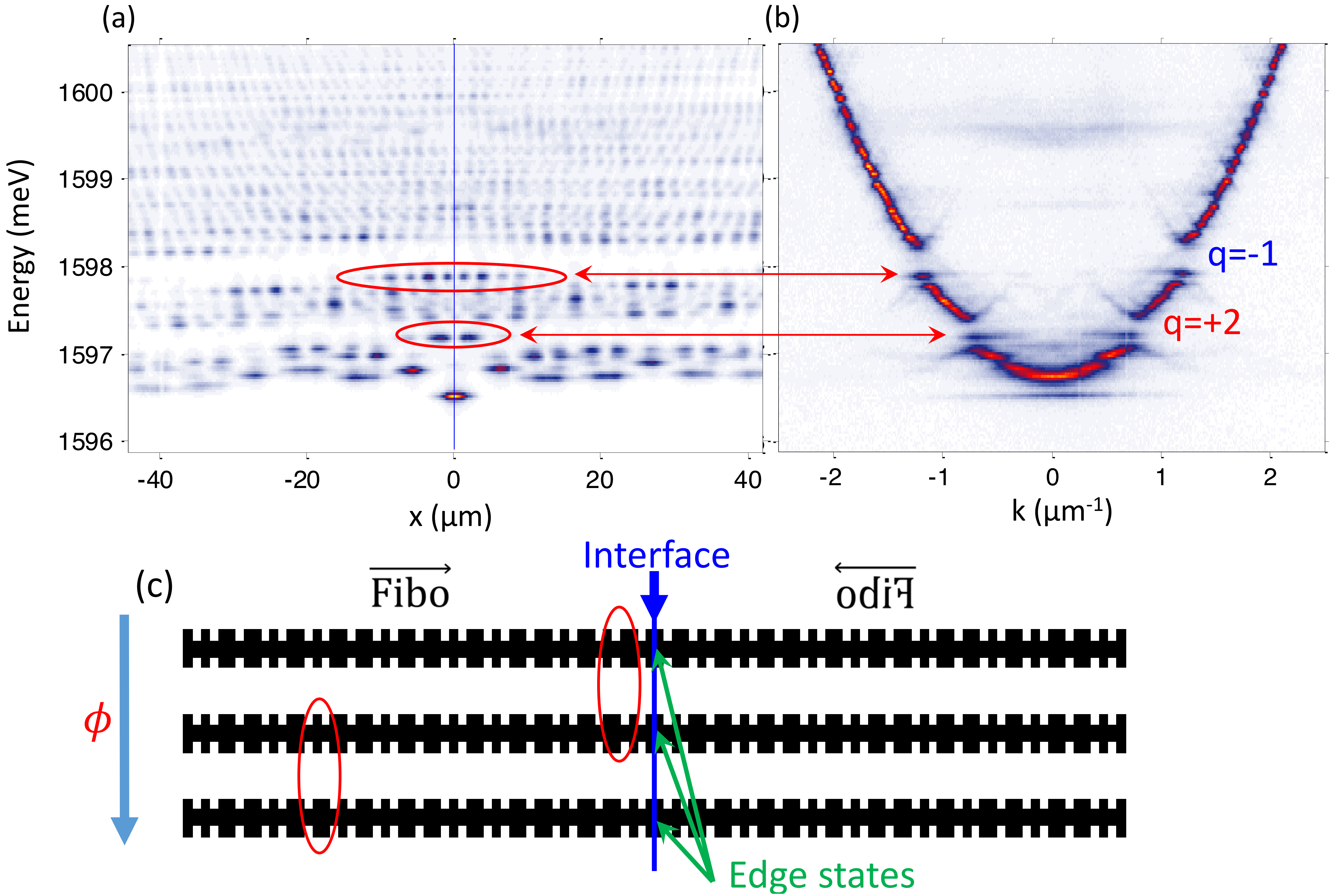
Measurement using cavity polaritons





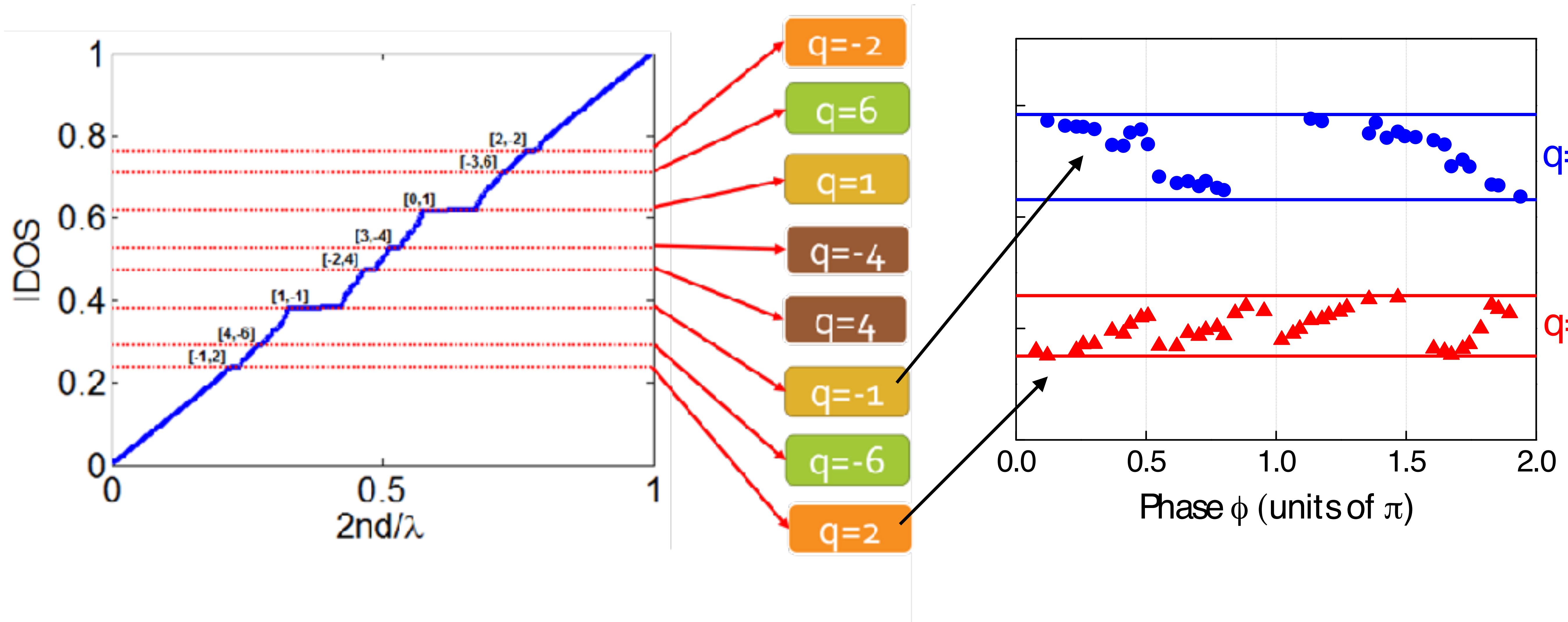
F. Baboux, E. Levy, J. Bloch, E.A, 2017

Measurement using cavity polaritons



F. Baboux, E. Levy et al., Phys. Rev. B **95**, 161114 (2017)

Measurement using cavity polaritons



F. Baboux, E. Levy, J. Bloch, E.A, 2016

Winding Relations

Two windings dependent on the same phason ϕ

Is there a relation?

Winding Relation

Structural – Spectral

$$2\mathcal{W}_{\phi}\left[\Theta\right] = 2q = \mathcal{W}_{\phi}\left[\alpha\right]$$

A Bloch Theorem ?

Two Phases - Winding numbers

- The Structural phase

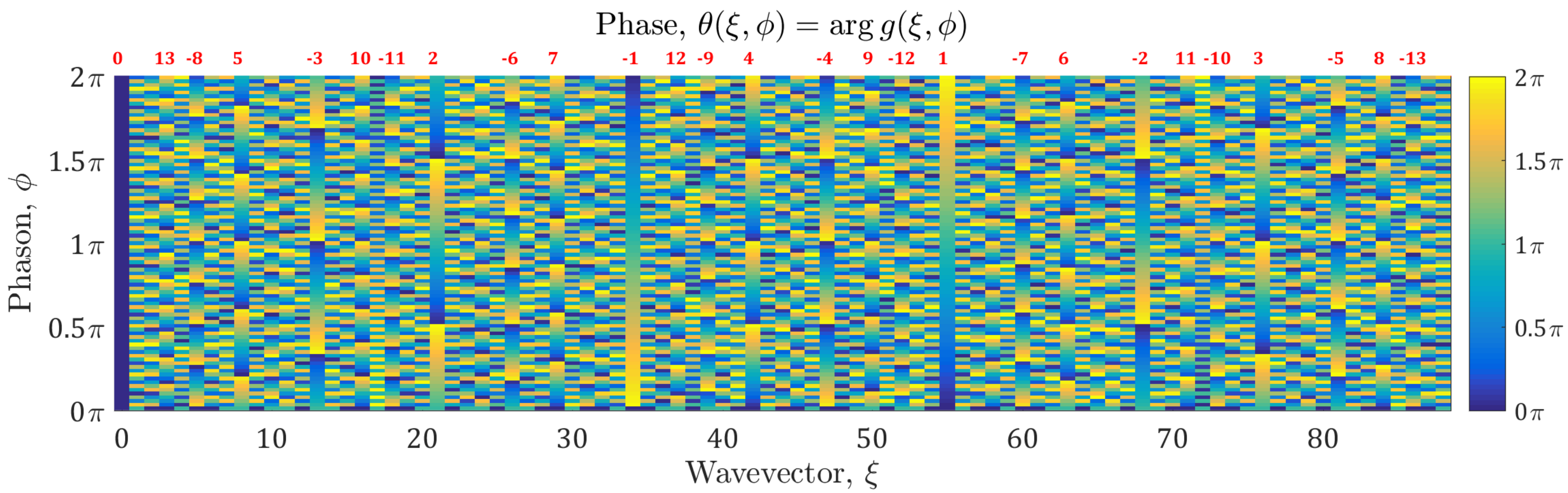
$$S(\xi,\phi) = |g(\xi,\phi)|^2, \quad \theta(\xi,\phi) = \arg g(\xi,\phi)$$

Two Phases - Winding numbers

- The Structural phase

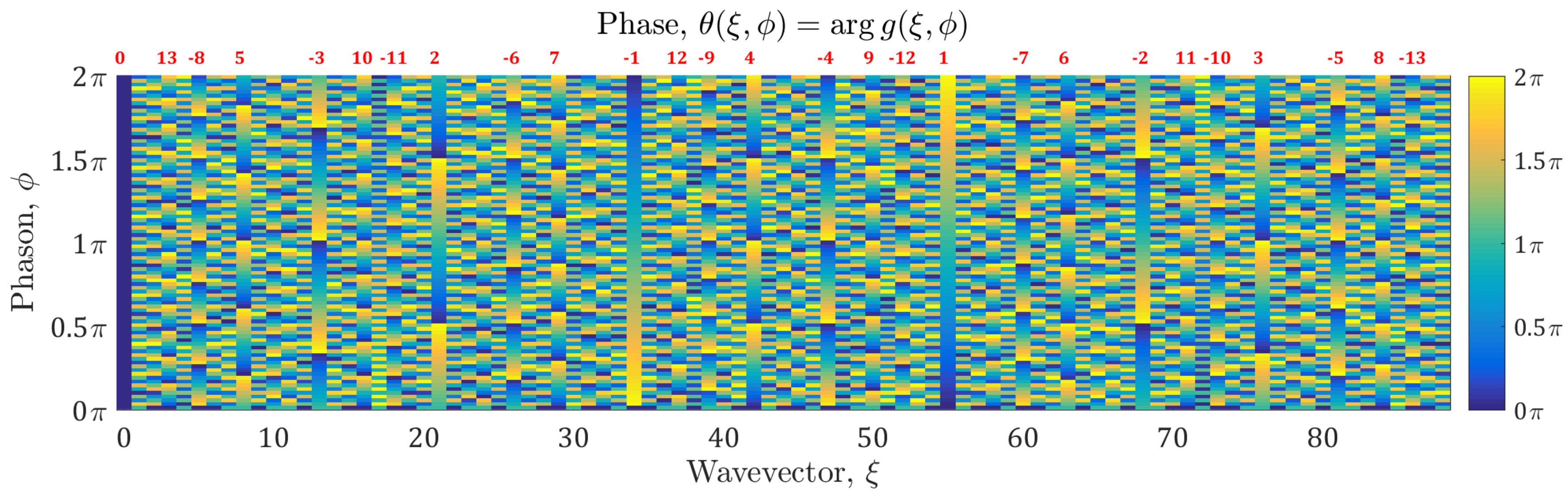
Winding number at a Bragg peak $\xi = \xi_0$

$$W_{\xi_0} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \theta \left(\xi = \xi_0, \phi\right)}{\partial \phi} \, \mathrm{d}\phi$$



Two Phases - Winding numbers

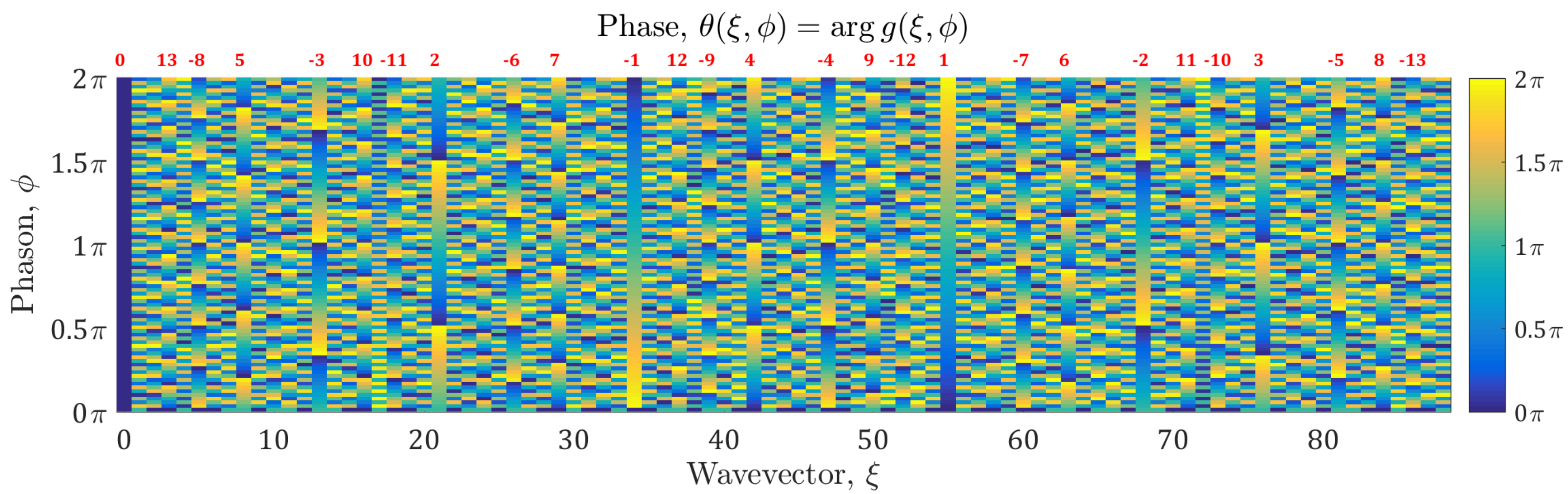
- The Structural phase



- The Chiral phase

Two Phases - Winding numbers

- The Structural phase



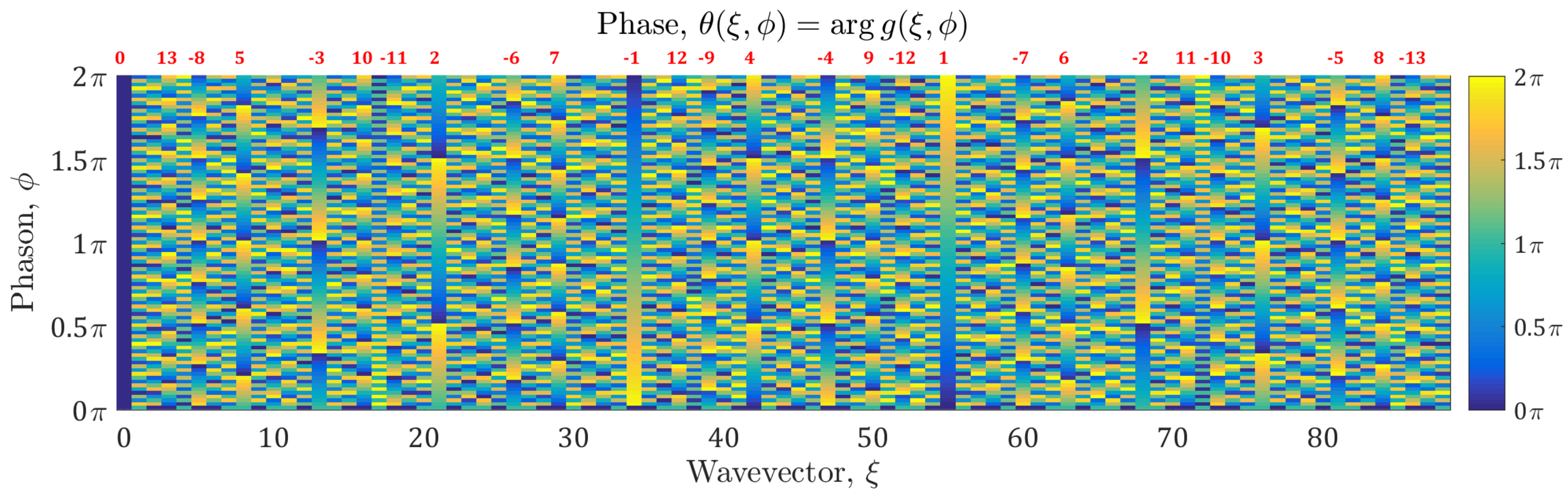
- The Chiral phase

Winding number at a spectral gap $k_{p,q}$

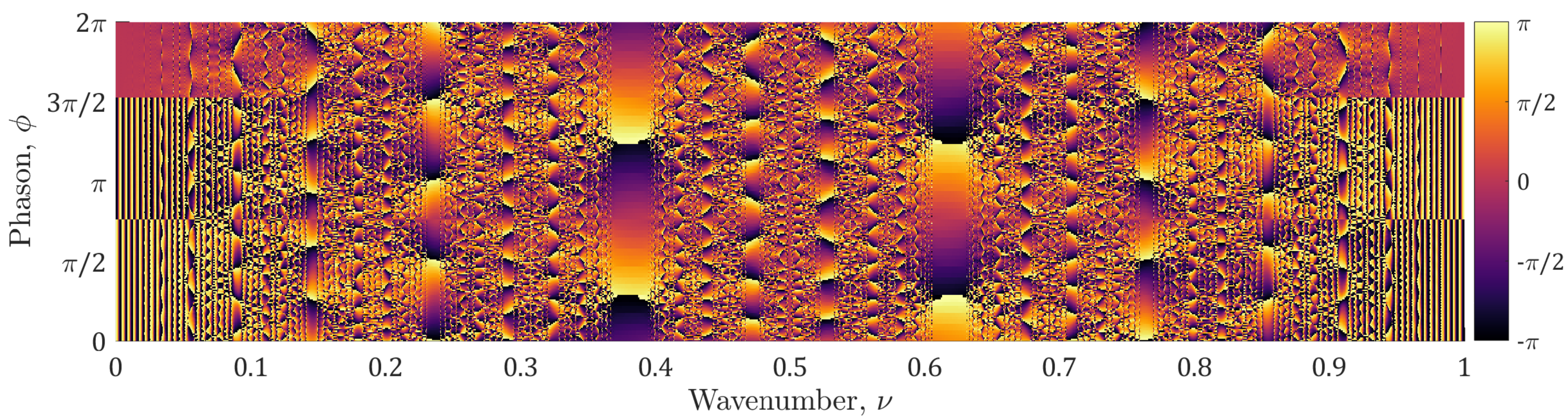
$$W_{\alpha_g} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \alpha \left(k = k_{p,q}, \phi\right)}{\partial \phi} \, \mathrm{d}\phi$$

Two Phases - Winding numbers

- The Structural phase

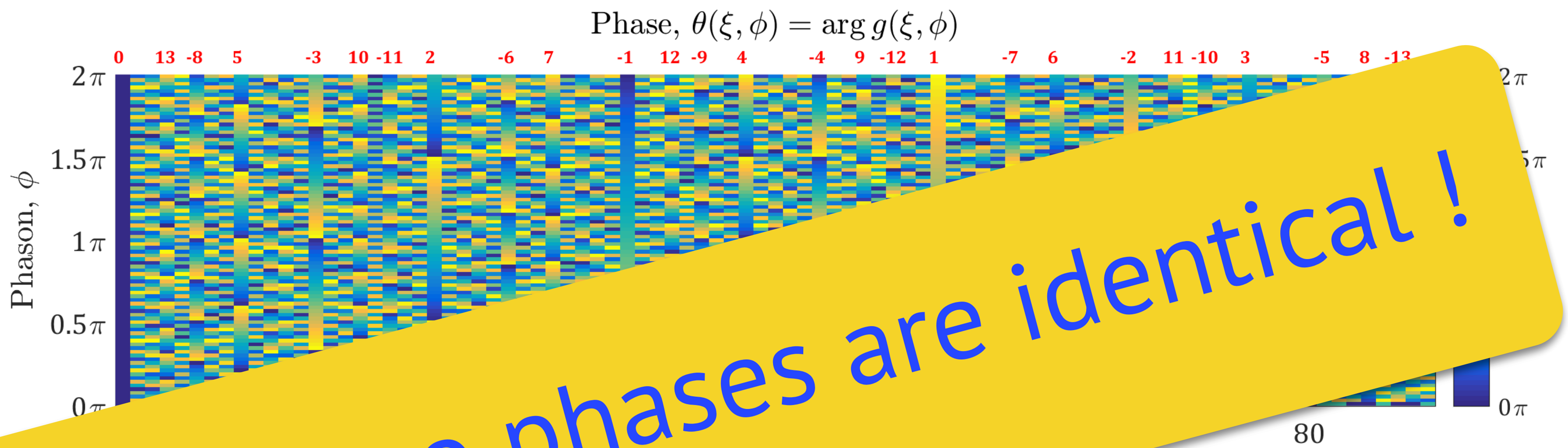


- The Chiral phase



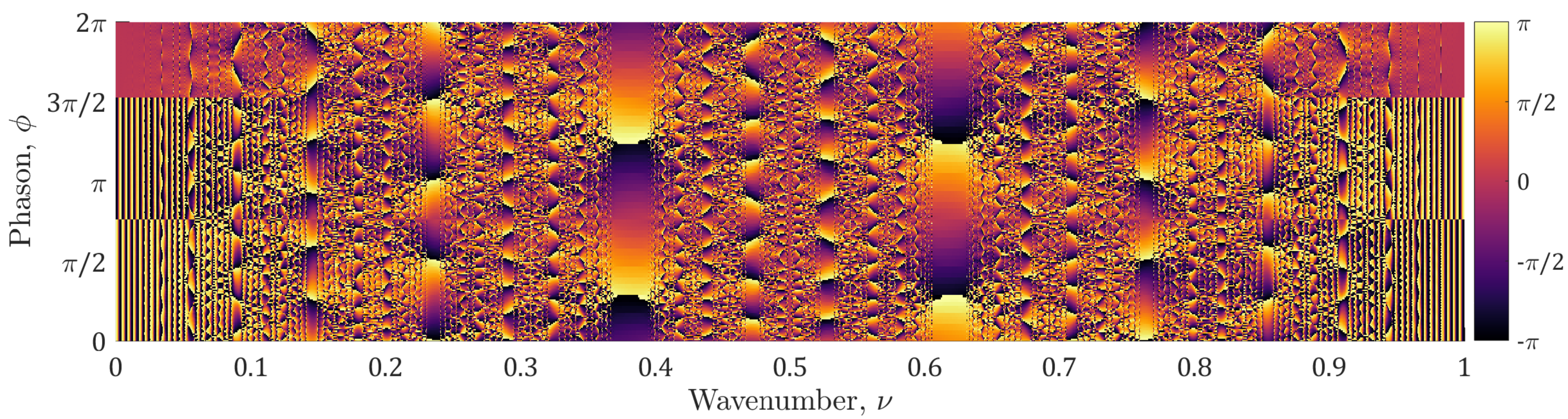
Two Phases - Winding numbers

- The Structural phase



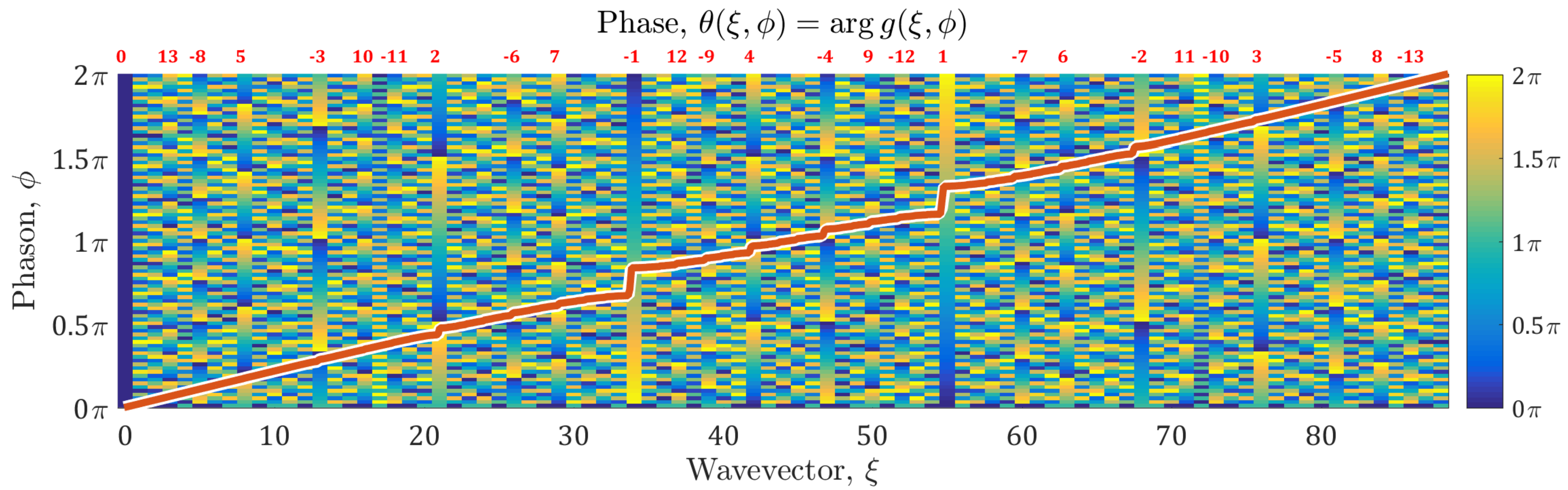
These two phases are identical !

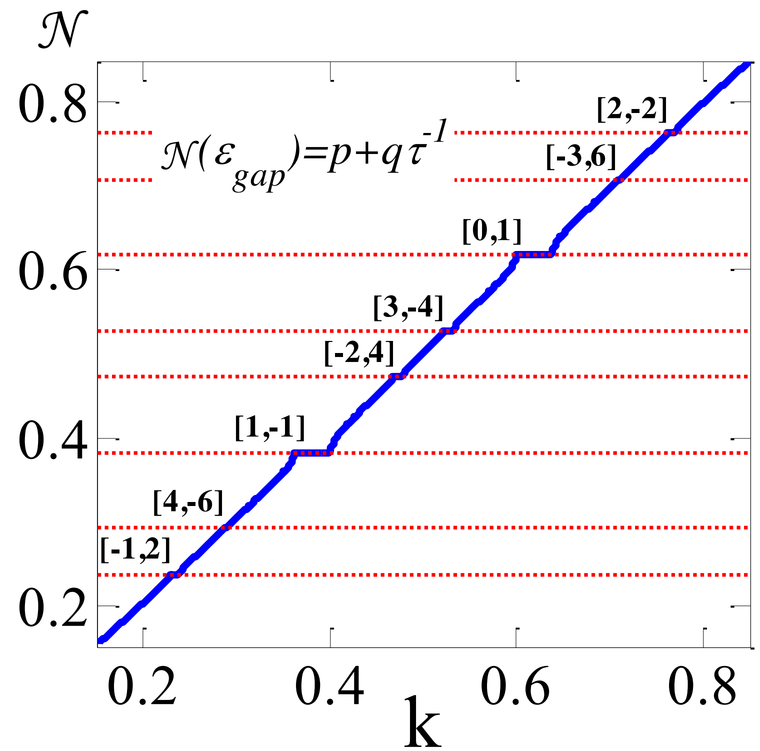
- Th hase



In case you are not yet convinced...

- The Structural phase





This establishes a relation between structure and spectrum.

A Bloch theorem for aperiodic tilings

$$k_b/k_0 = p + qs = \mathcal{N}\left(E_g\right), \quad p, q \in \mathbb{Z}.$$

$$2\mathcal{W}_\phi\left[\Theta_d\right] = \mathcal{W}_\phi\left[\Theta_S\right] = 2q$$

Structure **Spectrum**

$$\begin{array}{ccc} \mathbb{Z} \cong \mathcal{W}_\phi\left[\Theta\right] & \overset{\overset{!}{\cong}}{\longleftarrow} & \mathcal{W}_\phi\left[\alpha\right] \cong \mathbb{Z} \\ \uparrow & & \uparrow \\ \mathbb{Z}^2 \cong \check{H}^1_{\mathrm{C\&P}} & \longleftarrow & K_0^{\mathrm{C\&P}} \cong \mathbb{Z} \oplus \mathbb{Z} \\ \downarrow \tau_*^{\check{H}} & & \downarrow \tau_*^{K} \\ \mathbb{Z} \oplus s\mathbb{Z} & \overset{=}{\longleftrightarrow} & \mathbb{Z} \oplus s\mathbb{Z} \end{array}$$

- There is a topological content
  - independent of ϕ
- Our topological invariant has a name: the Čech Cohomology $\check{H}^1$
- Computable for many different tilings

Outline

Substitutions

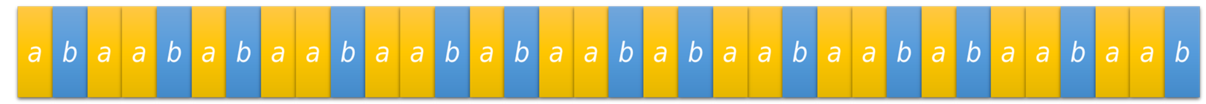
Periodic

- A simple rule: $\begin{cases} \sigma(a) = ab \\ \sigma(b) = ab \end{cases}$

- Resulting in...

$$a \mapsto ab \mapsto abab \mapsto$$
$$abab\,abab \mapsto$$
$$abab\,abab\,abab\,abab \mapsto \cdots$$

Fibonacci



- A simple rule: $\begin{cases} \sigma(a) = ab \\ \sigma(b) = a \end{cases}$

- Resulting in...

$$a \mapsto ab \mapsto aba \mapsto$$
$$abaab \mapsto abaab\,aba \mapsto$$
$$abaab\,aba\,abaab \mapsto \cdots$$

- Define substitution rules by

$$\begin{cases} \sigma(a) = a^{\alpha} b^{\beta} & \Leftrightarrow \quad a \mapsto a^{\alpha} b^{\beta} \\ \sigma(b) = a^{\gamma} b^{\delta} & \Leftrightarrow \quad b \mapsto a^{\gamma} b^{\delta} \end{cases}$$

  with $\alpha, \beta, \gamma, \delta \geq 0$.
- Acting on a word $w = \ell_1 \ell_2 \ldots \ell_k$ is a concatenation

$$\sigma(w) = \sigma(\ell_1)\,\sigma(\ell_2) \ldots \sigma(\ell_k)$$

- Associated occurrence matrix

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

- Consider only primitive matrices (substitutions)
  - $\exists N_0$ such that $\forall N > N_0$ all entries of M^N are strictly positive
  - Largest eigenvalue $\lambda_1 > 1$ (Perron-Frobenius)
  - Left and right first eigenvectors are strictly positive

Examples

Name	Rule		M	λ_*
Periodic	$a \mapsto ab$	$b \mapsto ab$	$\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$	2
Thue-Morse	$a \mapsto ab$	$b \mapsto ba$	$\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$	2
Fibonacci	$a \mapsto ab$	$b \mapsto a$	$\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$	$\tau = \frac{\sqrt{5}+1}{2}$
Fibonacci^2	$a \mapsto aab$	$b \mapsto ab$	$\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)$	τ^2
Non-Fibonacci^2	$a \mapsto aab$	$b \mapsto ba$	$\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)$	τ^2

- Representatives of 3 families:
  1. Periodic 2. Quasiperiodic 3. Aperiodic

Occurrence Matrix M

What can be done?

- Gap labeling Thm.

What cannot be done?

- Diffraction

How to calculate the Čech Cohomology $\check{H}^1$?

Supertiles (1D)

Infinite tiling: $w_\infty = \sigma^\infty(a)$

Supertiles (words): $\Gamma_n = \{w \in w_\infty \mid |w| = n\}$

Supertile rule: $\sigma_n : \Gamma_n \to \Gamma_n^{\mathbb{N}}$

Occurrence mat.: M_n (all with the same λ_*)

Example: Fibonacci, $n = 2$

$$\Gamma_1 = \{a, b\} \qquad \Gamma_2 = \{A, B, C\} = \{aa, ab, ba\}$$

$$\sigma_1 = \begin{cases} a \mapsto ab \\ b \mapsto a \end{cases} \qquad \sigma_2 = \begin{cases} A \mapsto BC \\ B \mapsto BC \\ C \mapsto A \end{cases}$$

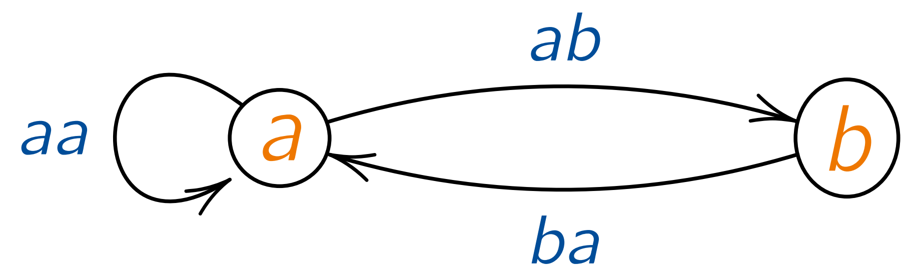
$$M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$w_\infty^{(1)} = abaab\,aba\,abaab \qquad w_\infty^{(2)} = BCABC\,BCA\,BCABC$$

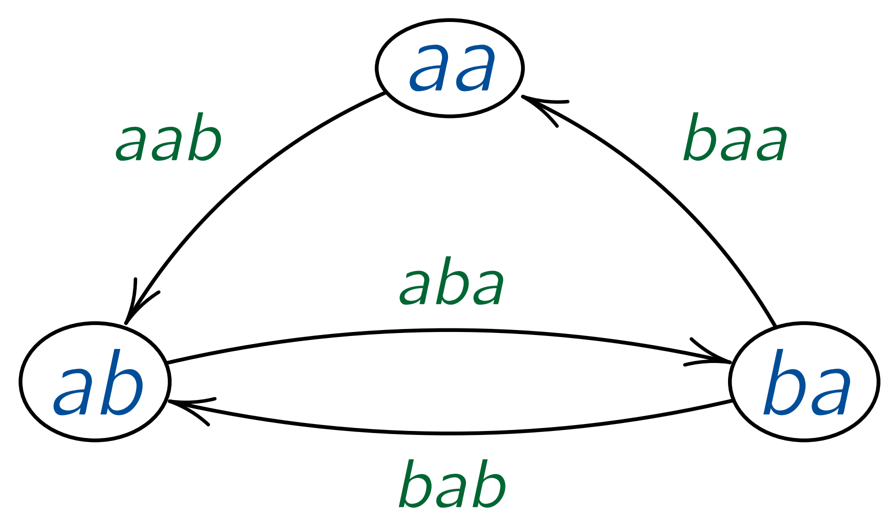
Supertiles (1D)

- Shift-maps $\gamma_n(L_i) = L_j$ if $L_i L_j$ exists in $w_\infty^{(n)}$
- Representation by planar graphs G_n
  (called Bratteli diagrams)

$G_1^{\text{Fib}} =$



$G_2^{\text{Fib}} =$



How to calculate the Čech Cohomology $\check{H}^1$?

$$\zeta(z) = \frac{\det\left(I - zA_0^{\mathsf{T}}\right)}{\det\left(I - zA_1^{\mathsf{T}}\right)} \doteq \frac{p_0(z)}{p_1(z)}.$$

$$p_k(z) = \prod_{i=1}^{I}(1 - c_i z)\prod_{j=1}^{J}\left(1 - d_j z - e_j z^2\right)$$

$$c_i, d_j, e_j \in \mathbb{Z}$$

$$\begin{aligned}\check{H}^k &\cong \bigoplus_{i=1}^{I}\mathbb{Z}[1/c_i] \;\oplus\; \bigoplus_{j=1}^{J}\mathbb{Z}^2[1/e_j] \\ &= \mathbb{Z}[c_1^{-1}] \oplus \cdots \oplus \mathbb{Z}[c_I^{-1}] \;\oplus\; \mathbb{Z}^2[e_1^{-1}] \oplus \cdots \oplus \mathbb{Z}[e_J^{-1}]\end{aligned}$$

$$\mathbb{Z}[1/c] = \{n/c^m \mid n, m \in \mathbb{Z}\}$$

How to calculate the Čech Cohomology $\check{H}^1$?

- Computable for many different tilings
- Distinguishes between families

Family	$\check{H}^1$	Diffraction peaks	Gap labeling
Periodic	$\mathbb{Z}$	$k_b = n/2$	$\mathcal{N}_g = 1/2$
Quasiperiodic	$\mathbb{Z}^2$	$k_b = p + q\,\varrho_b$	$\mathcal{N}_g = q\,\varrho_b$
Thue-Morse	$\mathbb{Z} \oplus \mathbb{Z}\left[\frac{1}{2}\right]$	$k_b = \frac{1}{2n+1}\,\frac{m}{2^N}$	$\mathcal{N}_g = \frac{1}{3}\,\frac{m}{2^N}$

Gap Labeling Theorem

- In $1D$ aperiodic substitutions, the possible gaps are

$$\mathcal{N}_{\text{gap}} \in \tau_*^K \left[K_0 \left(\mathcal{B} \right) \right]$$

- Explicitly ($k, N \in \mathbb{N}$),

$$\mathcal{N}_{\text{gap}} = \frac{1}{a} \frac{k}{\lambda_*^N} \pmod{1}$$

  - The normalization factor a is inferred by $\boldsymbol{v}_*, \boldsymbol{v}_*^{(2)}$

- In C&P tilings ($p, q \in \mathbb{Z}$)

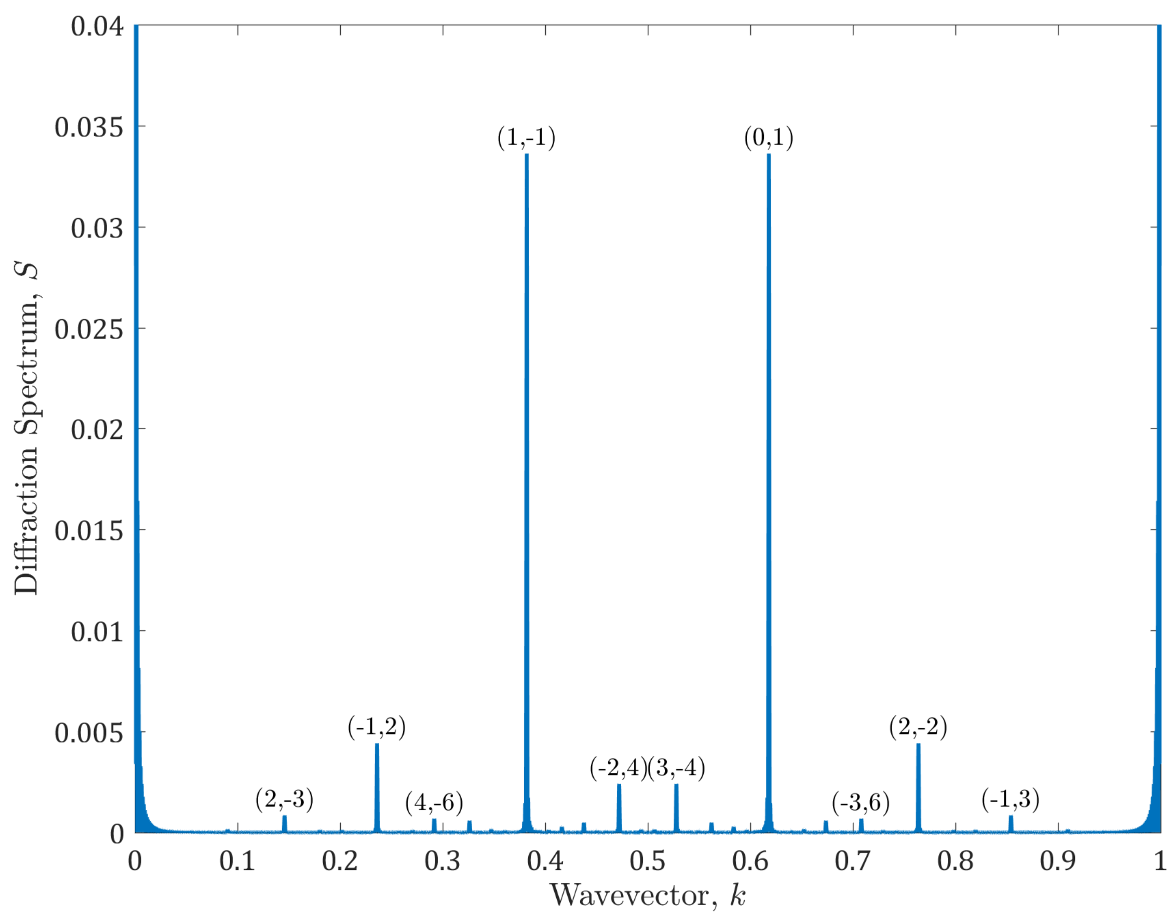
$$\mathcal{N}_{\text{gap}} = p + q\, s \pmod{1}$$

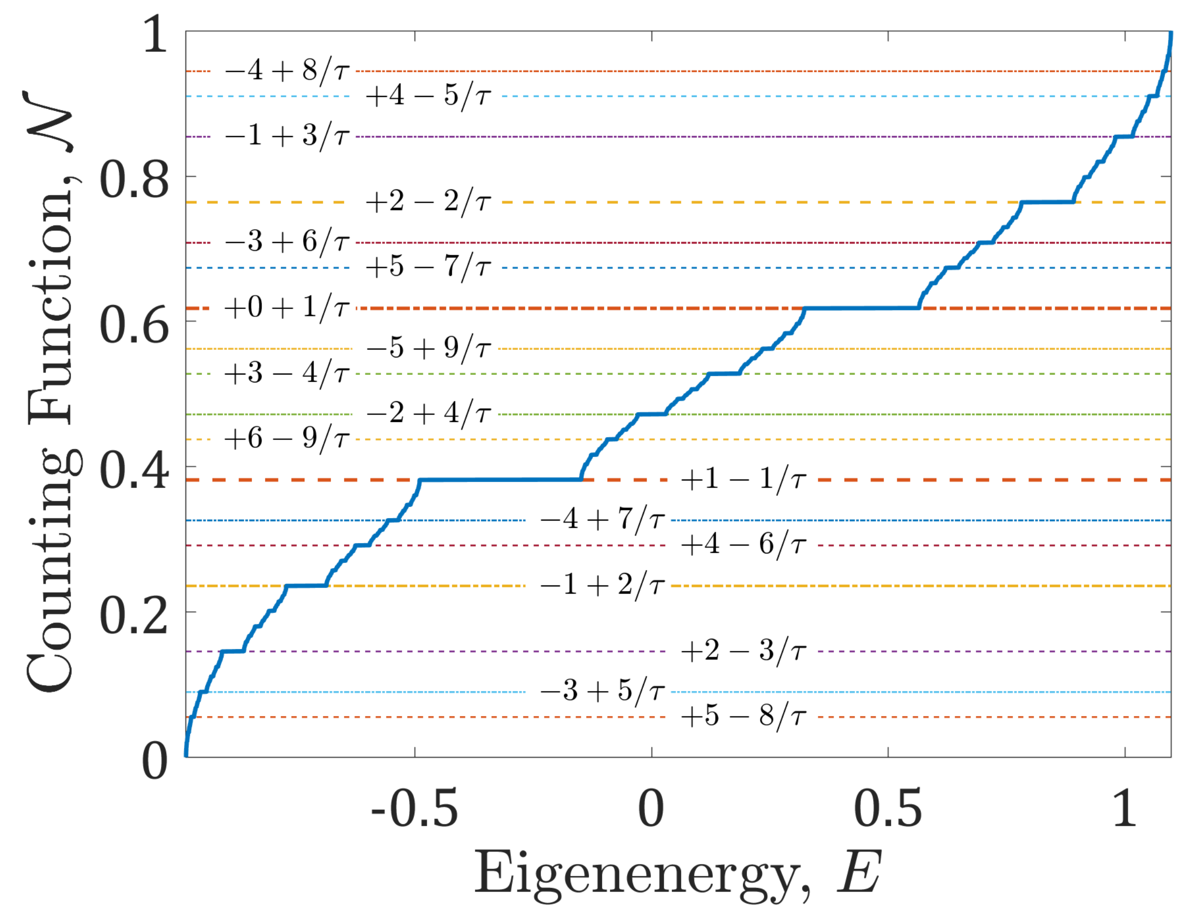
J. Bellissard, A. Bovier, J.-M. Ghez

Outline

Symmetry:

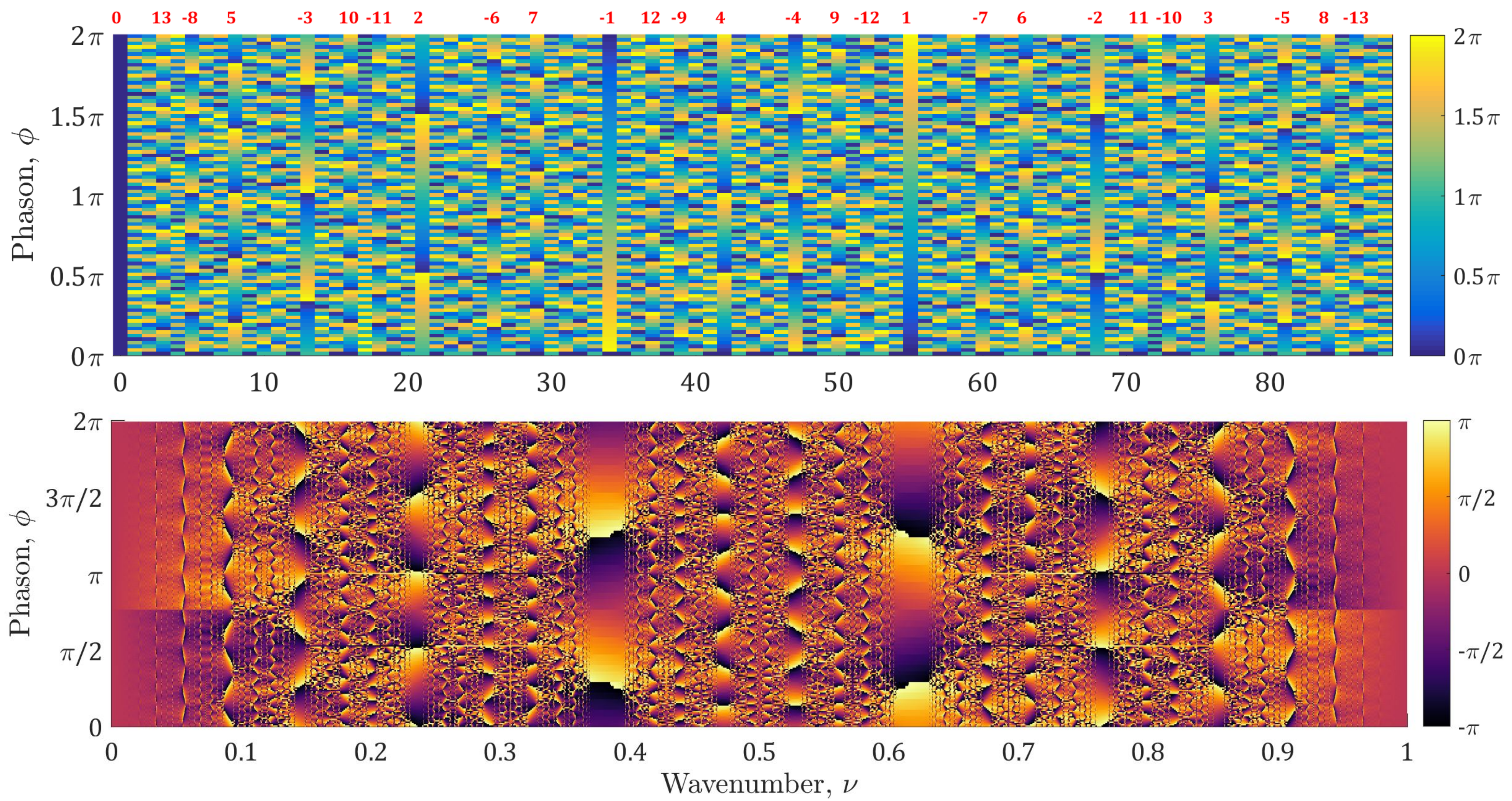
$$k_b = p + q\, s = \mathcal{N}\left(E_g\right)$$





Windings:

$$2\mathcal{W}_\phi\left[\Theta\right] = 2q = \mathcal{W}_\phi\left[\alpha\right]$$



Structure **Spectrum**

$$\begin{array}{ccc}
\mathbb{Z} \cong \mathcal{W}_\phi[\Theta] & \xleftarrow{\overset{!}{\cong}} & \mathcal{W}_\phi[\alpha] \cong \mathbb{Z} \\
\uparrow & & \uparrow \\
\mathbb{Z}^2 \cong \check{H}^1_{\mathrm{C\&P}} & \longleftarrow & K_0^{\mathrm{C\&P}} \cong \mathbb{Z} \oplus \mathbb{Z} \\
\downarrow \tau_*^{\check{H}} & & \downarrow \tau_*^{K} \\
\mathbb{Z} \oplus s\mathbb{Z} & \overset{=}{\longleftrightarrow} & \mathbb{Z} \oplus s\mathbb{Z}
\end{array}$$

Structure **Spectrum**

$$\begin{array}{ccc}
\mathbb{Z} \cong \mathcal{W}_\phi[\Theta] & \overset{\overset{!}{\cong}}{\longleftarrow} & \mathcal{W}_\phi[\alpha] \cong \mathbb{Z} \\
\uparrow & & \uparrow \\
\mathbb{Z}^2 \cong \check{H}^1_{\mathrm{C\&P}} & \longleftarrow & K_0^{\mathrm{C\&P}} \cong \mathbb{Z} \oplus \mathbb{Z} \\
\downarrow \tau_*^{\check{H}} & & \downarrow \tau_*^{K} \\
\mathbb{Z} \oplus s\mathbb{Z} & \overset{=}{\longleftrightarrow} & \mathbb{Z} \oplus s\mathbb{Z}
\end{array}$$

Theorem (Generalized Bloch)

For finite local complexity tilings with finitely many tile orientations, the following diagram commutes in dimensions $D \leq 3$

$$\begin{array}{ccc}
\check{H}^1 & \longleftarrow & K_0 \\
\downarrow \tau_*^{\check{H}} & & \downarrow \tau_*^{K} \\
\mathbb{R} & \overset{\mathrm{id}}{\longleftrightarrow} & \mathbb{R}
\end{array}$$

1. E. Akkermans, Y. Don, J. Rosenberg and C. L. Schochet, *Relating Diffraction and Spectral Data of Aperiodic Tilings: Towards a Bloch theorem*, J. Geom. Phys. **165**, 104217 (2021).

Implications

- Since the traces $\tau_*^{\check{H}}(\check{H}^1)$ and $\tau_*^{K}(K_0)$ commute, $\check{H}^1$ represents both spectral and structural properties
    - recall $\tau_*^{K}(K_0)$ is the GLT
- The trace $\tau_*^{\check{H}}(\check{H}^1)$ *does not* describe diffraction $S(k)$
    - except when $S(k)$ consists of Bragg peaks only

Generalized Bloch – Summary

Family	$\check{H}^1$	Diffraction peaks		$\tau_*^{\check{H}}(\check{H}^1)$	Spectral Gaps
Periodic	$\mathbb{Z}$	$k_n = n$	PP	$\mathbb{Z}$	$\mathcal{N} = \text{const}$
Fibonacci	$\mathbb{Z}^2$	$k_{p,q} = p + q/\tau$	PP	$\mathbb{Z} + \tau^{-1}\mathbb{Z}$	$\mathcal{N}_q = q/\tau$
Thue-Morse	$\mathbb{Z} \oplus \mathbb{Z}\left[\frac{1}{2}\right]$	$k_{n,m,N} = \frac{1}{2n+1}\frac{m}{2^N}$	SC+PP	$\frac{1}{3}\mathbb{Z}\left[\frac{1}{2}\right]$	$\mathcal{N}_{m,N} = \frac{1}{3}\frac{m}{2^N}$
Period Doubling	$\mathbb{Z} \oplus \mathbb{Z}\left[\frac{1}{2}\right]$	$k_{m,N} = \frac{m}{2^N}$	PP	$\frac{1}{3}\mathbb{Z}\left[\frac{1}{2}\right]$	$\mathcal{N}_{m,N} = \frac{1}{3}\frac{m}{2^N}$
Rudin-Shapiro	$\mathbb{Z} \oplus \mathbb{Z}\left[\frac{1}{2}\right] \oplus \mathbb{Z}^2\left[\frac{1}{2}\right]$	N/A	AC	$\mathbb{Z}\left[\frac{1}{2}\right]$	$\mathcal{N}_{m,N} = \frac{m}{2^N}$

Outline

Summary I

- C&P tilings exhibit topological content by winding numbers
  - a. diff. peaks and spectral gaps are analogous: $k_b = p + q\, s = \mathcal{N}(\nu_g)$
  - b. structural and spectral windings are related: $2\mathcal{W}_\phi[\Theta] = 2q = \mathcal{W}_\phi[\alpha]$
  - c. verified experimentally
- Aperiodic tilings are fully characterized by a topological invariant – the Čech cohomology $\check{H}^1$
  - a. the right mathematical tool to answer physical questions
  - b. allowing to: characterize tilings and count tiles; enumerate Bragg peaks; label spectral gaps
- The Bloch theorem is generalized to FLC tilings by $\check{H}^1$
  - a. highlights the connection b/w structural & spectral features of tilings
  - b. furthermore, for C&P tilings, relates structural and spectral windings
  - c. for non-C&P tilings, $\tau_*^{\check{H}}(\check{H}^1)$ is unrelated to diffraction $S(k)$

Summary II

1. A new description of diffraction using Bratteli diagrams
   a. Can be calculated for tilings with Bragg diffraction spectrum
   b. Closely related to windings on Bratteli diagrams
2. Diffraction of **Thue-Morse** tiling is carefully analyzed
   a. Characterization of peaks by their growth rate
   b. Inconclusive experimental results
3. Innovative portrayal of fractals employing tilings and substitutions
   a. Novel Gap Labeling Conjecture for fractals is presented
4. Topological phase transitions are found
   a. By flux in fractals
   b. By random substitution rules

Prospect

Future

- Bloch Theorem
  - Extend to include windings
  - Explore in dimensions ≥ 4
- Windings
  - Identify the topological numbers for all $1D$ tilings
  - Explore in $2D$ and $3D$
- Calculate diffraction $S(k)$ using Bratteli diagrams
  - for all $1D$ tilings
  - extend to $2D$ and beyond
- Fractals
  - Prove the Gap Labeling Conjecture
  - Identify the proper $\check{H}^1$

Name	Substitution		Substitution on Doublets		Self Properties		Cohomology		Zeta Function	Gap Labeling Theorem		Properties	
	Rule σ_1	Occurrence M_1	Rule σ_2	Occurrence M_2	Eigenvalue	Char. Polynomial	$H^0(G)$	$H^1(G)$	$\zeta(z)$			Pisot char.	Periodicity
Fibonacci	$\begin{matrix}0\mapsto 01\\ 1\mapsto 0\end{matrix}$	$\begin{pmatrix}1&1\\1&0\end{pmatrix}$	$\begin{matrix}a\mapsto bc\\ b\mapsto bc\\ c\mapsto a\end{matrix}$	$\begin{pmatrix}0&1&1\\0&1&1\\1&0&0\end{pmatrix}$	τ	$\lambda^2-\lambda-1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-z-z^2}$	$p+q\cdot\tau$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Cantor Set	$\begin{matrix}0\mapsto 010\\ 1\mapsto 111\end{matrix}$	$\begin{pmatrix}2&1\\0&3\end{pmatrix}$	$\begin{matrix}a\mapsto aba\\ b\mapsto ccb\\ c\mapsto ccc\end{matrix}$	$\begin{pmatrix}2&1&0\\0&1&2\\0&0&3\end{pmatrix}$	3	$\lambda^2-5\lambda+6=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{(1-2z)(1-3z)}$	$\frac{k}{3^N}$	$k,N\in\mathbb{Z}$	not primitive	limit-quasiperiodic
Non-Pisot	$\begin{matrix}0\mapsto 0001\\ 1\mapsto 011\end{matrix}$	$\begin{pmatrix}3&1\\1&2\end{pmatrix}$	$\begin{matrix}a\mapsto aabc\\ b\mapsto aabc\\ c\mapsto bdc\\ d\mapsto bdc\end{matrix}$	$\begin{pmatrix}2&1&1&0\\2&1&1&0\\0&1&1&1\\0&1&1&1\end{pmatrix}$	$\tau+2$	$\lambda^2-5\lambda+5=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{1-5z+5z^2}$	$\frac{p+q\cdot\tau}{5^N}$	$p,q,N\in\mathbb{Z}$	non-Pisot	limit-quasiperiodic
Periodic	$\begin{matrix}0\mapsto 01\\ 1\mapsto 01\end{matrix}$	$\begin{pmatrix}1&1\\1&1\end{pmatrix}$	$\begin{matrix}a\mapsto ab\\ b\mapsto ab\end{matrix}$	$\begin{pmatrix}1&1\\1&1\end{pmatrix}$	2	$\lambda^2-2\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^1$	$\frac{1-z}{1-2z}$	$\frac{k}{2}$	$k\in\mathbb{Z}$	Pisot	periodic
Thue-Morse	$\begin{matrix}0\mapsto 01\\ 1\mapsto 10\end{matrix}$	$\begin{pmatrix}1&1\\1&1\end{pmatrix}$	$\begin{matrix}a\mapsto bc\\ b\mapsto bd\\ c\mapsto ca\\ d\mapsto cb\end{matrix}$	$\begin{pmatrix}0&1&1&0\\0&1&0&1\\1&0&1&0\\0&1&1&0\end{pmatrix}$	2	$\lambda^2-2\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{(1-2z)(1+z)}$	$\frac{k}{3\cdot 2^N}$	$k,N\in\mathbb{Z}$	Pisot	aperiodic
Sierpiński	$\begin{matrix}0\mapsto 01010\\ 1\mapsto 11\end{matrix}$	$\begin{pmatrix}3&2\\0&2\end{pmatrix}$	$\begin{matrix}a\mapsto ababa\\ b\mapsto cb\\ c\mapsto cc\end{matrix}$	$\begin{pmatrix}3&2&0\\0&1&1\\0&0&2\end{pmatrix}$	3	$\lambda^2-5\lambda+6=0$	$\mathbb{Z}^1$	$\mathbb{Z}^4$	$\frac{1-z}{(1-2z)(1-3z)}$	$\frac{k}{3^N}$	$k,N\in\mathbb{Z}$	not primitive	limit-quasiperiodic
Degen. Sierpiński	$\begin{matrix}0\mapsto 0001\\ 1\mapsto 1112\\ 2\mapsto 1112\end{matrix}$	$\begin{pmatrix}3&1&0\\0&3&1\\0&3&1\end{pmatrix}$	$\begin{matrix}a\mapsto aabc\\ b\mapsto aabd\\ c\mapsto ddef\\ d\mapsto ddeg\\ e\mapsto ddeg\\ f\mapsto ddef\\ g\mapsto ddeg\end{matrix}$	$\begin{pmatrix}2&1&1&0&0&0&0\\2&1&0&1&0&0&0\\0&0&0&2&1&1&0\\0&0&0&2&1&0&1\\0&0&0&2&1&0&1\\0&0&0&2&1&1&0\\0&0&0&2&1&0&1\end{pmatrix}$	4	$\lambda^3-7\lambda^2+12\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{(1-3z)(1-4z)}$	$\frac{k}{4^N}$	$k,N\in\mathbb{Z}$	not primitive	aperiodic
Period Doubling	$\begin{matrix}0\mapsto 01\\ 1\mapsto 00\end{matrix}$	$\begin{pmatrix}1&1\\2&0\end{pmatrix}$	$\begin{matrix}a\mapsto bc\\ b\mapsto bc\\ c\mapsto aa\end{matrix}$	$\begin{pmatrix}0&1&1\\0&1&1\\2&0&0\end{pmatrix}$	2	$\lambda^2-\lambda-2=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{(1-2z)(1+z)}$	$\frac{k}{3\cdot 2^N}$	$k,N\in\mathbb{Z}$	non-Pisot	limit-quasiperiodic
Circle Sequence	$\begin{matrix}0\mapsto 202\\ 1\mapsto 02202\\ 2\mapsto 01202\end{matrix}$	$\begin{pmatrix}1&0&2\\2&0&3\\2&1&2\end{pmatrix}$	$\begin{matrix}a\mapsto dbd\\ b\mapsto dbd\\ c\mapsto bedbd\\ d\mapsto acdbe\\ e\mapsto acdbd\end{matrix}$	$\begin{pmatrix}0&1&0&2&0\\0&1&0&2&0\\0&2&0&2&1\\1&1&1&1&1\\1&1&1&2&0\end{pmatrix}$	τ^3	$\lambda^3-3\lambda^2-5\lambda-1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{(1+z)(1-4z-z^2)}$	$\frac{1}{2}(p+q\cdot\tau)$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Rudin-Shapiro	$\begin{matrix}0\mapsto 02\\ 1\mapsto 32\\ 2\mapsto 01\\ 3\mapsto 31\end{matrix}$	$\begin{pmatrix}1&0&1&0\\0&0&1&1\\1&1&0&0\\0&1&0&1\end{pmatrix}$	$\begin{matrix}a\mapsto bf\\ b\mapsto be\\ c\mapsto he\\ d\mapsto hf\\ e\mapsto ac\\ f\mapsto ad\\ g\mapsto gd\\ h\mapsto gc\end{matrix}$	$\begin{pmatrix}0&1&0&0&0&1&0&0\\0&1&0&0&1&0&0&0\\0&0&0&0&1&0&0&1\\0&0&0&0&0&1&0&1\\1&0&1&0&0&0&0&0\\1&0&0&1&0&0&0&0\\0&0&0&1&0&0&1&0\\0&0&1&0&0&0&1&0\end{pmatrix}$	2	$\lambda^4-2\lambda^3-2\lambda^2+4\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^9$	$\frac{1-z}{(1-2z)(1-2z^2)(1+z)}$	$\frac{k}{2^N}$	$k,N\in\mathbb{Z}$	non-Pisot	aperiodic
Skau Example #1	$\begin{matrix}0\mapsto 001\\ 1\mapsto 0101\end{matrix}$	$\begin{pmatrix}2&1\\2&2\end{pmatrix}$	$\begin{matrix}a\mapsto abc\\ b\mapsto abc\\ c\mapsto bcbc\end{matrix}$	$\begin{pmatrix}1&1&1\\1&1&1\\0&2&2\end{pmatrix}$	$\sqrt{2}+2$	$\lambda^2-4\lambda+2=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{1-4z+2z^2}$	$\frac{p+q\sqrt{2}}{2^N}$	$p,q,N\in\mathbb{Z}$	Pisot	limit-quasiperiodic
Skau Example #2	$\begin{matrix}0\mapsto 010\\ 1\mapsto 01\end{matrix}$	$\begin{pmatrix}2&1\\1&1\end{pmatrix}$	$\begin{matrix}a\mapsto bca\\ b\mapsto bca\\ c\mapsto bc\end{matrix}$	$\begin{pmatrix}1&1&1\\1&1&1\\0&1&1\end{pmatrix}$	$\tau+1$	$\lambda^2-3\lambda+1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-3z+z^2}$	$p+q\cdot\tau$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Skau Example #3	$\begin{matrix}0\mapsto 001\\ 1\mapsto 10\end{matrix}$	$\begin{pmatrix}2&1\\1&1\end{pmatrix}$	$\begin{matrix}a\mapsto abc\\ b\mapsto abd\\ c\mapsto ca\\ d\mapsto cb\end{matrix}$	$\begin{pmatrix}1&1&1&0\\1&1&0&1\\1&0&1&0\\0&1&1&0\end{pmatrix}$	$\tau+1$	$\lambda^2-3\lambda+1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^5$	$\frac{1-z}{(1+z)(1-3z+z^2)}$	$\frac{p+q\cdot\tau}{5}$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Skau Example #4	$\begin{matrix}0\mapsto 010\\ 1\mapsto 1001\end{matrix}$	$\begin{pmatrix}2&1\\2&2\end{pmatrix}$	$\begin{matrix}a\mapsto bca\\ b\mapsto bcb\\ c\mapsto cabc\end{matrix}$	$\begin{pmatrix}1&1&1\\0&2&1\\1&1&2\end{pmatrix}$	$\sqrt{2}+2$	$\lambda^2-4\lambda+2=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-4z+2z^2}$	$\frac{p+q\sqrt{2}}{2^N}$	$p,q,N\in\mathbb{Z}$	Pisot	limit-quasiperiodic
Chacon	$\begin{matrix}0\mapsto 0010\\ 1\mapsto 1\end{matrix}$	$\begin{pmatrix}3&1\\0&1\end{pmatrix}$	$\begin{matrix}a\mapsto abca\\ b\mapsto abcb\\ c\mapsto c\end{matrix}$	$\begin{pmatrix}2&1&1\\1&2&1\\0&0&1\end{pmatrix}$	3	$\lambda^2-4\lambda+3=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1}{1-3z}$	$\frac{k}{3^N}$	$k,N\in\mathbb{Z}$	not primitive	limit-quasiperiodic
Golden Mean Squared	$\begin{matrix}0\mapsto 100\\ 1\mapsto 10\end{matrix}$	$\begin{pmatrix}2&1\\1&1\end{pmatrix}$	$\begin{matrix}a\mapsto cab\\ b\mapsto cab\\ c\mapsto cb\end{matrix}$	$\begin{pmatrix}1&1&1\\1&1&1\\0&1&1\end{pmatrix}$	$\tau+1$	$\lambda^2-3\lambda+1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-3z+z^2}$	$p+q\cdot\lambda_1$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Silver Mean Squared	$\begin{matrix}0\mapsto 1001000\\ 1\mapsto 100\end{matrix}$	$\begin{pmatrix}5&2\\2&1\end{pmatrix}$	$\begin{matrix}a\mapsto cabcaab\\ b\mapsto cabcaab\\ c\mapsto cab\end{matrix}$	$\begin{pmatrix}3&2&2\\3&2&2\\1&1&1\end{pmatrix}$	$2\sqrt{2}+3$	$\lambda^2-6\lambda+1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-6z+z^2}$	$p+q\cdot\lambda_1$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Copper Mean Squared	$\begin{matrix}0\mapsto 1000100010000\\ 1\mapsto 1000\end{matrix}$	$\begin{pmatrix}10&3\\3&1\end{pmatrix}$	$\begin{matrix}a\mapsto caabcaabcaaab\\ b\mapsto caabcaabcaaab\\ c\mapsto caab\end{matrix}$	$\begin{pmatrix}7&3&3\\7&3&3\\2&1&1\end{pmatrix}$	$\frac{3\sqrt{13}}{2}+\frac{11}{2}$	$\lambda^2-11\lambda+1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-11z+z^2}$	$p+q\cdot\lambda_1$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Luck Ternary #1	$\begin{matrix}0\mapsto 01\\ 1\mapsto 02\\ 2\mapsto 012\end{matrix}$	$\begin{pmatrix}1&1&0\\1&0&1\\1&1&1\end{pmatrix}$	$\begin{matrix}a\mapsto ac\\ b\mapsto ac\\ c\mapsto be\\ d\mapsto be\\ e\mapsto ade\end{matrix}$	$\begin{pmatrix}1&0&1&0&0\\1&0&1&0&0\\0&1&0&0&1\\0&1&0&0&1\\1&0&0&1&1\end{pmatrix}$	2.247	$\lambda^3-2\lambda^2-\lambda+1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{1-2z-z^2+z^3}$	$p+q\cdot\lambda_1+r\cdot\lambda_1^2$	$p,q,r\in\mathbb{Z}$	Pisot	quasiperiodic
Luck Ternary #2	$\begin{matrix}0\mapsto 2\\ 1\mapsto 0\\ 2\mapsto 12\end{matrix}$	$\begin{pmatrix}0&0&1\\1&0&0\\0&1&1\end{pmatrix}$	$\begin{matrix}a\mapsto c\\ b\mapsto a\\ c\mapsto be\\ d\mapsto bc\\ e\mapsto bd\end{matrix}$	$\begin{pmatrix}0&0&1&0&0\\1&0&0&0&0\\0&1&0&0&1\\0&1&1&0&0\\0&1&0&1&0\end{pmatrix}$	1.4656	$\lambda^3-\lambda^2-1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{1-z-z^3}$	$p+q\cdot\lambda_1+r\cdot\lambda_1^2$	$p,q,r\in\mathbb{Z}$	Pisot	quasiperiodic
Periodic 1-2	$\begin{matrix}0\mapsto 011\\ 1\mapsto 011\end{matrix}$	$\begin{pmatrix}1&2\\1&2\end{pmatrix}$	$\begin{matrix}a\mapsto acb\\ b\mapsto acb\\ c\mapsto acb\end{matrix}$	$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$	3	$\lambda^2-3\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^1$	$\frac{1-z}{1-3z}$	$\frac{k}{3}$	$k\in\mathbb{Z}$	Pisot	periodic
Periodic 1-3	$\begin{matrix}0\mapsto 0111\\ 1\mapsto 0111\end{matrix}$	$\begin{pmatrix}1&3\\1&3\end{pmatrix}$	$\begin{matrix}a\mapsto accb\\ b\mapsto accb\\ c\mapsto accb\end{matrix}$	$\begin{pmatrix}1&1&2\\1&1&2\\1&1&2\end{pmatrix}$	4	$\lambda^2-4\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^1$	$\frac{1-z}{1-4z}$	$\frac{k}{4}$	$k\in\mathbb{Z}$	Pisot	periodic
Periodic 1-4	$\begin{matrix}0\mapsto 01111\\ 1\mapsto 01111\end{matrix}$	$\begin{pmatrix}1&4\\1&4\end{pmatrix}$	$\begin{matrix}a\mapsto acccb\\ b\mapsto acccb\\ c\mapsto acccb\end{matrix}$	$\begin{pmatrix}1&1&3\\1&1&3\\1&1&3\end{pmatrix}$	5	$\lambda^2-5\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^1$	$\frac{1-z}{1-5z}$	$\frac{k}{5}$	$k\in\mathbb{Z}$	Pisot	periodic
Periodic 2-3	$\begin{matrix}0\mapsto 00111\\ 1\mapsto 00111\end{matrix}$	$\begin{pmatrix}2&3\\2&3\end{pmatrix}$	$\begin{matrix}a\mapsto abddc\\ b\mapsto abddc\\ c\mapsto abddc\\ d\mapsto abddc\end{matrix}$	$\begin{pmatrix}1&1&1&2\\1&1&1&2\\1&1&1&2\\1&1&1&2\end{pmatrix}$	5	$\lambda^2-5\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^1$	$\frac{1-z}{1-5z}$	$\frac{k}{5}$	$k\in\mathbb{Z}$	Pisot	periodic
Periodic 2-5	$\begin{matrix}0\mapsto 0011111\\ 1\mapsto 0011111\end{matrix}$	$\begin{pmatrix}2&5\\2&5\end{pmatrix}$	$\begin{matrix}a\mapsto abddddc\\ b\mapsto abddddc\\ c\mapsto abddddc\\ d\mapsto abddddc\end{matrix}$	$\begin{pmatrix}1&1&1&4\\1&1&1&4\\1&1&1&4\\1&1&1&4\end{pmatrix}$	7	$\lambda^2-7\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^1$	$\frac{1-z}{1-7z}$	$\frac{k}{7}$	$k\in\mathbb{Z}$	Pisot	periodic
Golden Mean	$\begin{matrix}0\mapsto 10\\ 1\mapsto 0\end{matrix}$	$\begin{pmatrix}1&1\\1&0\end{pmatrix}$	$\begin{matrix}a\mapsto cb\\ b\mapsto ca\\ c\mapsto b\end{matrix}$	$\begin{pmatrix}0&1&1\\1&0&1\\0&1&0\end{pmatrix}$	τ	$\lambda^2-\lambda-1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-z-z^2}$	$p+q\cdot\lambda_1$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Silver Mean	$\begin{matrix}0\mapsto 100\\ 1\mapsto 0\end{matrix}$	$\begin{pmatrix}2&1\\1&0\end{pmatrix}$	$\begin{matrix}a\mapsto cab\\ b\mapsto caa\\ c\mapsto b\end{matrix}$	$\begin{pmatrix}1&1&1\\2&0&1\\0&1&0\end{pmatrix}$	$\sqrt{2}+1$	$\lambda^2-2\lambda-1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-2z-z^2}$	$p+q\cdot\lambda_1$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Copper Mean	$\begin{matrix}0\mapsto 1000\\ 1\mapsto 0\end{matrix}$	$\begin{pmatrix}3&1\\1&0\end{pmatrix}$	$\begin{matrix}a\mapsto caab\\ b\mapsto caaa\\ c\mapsto b\end{matrix}$	$\begin{pmatrix}2&1&1\\3&0&1\\0&1&0\end{pmatrix}$	$\frac{\sqrt{13}}{2}+\frac{3}{2}$	$\lambda^2-3\lambda-1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-3z-z^2}$	$p+q\cdot\lambda_1$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Marginal	$\begin{matrix}0\mapsto 001\\ 1\mapsto 011\end{matrix}$	$\begin{pmatrix}2&1\\1&2\end{pmatrix}$	$\begin{matrix}a\mapsto abc\\ b\mapsto abc\\ c\mapsto bdc\\ d\mapsto bdc\end{matrix}$	$\begin{pmatrix}1&1&1&0\\1&1&1&0\\0&1&1&1\\0&1&1&1\end{pmatrix}$	3	$\lambda^2-4\lambda+3=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1}{1-3z}$	$\frac{k}{2\cdot 3^N}$	$k,N\in\mathbb{Z}$	non-Pisot	limit-quasiperiodic
Luck non-Pisot	$\begin{matrix}0\mapsto 0001\\ 1\mapsto 110\end{matrix}$	$\begin{pmatrix}3&1\\1&2\end{pmatrix}$	$\begin{matrix}a\mapsto aabc\\ b\mapsto aabd\\ c\mapsto dca\\ d\mapsto dcb\end{matrix}$	$\begin{pmatrix}2&1&1&0\\2&1&0&1\\1&0&1&1\\0&1&1&1\end{pmatrix}$	$\tau+2$	$\lambda^2-5\lambda+5=0$	$\mathbb{Z}^1$	$\mathbb{Z}^5$	$\frac{1-z}{(z+1)(1-5z+5z^2)}$	$\frac{p+q\cdot\tau}{11\cdot 5^N}$	$p,q,N\in\mathbb{Z}$	non-Pisot	limit-quasiperiodic
Binary non-Pisot	$\begin{matrix}0\mapsto 01\\ 1\mapsto 000\end{matrix}$	$\begin{pmatrix}1&1\\3&0\end{pmatrix}$	$\begin{matrix}a\mapsto bc\\ b\mapsto bc\\ c\mapsto aaa\end{matrix}$	$\begin{pmatrix}0&1&1\\0&1&1\\3&0&0\end{pmatrix}$	$\frac{\sqrt{13}}{2}+\frac{1}{2}$	$\lambda^2-\lambda-3=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{1-z-3z^2}$	$\frac{p+q\cdot\lambda_1}{3^N}$	$p,q,N\in\mathbb{Z}$	non-Pisot	limit-quasiperiodic
Ternary non-Pisot	$\begin{matrix}0\mapsto 2\\ 1\mapsto 0\\ 2\mapsto 101\end{matrix}$	$\begin{pmatrix}0&0&1\\1&0&0\\1&2&0\end{pmatrix}$	$\begin{matrix}a\mapsto e\\ b\mapsto f\\ c\mapsto b\\ d\mapsto a\\ e\mapsto cad\\ f\mapsto cac\end{matrix}$	$\begin{pmatrix}0&0&0&0&1&0\\0&0&0&0&0&1\\0&1&0&0&0&0\\1&0&0&0&0&0\\1&0&1&1&0&0\\1&0&2&0&0&0\end{pmatrix}$	1.5214	$\lambda^3-\lambda-2=0$	$\mathbb{Z}^1$	$\mathbb{Z}^6$	$\frac{1}{1-z^2-2z^3}$	$\frac{p+q\cdot\lambda_1+r\cdot\lambda_1^2}{2^N}$	$p,q,r,N\in\mathbb{Z}$	non-Pisot	limit-quasiperiodic

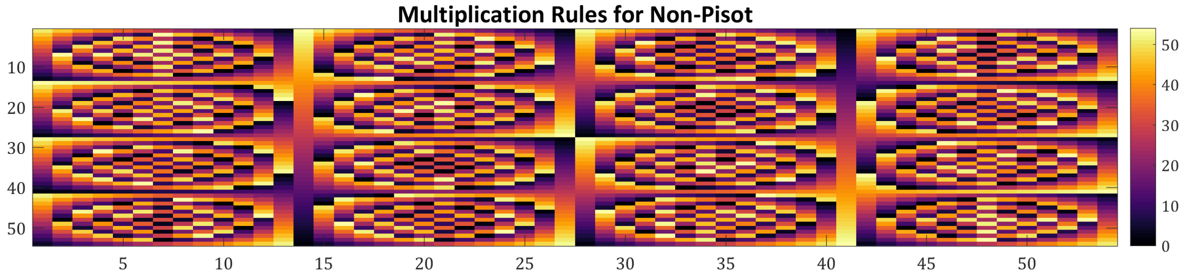


Fig. A.1: Multiplication matrix $\Lambda(q, r)$ for the Non-Pisot substitution.

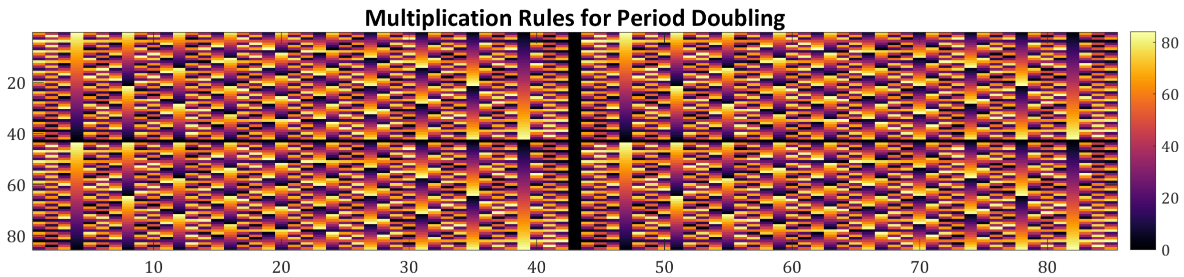


Fig. A.2: Multiplication matrix $\Lambda(q, r)$ for the Period Doubling substitution.

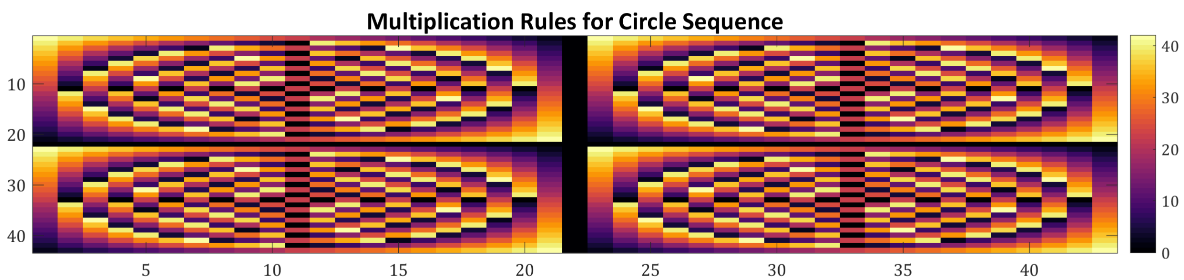


Fig. A.3: Multiplication matrix $\Lambda(q, r)$ for the Circle Sequence substitution.

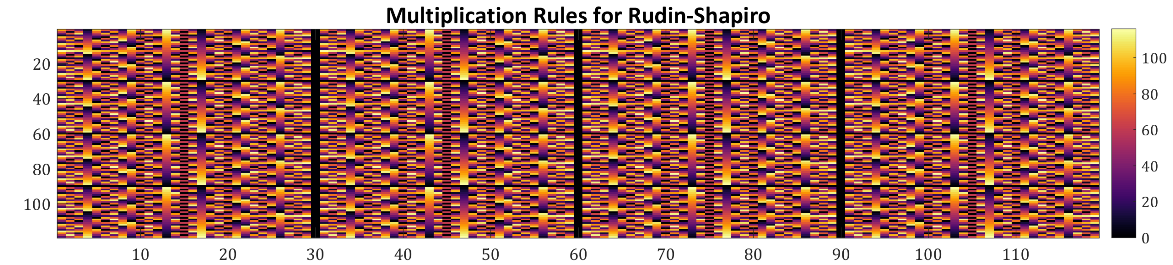


Fig. A.4: Multiplication matrix $\Lambda(q, r)$ for the Rudin-Shapiro substitution.

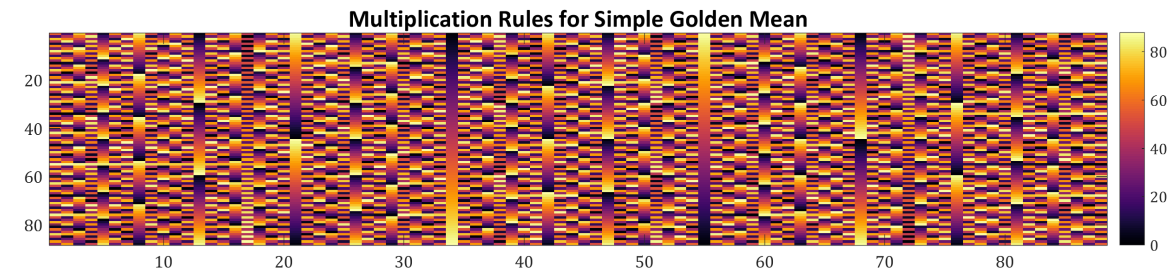


Fig. A.5: Multiplication matrix $\Lambda(q, r)$ for the Golden Mean substitution.

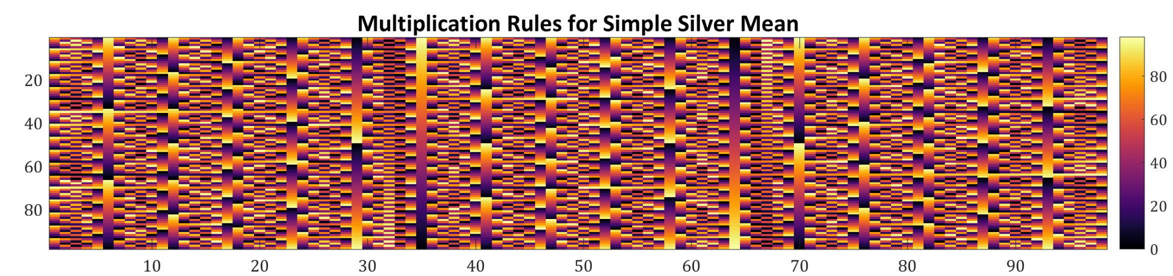


Fig. A.6: Multiplication matrix $\Lambda(q, r)$ for the Silver Mean substitution.

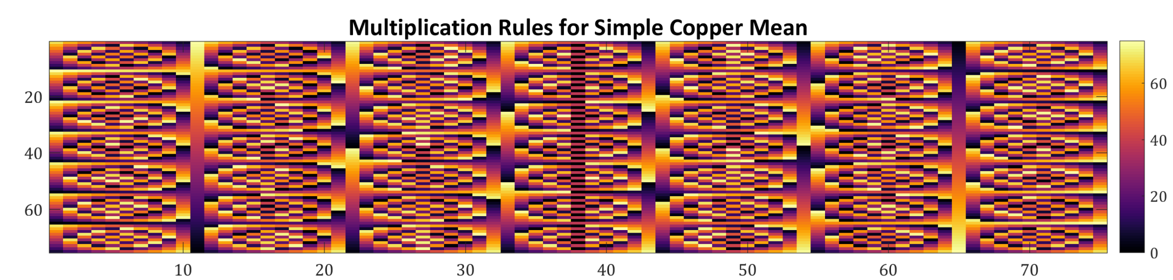


Fig. A.7: Multiplication matrix $\Lambda(q, r)$ for the Copper Mean substitution.

Thank you for your attention

Most of results and details are available at :

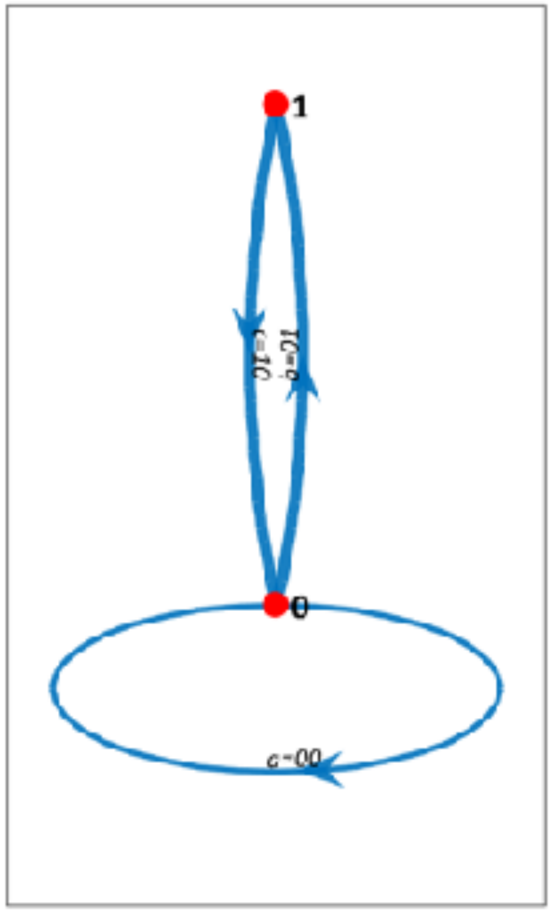
https://phsites.technion.ac.il/eric/

Summary - the untold part

- Given a topological meaning to the integers labelling the gaps of the fractal spectrum.
- Proposed a complete algebraic structure to account for the topological integers (Abelian group structure isomorphic to $\mathbb{Z}/F_N\mathbb{Z}$
- This Abelian group is isomorphic to the cohomology group $H^{(1)}$ defined on (Bratelli) graphs associated to the quasi periodic structures.

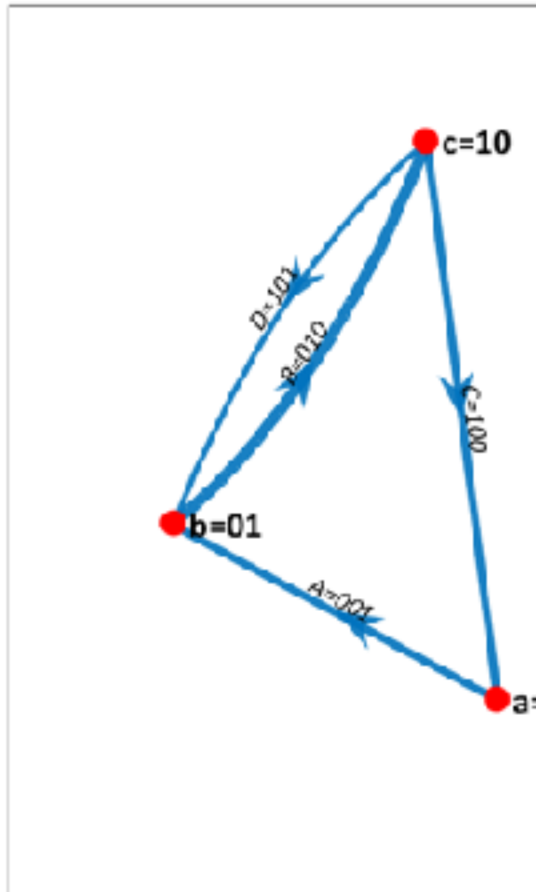
Name	Substitution		Substitution on Doublets		Self Properties		Cohomology		Zeta Function	Gap Labeling Theorem		Properties	
	Rule σ_1	Occurrence M_1	Rule σ_2	Occurrence M_2	Eigenvalue	Char. Polynomial	$H^0(G)$	$H^1(G)$	$\zeta(z)$			Pisot char.	Periodicity
Fibonacci	$0 \mapsto 01$ $1 \mapsto 0$	$\begin{pmatrix}1&1\\1&0\end{pmatrix}$	$a \mapsto bc$ $b \mapsto bc$ $c \mapsto a$	$\begin{pmatrix}0&1&1\\0&1&1\\1&0&0\end{pmatrix}$	τ	$\lambda^2-\lambda-1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-z-z^2}$	$p+q\cdot\tau$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Cantor Set	$0 \mapsto 010$ $1 \mapsto 111$	$\begin{pmatrix}2&1\\0&3\end{pmatrix}$	$a \mapsto aba$ $b \mapsto ccb$ $c \mapsto ccc$	$\begin{pmatrix}2&1&0\\0&1&2\\0&0&3\end{pmatrix}$	3	$\lambda^2-5\lambda+6=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{(1-2z)(1-3z)}$	$\frac{k}{3^N}$	$k,N\in\mathbb{Z}$	not primitive	limit-quasiperiodic
Non-Pisot	$0 \mapsto 0001$ $1 \mapsto 011$	$\begin{pmatrix}3&1\\1&2\end{pmatrix}$	$a \mapsto aabc$ $b \mapsto aabc$ $c \mapsto bdc$ $d \mapsto bdc$	$\begin{pmatrix}2&1&1&0\\2&1&1&0\\0&1&1&1\\0&1&1&1\end{pmatrix}$	$\tau+2$	$\lambda^2-5\lambda+5=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{1-5z+5z^2}$	$\frac{p+q\cdot\tau}{5^N}$	$p,q,N\in\mathbb{Z}$	non-Pisot	limit-quasiperiodic
Periodic	$0 \mapsto 01$ $1 \mapsto 01$	$\begin{pmatrix}1&1\\1&1\end{pmatrix}$	$a \mapsto ab$ $b \mapsto ab$	$\begin{pmatrix}1&1\\1&1\end{pmatrix}$	2	$\lambda^2-2\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^1$	$\frac{1-z}{1-2z}$	$\frac{k}{2}$	$k\in\mathbb{Z}$	Pisot	periodic
Thue-Morse	$0 \mapsto 01$ $1 \mapsto 10$	$\begin{pmatrix}1&1\\1&1\end{pmatrix}$	$a \mapsto bc$ $b \mapsto bd$ $c \mapsto ca$ $d \mapsto cb$	$\begin{pmatrix}0&1&1&0\\0&1&0&1\\1&0&1&0\\0&1&1&0\end{pmatrix}$	2	$\lambda^2-2\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{(1-2z)(1+z)}$	$\frac{k}{3\cdot 2^N}$	$k,N\in\mathbb{Z}$	Pisot	aperiodic
Sierpiński	$0 \mapsto 01010$ $1 \mapsto 11$	$\begin{pmatrix}3&2\\0&2\end{pmatrix}$	$a \mapsto ababa$ $b \mapsto cb$ $c \mapsto cc$	$\begin{pmatrix}3&2&0\\0&1&1\\0&0&2\end{pmatrix}$	3	$\lambda^2-5\lambda+6=0$	$\mathbb{Z}^1$	$\mathbb{Z}^4$	$\frac{1-z}{(1-2z)(1-3z)}$	$\frac{k}{3^N}$	$k,N\in\mathbb{Z}$	not primitive	limit-quasiperiodic
Degen. Sierpiński	$0 \mapsto 0001$ $1 \mapsto 1112$ $2 \mapsto 1112$	$\begin{pmatrix}3&1&0\\0&3&1\\0&3&1\end{pmatrix}$	$a \mapsto$ [illegible] $b \mapsto aabd$ $c \mapsto ddef$ $d \mapsto ddeg$ $e \mapsto ddeg$ $f \mapsto ddef$ $g \mapsto ddeg$	$\begin{pmatrix}2&1&1&0&0&0&0\\2&1&0&1&0&0&0\\0&0&0&2&1&1&0\\0&0&0&2&1&0&1\\0&0&0&2&1&0&1\\0&0&0&2&1&1&0\\0&0&0&2&1&0&1\end{pmatrix}$	4	$\lambda^3-7\lambda^2+12\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{(1-3z)(1-4z)}$	$\frac{k}{4^N}$	$k,N\in\mathbb{Z}$	not primitive	aperiodic
Period Doubling	$0 \mapsto 01$ $1 \mapsto 00$	$\begin{pmatrix}1&1\\2&0\end{pmatrix}$	$a \mapsto bc$ $b \mapsto bc$ $c \mapsto aa$	$\begin{pmatrix}0&1&1\\0&1&1\\2&0&0\end{pmatrix}$	2	$\lambda^2-\lambda-2=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{(1-2z)(1+z)}$	$\frac{k}{3\cdot 2^N}$	$k,N\in\mathbb{Z}$	non-Pisot	limit-quasiperiodic
Circle Sequence	$0 \mapsto 202$ $1 \mapsto 02202$ $2 \mapsto 01202$	$\begin{pmatrix}1&0&2\\2&0&3\\2&1&2\end{pmatrix}$	$a \mapsto dbd$ $b \mapsto dbd$ $c \mapsto bedbd$ $d \mapsto acdbe$ $e \mapsto acdbd$	$\begin{pmatrix}0&1&0&2&0\\0&1&0&2&0\\0&2&0&2&1\\1&1&1&1&1\\1&1&1&2&0\end{pmatrix}$	τ^3	$\lambda^3-3\lambda^2-5\lambda-1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{(1+z)(1-4z-z^2)}$	$\frac{1}{2}(p+q\cdot\tau)$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Rudin-Shapiro	$0 \mapsto 02$ $1 \mapsto 32$ $2 \mapsto 01$ $3 \mapsto 31$	$\begin{pmatrix}1&0&1&0\\0&0&1&1\\1&1&0&0\\0&1&0&1\end{pmatrix}$	$a \mapsto bf$ $b \mapsto be$ $c \mapsto he$ $d \mapsto hf$ $e \mapsto ac$ $f \mapsto ad$ $g \mapsto gd$ $h \mapsto gc$	$\begin{pmatrix}0&1&0&0&0&1&0&0\\0&1&0&0&1&0&0&0\\0&0&0&0&1&0&0&1\\0&0&0&0&0&1&0&1\\1&0&1&0&0&0&0&0\\1&0&0&1&0&0&0&0\\0&0&0&1&0&0&1&0\\0&0&1&0&0&0&1&0\end{pmatrix}$	2	$\lambda^4-2\lambda^3-2\lambda^2+4\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^9$	$\frac{1-z}{(1-2z)(1-2z^2)(1+z)}$	$\frac{k}{2^N}$	$k,N\in\mathbb{Z}$	non-Pisot	aperiodic
Skau Example #1	$0 \mapsto 001$ $1 \mapsto 0101$	$\begin{pmatrix}2&1\\2&2\end{pmatrix}$	$a \mapsto abc$ $b \mapsto abc$ $c \mapsto bcbc$	$\begin{pmatrix}1&1&1\\1&1&1\\0&2&2\end{pmatrix}$	$\sqrt{2}+2$	$\lambda^2-4\lambda+2=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{1-4z+2z^2}$	$\frac{p+q\sqrt{2}}{2^N}$	$p,q,N\in\mathbb{Z}$	Pisot	limit-quasiperiodic
Skau Example #2	$0 \mapsto 010$ $1 \mapsto 01$	$\begin{pmatrix}2&1\\1&1\end{pmatrix}$	$a \mapsto bca$ $b \mapsto bca$ $c \mapsto bc$	$\begin{pmatrix}1&1&1\\1&1&1\\0&1&1\end{pmatrix}$	$\tau+1$	$\lambda^2-3\lambda+1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-3z+z^2}$	$p+q\cdot\tau$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Skau Example #3	$0 \mapsto 001$ $1 \mapsto 10$	$\begin{pmatrix}2&1\\1&1\end{pmatrix}$	$a \mapsto abc$ $b \mapsto abd$ $c \mapsto ca$ $d \mapsto cb$	$\begin{pmatrix}1&1&1&0\\1&1&0&1\\1&0&1&0\\0&1&1&0\end{pmatrix}$	$\tau+1$	$\lambda^2-3\lambda+1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^5$	$\frac{1-z}{(1+z)(1-3z+z^2)}$	$\frac{p+q\cdot\tau}{5}$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Skau Example #4	$0 \mapsto 010$ $1 \mapsto 1001$	$\begin{pmatrix}2&1\\2&2\end{pmatrix}$	$a \mapsto bca$ $b \mapsto bcb$ $c \mapsto cabc$	$\begin{pmatrix}1&1&1\\0&2&1\\1&1&2\end{pmatrix}$	$\sqrt{2}+2$	$\lambda^2-4\lambda+2=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-4z+2z^2}$	$\frac{p+q\sqrt{2}}{2^N}$	$p,q,N\in\mathbb{Z}$	Pisot	limit-quasiperiodic
Chacon	$0 \mapsto 0010$ $1 \mapsto 1$	$\begin{pmatrix}3&1\\0&1\end{pmatrix}$	$a \mapsto abca$ $b \mapsto abcb$ $c \mapsto c$	$\begin{pmatrix}2&1&1\\1&2&1\\0&0&1\end{pmatrix}$	3	$\lambda^2-4\lambda+3=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1}{1-3z}$	$\frac{k}{3^N}$	$k,N\in\mathbb{Z}$	not primitive	limit-quasiperiodic
Golden Mean Squared	$0 \mapsto 100$ $1 \mapsto 10$	$\begin{pmatrix}2&1\\1&1\end{pmatrix}$	$a \mapsto cab$ $b \mapsto cab$ $c \mapsto cb$	$\begin{pmatrix}1&1&1\\1&1&1\\0&1&1\end{pmatrix}$	$\tau+1$	$\lambda^2-3\lambda+1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-3z+z^2}$	$p+q\cdot\lambda_1$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Silver Mean Squared	$0 \mapsto 1001000$ $1 \mapsto 100$	$\begin{pmatrix}5&2\\2&1\end{pmatrix}$	$a \mapsto cabcaab$ $b \mapsto cabcaab$ $c \mapsto cab$	$\begin{pmatrix}3&2&2\\3&2&2\\1&1&1\end{pmatrix}$	$2\sqrt{2}+3$	$\lambda^2-6\lambda+1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-6z+z^2}$	$p+q\cdot\lambda_1$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Copper Mean Squared	$0 \mapsto 1000100010000$ $1 \mapsto 1000$	$\begin{pmatrix}10&3\\3&1\end{pmatrix}$	$a \mapsto caabcaabcaaab$ $b \mapsto caabcaabcaaab$ $c \mapsto caab$	$\begin{pmatrix}7&3&3\\7&3&3\\2&1&1\end{pmatrix}$	$\frac{3\sqrt{13}}{2}+\frac{11}{2}$	$\lambda^2-11\lambda+1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-11z+z^2}$	$p+q\cdot\lambda_1$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Luck Ternary #1	$0 \mapsto 01$ $1 \mapsto 02$ $2 \mapsto 012$	$\begin{pmatrix}1&1&0\\1&0&1\\1&1&1\end{pmatrix}$	$a \mapsto ac$ $b \mapsto ac$ $c \mapsto be$ $d \mapsto be$ $e \mapsto ade$	$\begin{pmatrix}1&0&1&0&0\\1&0&1&0&0\\0&1&0&0&1\\0&1&0&0&1\\1&0&0&1&1\end{pmatrix}$	2.247	$\lambda^3-2\lambda^2-\lambda+1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{1-2z-z^2+z^3}$	$p+q\cdot\lambda_1+r\cdot\lambda_1^2$	$p,q,r\in\mathbb{Z}$	Pisot	quasiperiodic
Luck Ternary #2	$0 \mapsto 2$ $1 \mapsto 0$ $2 \mapsto 12$	$\begin{pmatrix}0&0&1\\1&0&0\\0&1&1\end{pmatrix}$	$a \mapsto c$ $b \mapsto a$ $c \mapsto be$ $d \mapsto bc$ $e \mapsto bd$	$\begin{pmatrix}0&0&1&0&0\\1&0&0&0&0\\0&1&0&0&1\\0&1&1&0&0\\0&1&0&1&0\end{pmatrix}$	1.4656	$\lambda^3-\lambda^2-1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{1-z-z^3}$	$p+q\cdot\lambda_1+r\cdot\lambda_1^2$	$p,q,r\in\mathbb{Z}$	Pisot	quasiperiodic
Periodic 1-2	$0 \mapsto 011$ $1 \mapsto 011$	$\begin{pmatrix}1&2\\1&2\end{pmatrix}$	$a \mapsto acb$ $b \mapsto acb$ $c \mapsto acb$	$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$	3	$\lambda^2-3\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^1$	$\frac{1-z}{1-3z}$	$\frac{k}{3}$	$k\in\mathbb{Z}$	Pisot	periodic
Periodic 1-3	$0 \mapsto 0111$ $1 \mapsto 0111$	$\begin{pmatrix}1&3\\1&3\end{pmatrix}$	$a \mapsto accb$ $b \mapsto accb$ $c \mapsto accb$	$\begin{pmatrix}1&1&2\\1&1&2\\1&1&2\end{pmatrix}$	4	$\lambda^2-4\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^1$	$\frac{1-z}{1-4z}$	$\frac{k}{4}$	$k\in\mathbb{Z}$	Pisot	periodic
Periodic 1-4	$0 \mapsto 01111$ $1 \mapsto 01111$	$\begin{pmatrix}1&4\\1&4\end{pmatrix}$	$a \mapsto acccb$ $b \mapsto acccb$ $c \mapsto acccb$	$\begin{pmatrix}1&1&3\\1&1&3\\1&1&3\end{pmatrix}$	5	$\lambda^2-5\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^1$	$\frac{1-z}{1-5z}$	$\frac{k}{5}$	$k\in\mathbb{Z}$	Pisot	periodic
Periodic 2-3	$0 \mapsto 00111$ $1 \mapsto 00111$	$\begin{pmatrix}2&3\\2&3\end{pmatrix}$	$a \mapsto abddc$ $b \mapsto abddc$ $c \mapsto abddc$ $d \mapsto abddc$	$\begin{pmatrix}1&1&1&2\\1&1&1&2\\1&1&1&2\\1&1&1&2\end{pmatrix}$	5	$\lambda^2-5\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^1$	$\frac{1-z}{1-5z}$	$\frac{k}{5}$	$k\in\mathbb{Z}$	Pisot	periodic
Periodic 2-5	$0 \mapsto 0011111$ $1 \mapsto 0011111$	$\begin{pmatrix}2&5\\2&5\end{pmatrix}$	$a \mapsto abddddc$ $b \mapsto abddddc$ $c \mapsto abddddc$ $d \mapsto abddddc$	$\begin{pmatrix}1&1&1&4\\1&1&1&4\\1&1&1&4\\1&1&1&4\end{pmatrix}$	7	$\lambda^2-7\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^1$	$\frac{1-z}{1-7z}$	$\frac{k}{7}$	$k\in\mathbb{Z}$	Pisot	periodic
Golden Mean	$0 \mapsto 10$ $1 \mapsto 0$	$\begin{pmatrix}1&1\\1&0\end{pmatrix}$	$a \mapsto cb$ $b \mapsto ca$ $c \mapsto b$	$\begin{pmatrix}0&1&1\\1&0&1\\0&1&0\end{pmatrix}$	τ	$\lambda^2-\lambda-1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-z-z^2}$	$p+q\cdot\lambda_1$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Silver Mean	$0 \mapsto 100$ $1 \mapsto 0$	$\begin{pmatrix}2&1\\1&0\end{pmatrix}$	$a \mapsto cab$ $b \mapsto caa$ $c \mapsto b$	$\begin{pmatrix}1&1&1\\2&0&1\\0&1&0\end{pmatrix}$	$\sqrt{2}+1$	$\lambda^2-2\lambda-1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-2z-z^2}$	$p+q\cdot\lambda_1$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Copper Mean	$0 \mapsto 1000$ $1 \mapsto 0$	$\begin{pmatrix}3&1\\1&0\end{pmatrix}$	$a \mapsto caab$ $b \mapsto caaa$ $c \mapsto b$	$\begin{pmatrix}2&1&1\\3&0&1\\0&1&0\end{pmatrix}$	$\frac{\sqrt{13}}{2}+\frac{3}{2}$	$\lambda^2-3\lambda-1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-3z-z^2}$	$p+q\cdot\lambda_1$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Marginal	$0 \mapsto 001$ $1 \mapsto 011$	$\begin{pmatrix}2&1\\1&2\end{pmatrix}$	$a \mapsto abc$ $b \mapsto abc$ $c \mapsto bdc$ $d \mapsto bdc$	$\begin{pmatrix}1&1&1&0\\1&1&1&0\\0&1&1&1\\0&1&1&1\end{pmatrix}$	3	$\lambda^2-4\lambda+3=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1}{1-3z}$	$\frac{k}{2\cdot 3^N}$	$k,N\in\mathbb{Z}$	non-Pisot	limit-quasiperiodic
Luck non-Pisot	$0 \mapsto 0001$ $1 \mapsto 110$	$\begin{pmatrix}3&1\\1&2\end{pmatrix}$	$a \mapsto aabc$ $b \mapsto aabd$ $c \mapsto dca$ $d \mapsto dcb$	$\begin{pmatrix}2&1&1&0\\2&1&0&1\\1&0&1&1\\0&1&1&1\end{pmatrix}$	$\tau+2$	$\lambda^2-5\lambda+5=0$	$\mathbb{Z}^1$	$\mathbb{Z}^5$	$\frac{1-z}{(z+1)(1-5z+5z^2)}$	$\frac{p+q\cdot\tau}{11\cdot 5^N}$	$p,q,N\in\mathbb{Z}$	non-Pisot	limit-quasiperiodic
Binary non-Pisot	$0 \mapsto 01$ $1 \mapsto 000$	$\begin{pmatrix}1&1\\3&0\end{pmatrix}$	$a \mapsto bc$ $b \mapsto bc$ $c \mapsto aaa$	$\begin{pmatrix}0&1&1\\0&1&1\\3&0&0\end{pmatrix}$	$\frac{\sqrt{13}}{2}+\frac{1}{2}$	$\lambda^2-\lambda-3=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{1-z-3z^2}$	$\frac{p+q\cdot\lambda_1}{3^N}$	$p,q,N\in\mathbb{Z}$	non-Pisot	limit-quasiperiodic
Ternary non-Pisot	$0 \mapsto 2$ $1 \mapsto 0$ $2 \mapsto 101$	$\begin{pmatrix}0&0&1\\1&0&0\\1&2&0\end{pmatrix}$	$a \mapsto e$ $b \mapsto f$ $c \mapsto b$ $d \mapsto a$ $e \mapsto cad$ $f \mapsto cac$	$\begin{pmatrix}0&0&0&0&1&0\\0&0&0&0&0&1\\0&1&0&0&0&0\\1&0&0&0&0&0\\1&0&1&1&0&0\\1&0&2&0&0&0\end{pmatrix}$	1.5214	$\lambda^3-\lambda-2=0$	$\mathbb{Z}^1$	$\mathbb{Z}^6$	$\frac{1}{1-z^2-2z^3}$	$\frac{p+q\cdot\lambda_1+r\cdot\lambda_1^2}{2^N}$	$p,q,r,N\in\mathbb{Z}$	non-Pisot	limit-quasiperiodic

Fibonacci for Letters



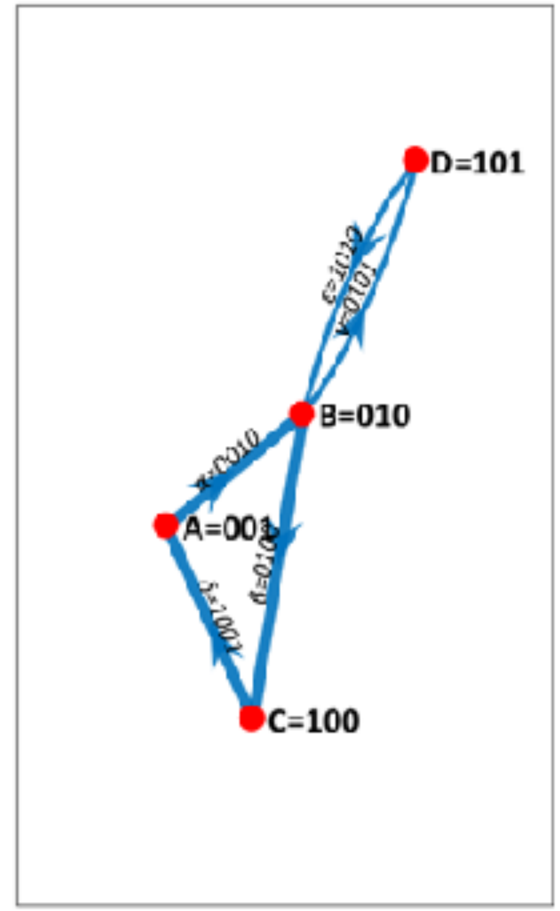
$H^0 \cong \mathbb{Z}^1, \; H^1 \cong \mathbb{Z}^2$

Fibonacci for Nodes



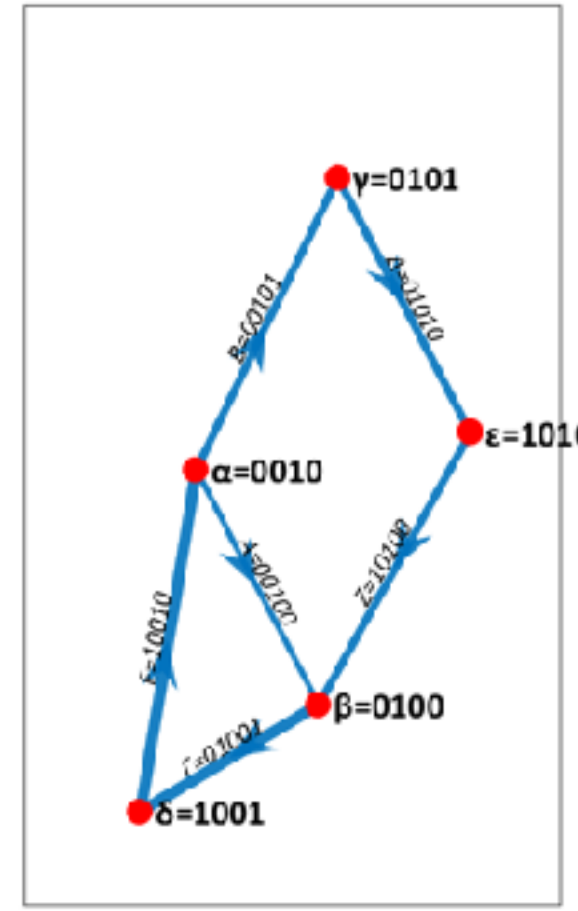
$H^0 \cong \mathbb{Z}^1, \; H^1 \cong \mathbb{Z}^2$

Fibonacci for Edges



$H^0 \cong \mathbb{Z}^1, \; H^1 \cong \mathbb{Z}^2$

Fibonacci for Frames



$H^0 \cong \mathbb{Z}^1, \; H^1 \cong \mathbb{Z}^2$

$M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{matrix} \sigma_1(0) = 01 \\ \sigma_1(1) = 0 \end{matrix}$

$\lambda_1^{(1)} = \tau, \quad v_1^{(1)} = \begin{pmatrix} \tau - 1 & 2 - \tau \end{pmatrix}$

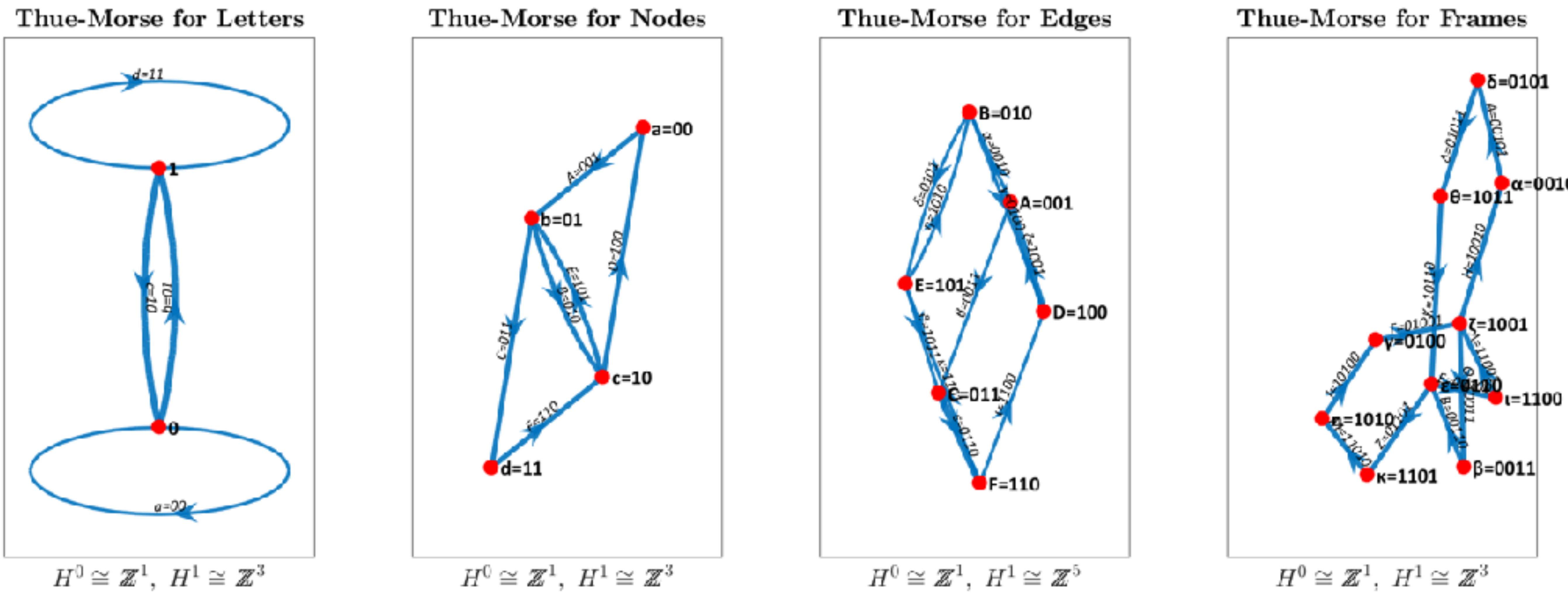
$M_2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{matrix} \sigma_2(a) = bc \\ \sigma_2(b) = bc \\ \sigma_2(c) = a \end{matrix}$

$\lambda_1^{(2)} = \tau, \quad v_1^{(2)} = \begin{pmatrix} 2\tau - 3 & 2 - \tau & 2 - \tau \end{pmatrix}$

Gaps: $p + q \cdot \tau \cap [0, 1)$ for $p, q \in \mathbb{Z}$

Pisot ($|\lambda_2^{(1)}| = \tau - 1$), quasiperiodic

Inflation: $\zeta(z) = \dfrac{1 - z}{-z^2 - z + 1}$



$$M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \sigma_1(0) = 01, \quad \sigma_1(1) = 10$$

$$\lambda_1^{(1)} = 2, \quad v_1^{(1)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

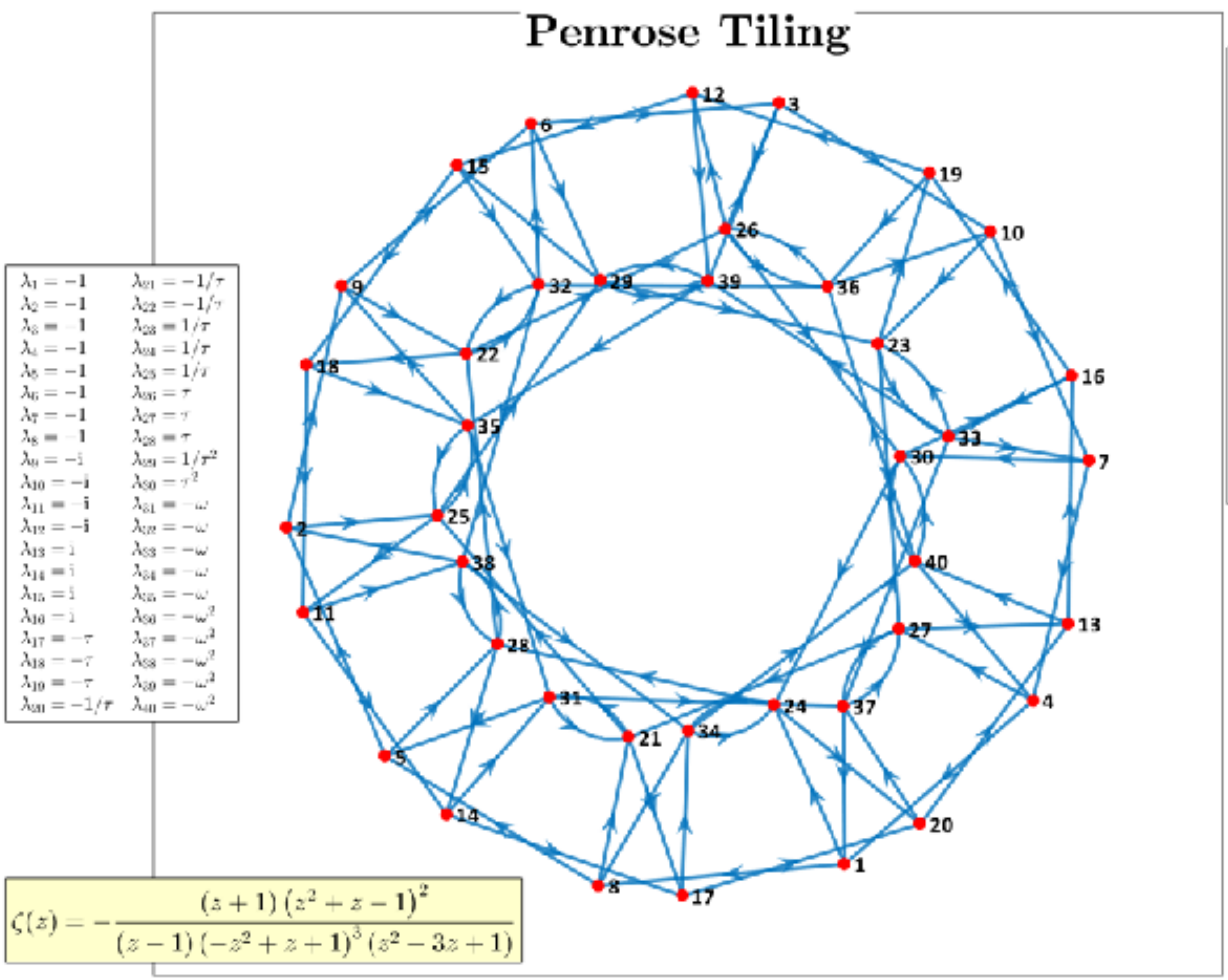
$$M_2 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \qquad \sigma_2(a) = bc, \quad \sigma_2(b) = bd, \quad \sigma_2(c) = ca, \quad \sigma_2(d) = cb$$

$$\lambda_1^{(2)} = 2, \quad v_1^{(2)} = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{pmatrix}$$

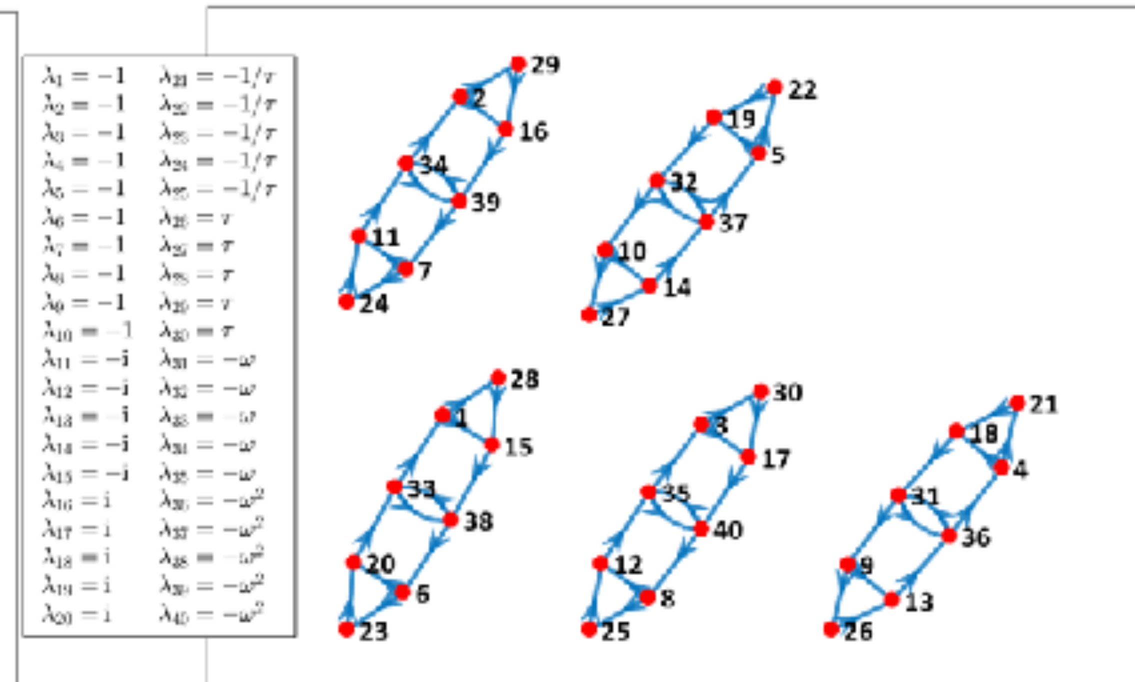
Caps: $\frac{k}{3 \cdot 2^N} \cap [0, 1)$ for $k, N \in \mathbb{Z}$

Pisot ($|\lambda_2^{(1)}| = 0$), aperiodic

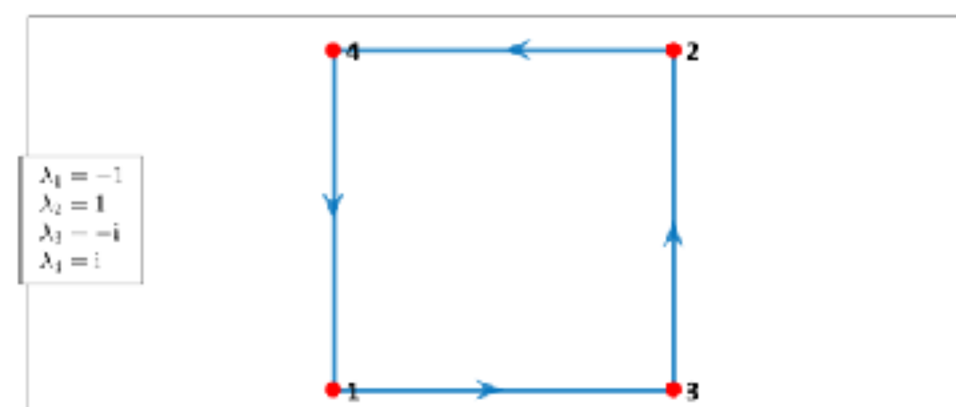
Inflation: $\zeta(z) = \frac{1 - z}{-(2z - 1)(z + 1)}$



Inflation for Faces, $H^{(2)} = \mathbb{Z}^8$



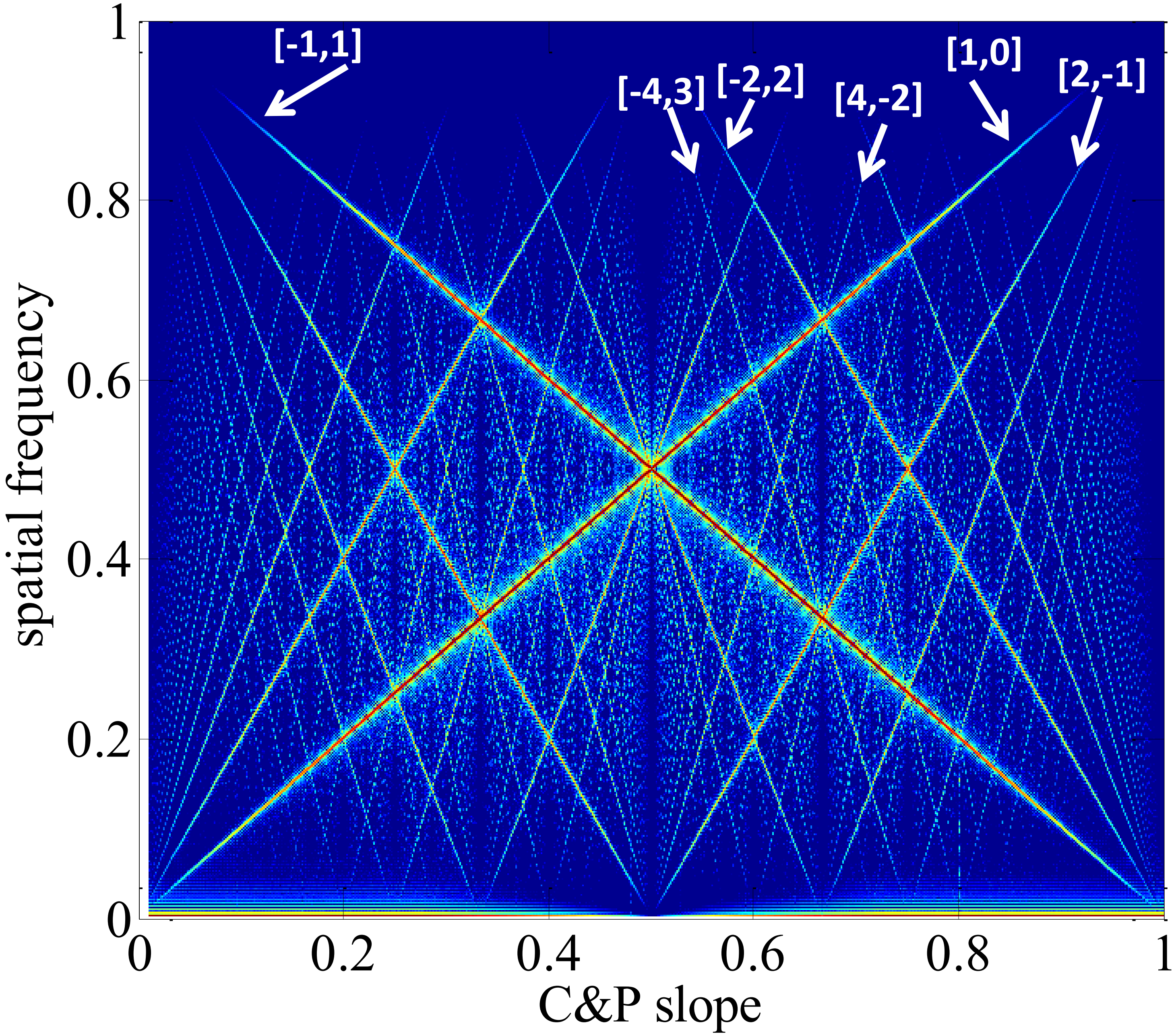
Inflation for Edges, $H^{(1)} = \mathbb{Z}^5$



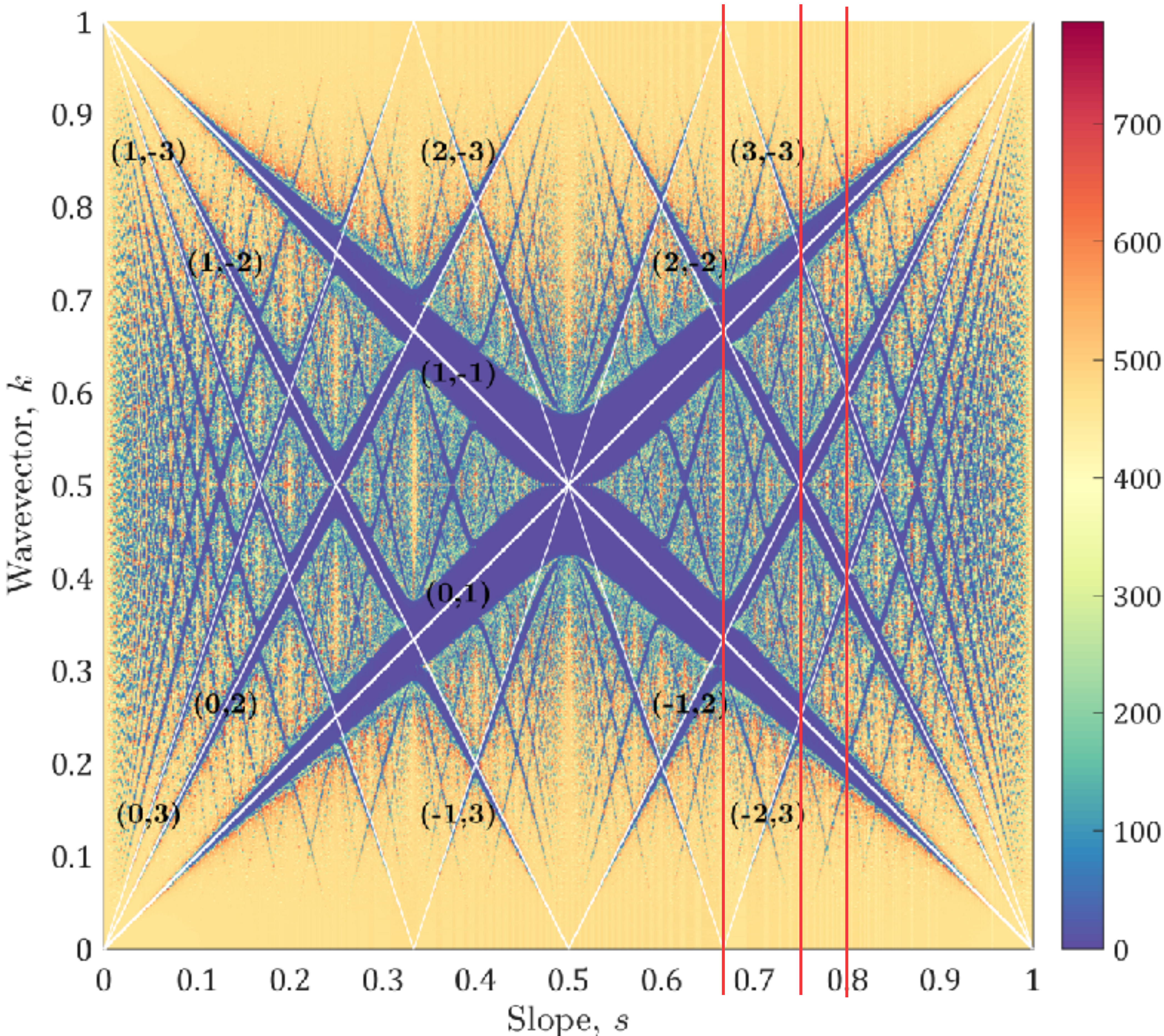
Inflation for Vertices, $H^{(0)} = \mathbb{Z}^1$

Summary - the untold part

- Given a topological meaning to the integers labelling the gaps of the fractal spectrum.
- Proposed a complete algebraic structure to account for the topological integers (Abelian group structure isomorphic to $\mathbb{Z}/F_N\mathbb{Z}$
- Generalisation to other substitutions
- Generalized Cut&Project

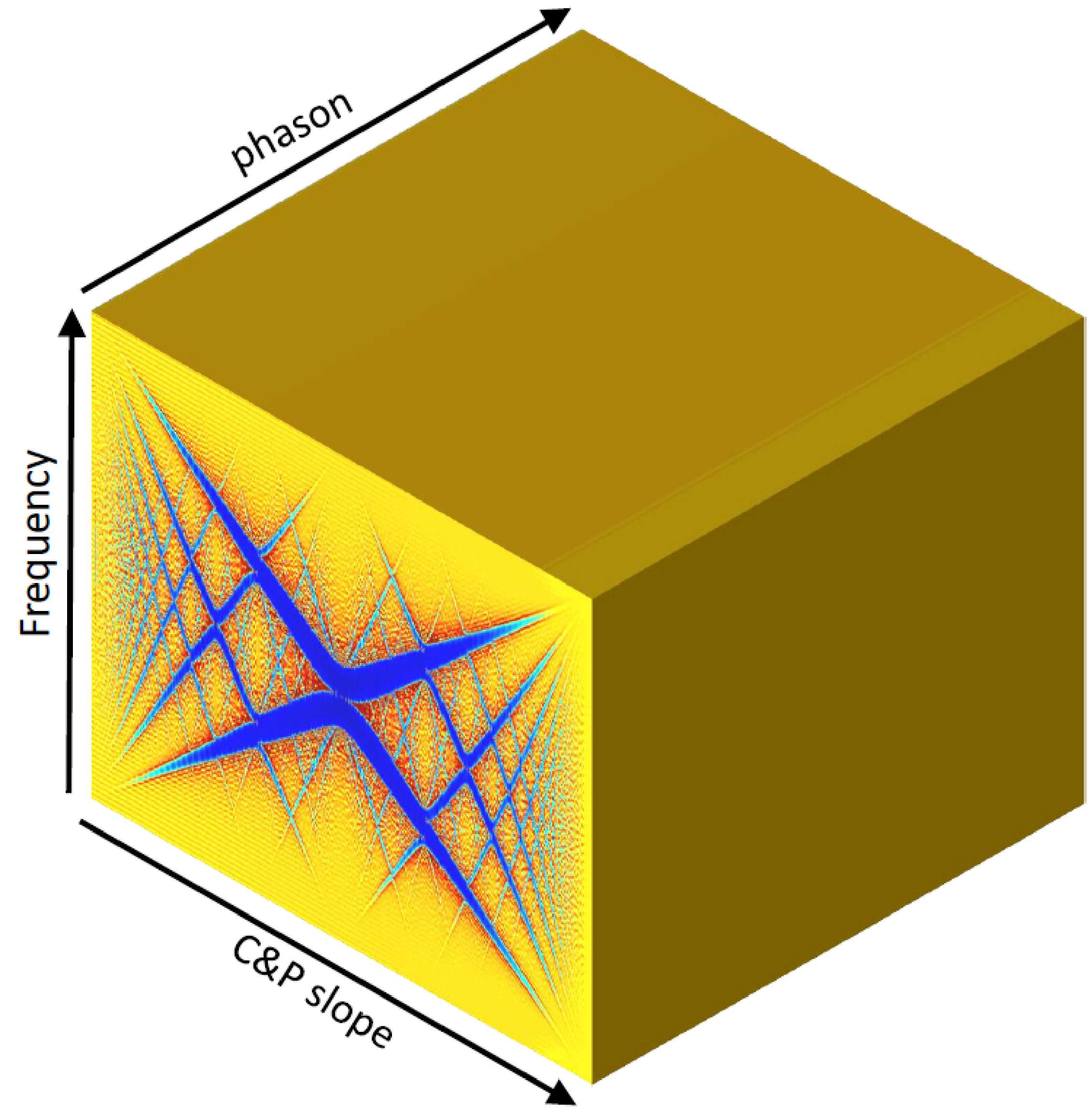


[-1,1]
[-4,3]
[-2,2]
[4,-2]
[1,0]
[2,-1]
spatial frequency
C&P slope
0
0.2
0.4
0.6
0.8
1

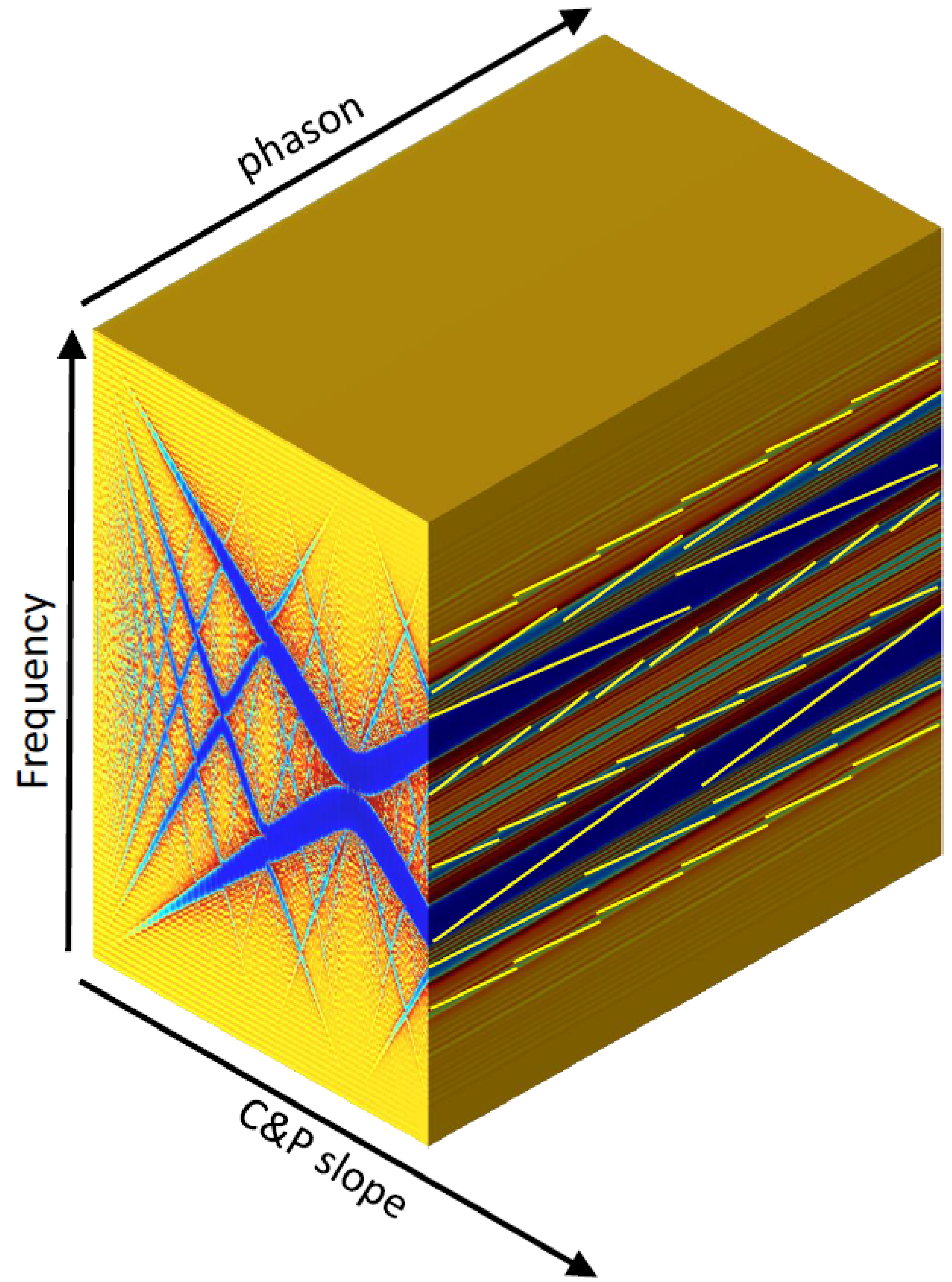


Wavevector, k
Slope, s
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
(1,-3)
(2,-3)
(3,-3)
(1,-2)
(2,-2)
(1,-1)
(0,1)
(0,2)
(-1,2)
(0,3)
(-1,3)
(-2,3)
100
200
300
400
500
600
700

- The λ_1 is fixed per substitution
- How to change λ_1 continuously? By Cut and Project



- The λ_1 is fixed per substitution
- How to change λ_1 continuously? By Cut and Project



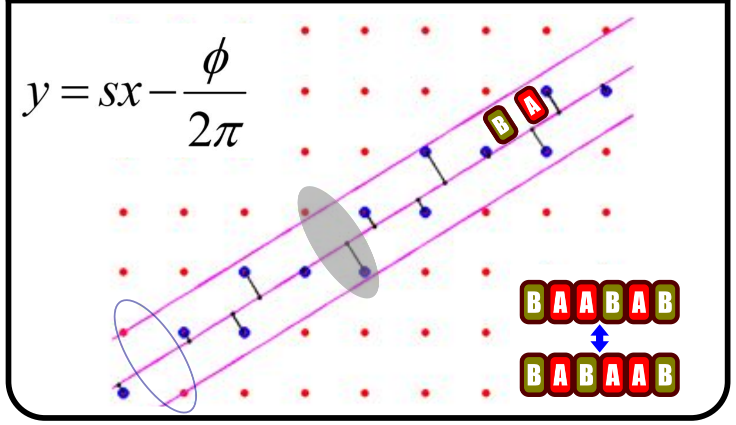
Wannier Butterfly

E. Levy, F. Mortessagne, U. Kuhl, E.A, 2016

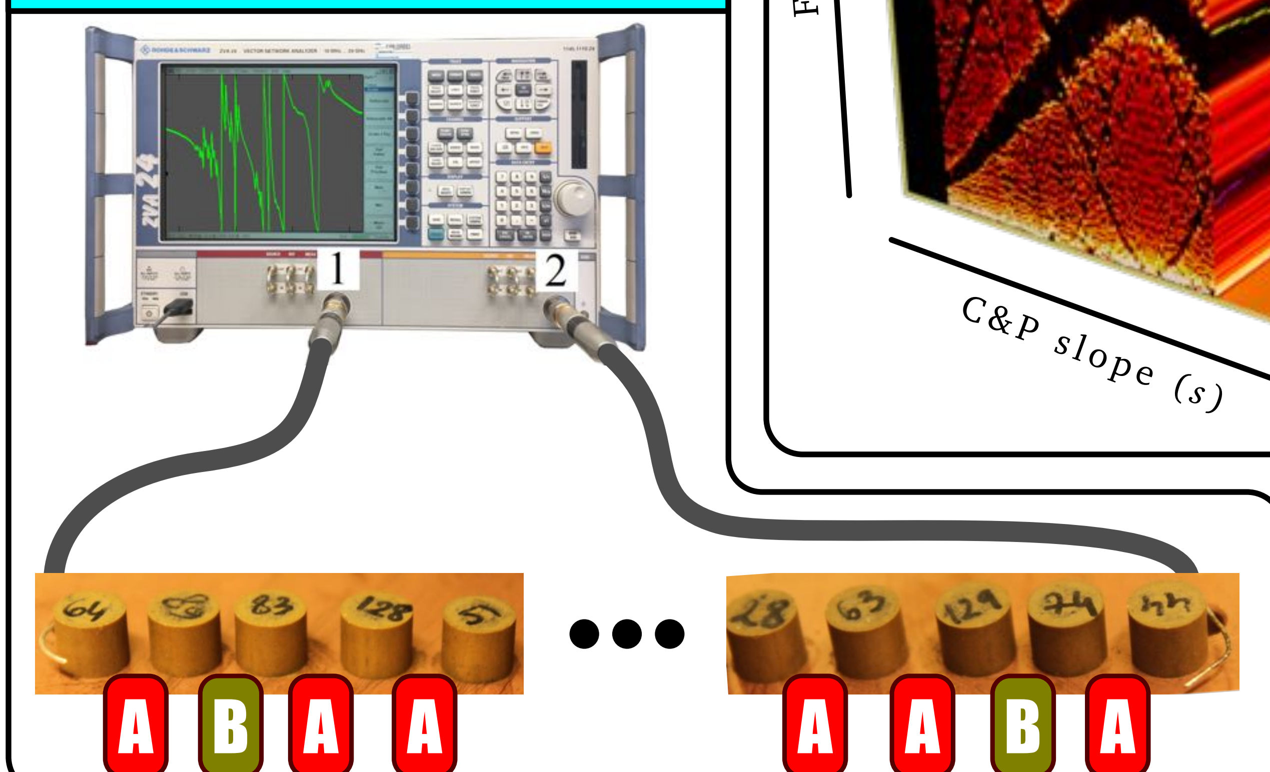
Wannier Butterfly

Cut & Paste

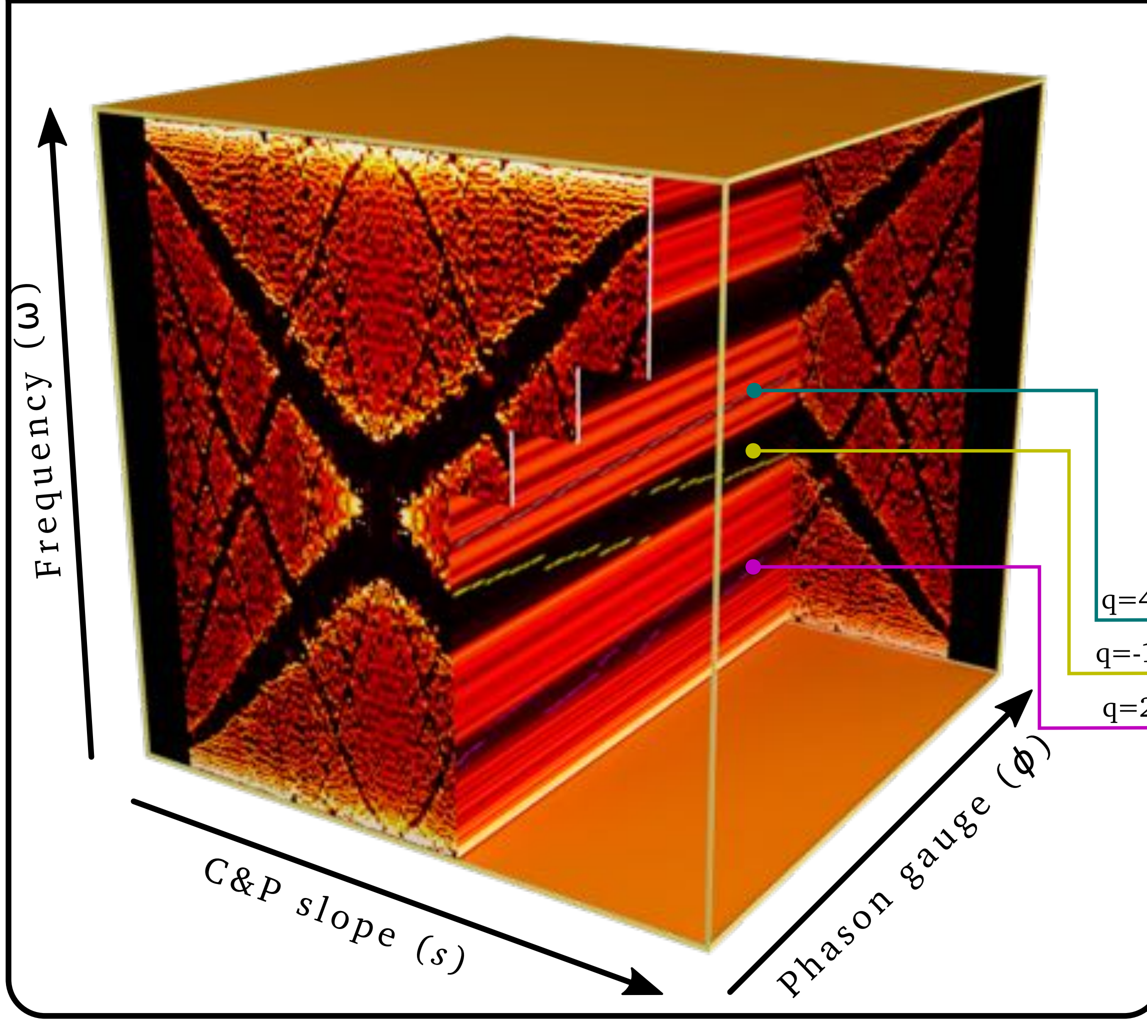
$$y = sx - \frac{\phi}{2\pi}$$



Scattering phases: δ(ω), α(ω)



Phase space

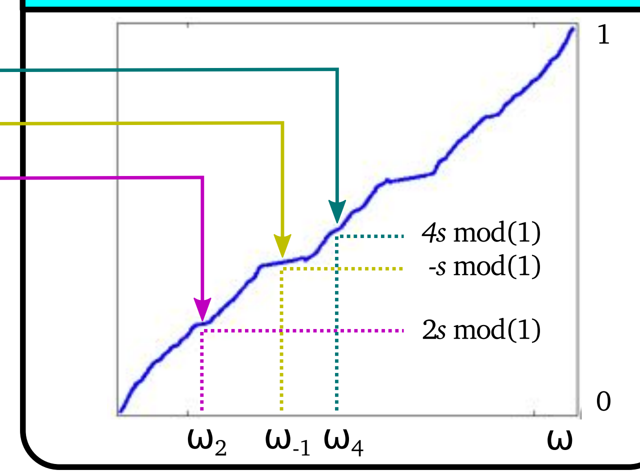


Gap labeling theorem

$$IDOS(\omega_q) = \frac{1}{\pi}\delta(\omega_q) = \\ = qs \mod(1)$$

$q \in \mathbb{Z}$

IDOS (ω)



Topological winding

$$\mathcal{W}(\alpha_q) = \frac{1}{2\pi}\int_0^{2\pi} d\phi \, \frac{d\alpha_q(\phi)}{d\phi} = 2q$$

E. Levy, F. Mortessagne, U. Kuhl, E.A, 2016

Ambiguity for winding numbers