$L^p$ Poincaré inequalities on some fractals

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Poincaré inequalities: In many situations a \((q, p)\)-Poincaré inequality

\[
\inf_{a} \int_{B(x, r)} |f - a|^q d\mu \leq C(r) \int_{B(x, \lambda r)} |\nabla f|^p d\mu
\]

gives useful information about the space.

Example: On Euclidean space, Riemannian manifolds with non-negative Ricci curvature,

\[
\int_{B(x, r)} |f - f_{B(x, r)}|^2 d\mu \leq Cr^2 \int_{B(x, r)} |\nabla f|^2 d\mu.
\]

In this talk, we will discuss Poincaré inequalities on some tree-like fractals for which an illustrative example is the Vicsek set.
Let $V_0 = \{q_1, q_2, q_3, q_4, q_5\}$ be the 4 corners of a unit square and the center. Define

$$\psi_i(z) = \frac{1}{3} (z - q_i) + q_i, \quad 1 \leq i \leq 5.$$  

The Vicsek set is the unique non-empty compact set such that

$$K = \bigcup_{i=1}^{5} \psi_i(K) =: \Psi(K).$$

Denote $W_n = \{1, 2, \cdots, 5\}^n$. For any $w = \{i_1, \cdots, i_n\} \in W_n$, the set $K_w := \Psi_w(K) = \psi_{i_1} \circ \cdots \circ \psi_{i_n}(K)$ is an $n$-simplex.
Vicsek (metric) graph

Vicsek graphs Define the set of vertices inductively by

\[ V_{n+1} = \Psi(V_n), \quad n \geq 0. \]

The Vicsek graph \( G_n \) is equipped with the set of vertices \( V_n \) and the set of edges \( E_n \) for which each edge has length \( 3^{-n} \).

Vicsek metric graphs Let \( \bar{V}_0 \) be the metric graph with vertices \( \bar{V}_0 \) which consists of the union of the two diagonals of the unit square. Define Vicsek metric graphs inductively by

\[ \bar{V}_{n+1} = \Psi(\bar{V}_n), \quad n \geq 0. \]

Figure 2: Vicsek graphs \( V_0, V_1, V_2 \)
Unbounded Vicsek set

By unbounded Vicsek set we mean blow-ups of compact Vicsek set. For instance, assume that $\psi_1(x) = x/3$ without loss of generality, then the unbounded nested fractal $X$ defined by

$$X = \bigcup_{n=1}^{\infty} K^{\langle n \rangle} := \bigcup_{n=1}^{\infty} 3^n K.$$ 

Figure 3: Infinite Vicsek set
Dirichlet form on the Vicsek set

\( \mu \): normalized Hausdorff measure; \( d \): Euclidean distance.

**Dirichlet form on** \( L^2(K, \mu) \):

\[
\mathcal{E}_K(f, f) = \lim_{n \to \infty} 3^n \sum_{w \in W_n} \sum_{e(x, y) \in E_0} (f \circ \psi_w(x) - f \circ \psi_w(y))^2.
\]

Then \( \mathcal{E}_K \) can be extended to \( X \) through dilation and limiting procedures. (see J. Kigami, M. Barlow, Fitzsimmons & Hambly & Kumagai, R. Strichartz et al).

**Energy measure**: for any \( f, g \in \mathcal{F} \), there exists a measure \( d\Gamma(f, g) \) in the sense of Beurling-Deny:

\[
\mathcal{E}(f, g) = \int_X d\Gamma(f, g).
\]

\( d\Gamma \) satisfies **Leibniz type rule**: 

\[
d\Gamma(fg, h) = f d\Gamma(g, h) + g d\Gamma(f, h).
\]

\( d\Gamma(f, g) \) is **not absolutely continuous** with respect to \( \mu \).
Volume and heat kernel bounds on the Vicsek set

Associated with the Dirichlet form there are a Laplacian type operator $\mathcal{L}$ and a Markov semigroup $P_t = \exp(t\mathcal{L})$ satisfying

$$\mathcal{E}(f, f) = -\langle \mathcal{L}f, f \rangle = \lim_{t \to 0^+} \frac{1}{t} \langle (I - P_t)f, f \rangle.$$ 

The heat semigroup $P_t$ admits a density $p_t(x, y)$ w.r.t. $\mu$.

- **Ahlfors $d_h$-regularity**: $\mu(B(x, r)) \asymp r^{d_h}$, where $d_h = \frac{\log 5}{\log 3}$ is the Hausdorff dimension.

- **Sub-Gaussian heat kernel (Neumann) estimate**:

  $$p^K_t(x, y) \asymp Ct^{-d_h/d_w} \exp \left( -c \left( d(x, y)^{d_w / t} \right)^{1/(d_w - 1)} \right), \quad \forall t \in (0, 1),$$

  where $d_w = \frac{\log 15}{\log 3} = d_h + 1$ is the walk dimension.
Our goal

The $L^2$ Poincaré inequality $\text{PI}(d_w)$ holds on $X$ (or a local version on $K$):

$$
\int_{B(x,r)} |u - u_{B(x,r)}|^2 \, d\mu \leq Cr^{d_w} \int_{B(x,cr)} d\Gamma(u, u), \quad \forall x \in X, r > 0.
$$

In fact, the sub-Gaussian heat kernel bound implies the volume doubling property and $\text{PI}(d_w)$ (see the work of Barlow & Bass & Kumagai, Hebisch & Saloff-Coste, Grigor’y’an & Telcs in 2000s).

**Question**

Can we extend the $\text{PI}(d_w)$ to $L^p$ type Poincaré inequalities on tree-like fractals such as Vicsek set?
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**Question**

Can we extend the $\text{PI}(d_w)$ to $L^p$ type Poincaré inequalities on tree-like fractals such as Vicsek set?

**Difficulty**: no analogue of curvature; no differential structure and $L^p$ version of energy measure.
Positive evidences

- On infinite Vicsek graph, there holds for \( p \geq 1 \) ([C. 2019])

\[
\| f - f_{B(x,n)} \|_{\ell^p(B(x,n))} \leq C n^{1 - \frac{1}{p} + \frac{d}{p}} \| \nabla f \|_{\ell^p(B(x,2n))}.
\]

- Weak Bakry-Émery type condition on Vicsek set proved in [ABCRST III] (following Barlow):

\[
|P_t f(x) - P_t f(y)| \leq C \frac{d(x, y)}{t^{1/d_w}} \| f \|_{\infty}.
\]

- Recent development of \( L^p \) Korevaar-Schoen and BV spaces on fractals using heat kernel methods in [ABCRST I, III, Alonso-Ruiz & Baudoin].
For any $p > 1$, the Korevaar-Schoen-Sobolev space $W^{1,p}(X)$ is the space of functions $f \in L^p(X, \mu)$ such that
\[
\limsup_{r \to 0^+} \frac{1}{r^{\alpha_p}} \left( \int_X \int_{B(x,r)} \frac{|f(y) - f(x)|^p}{\mu(B(x,r))} \, d\mu(y) \, d\mu(x) \right)^{1/p} < +\infty,
\]
where $\alpha_p = 1 - \frac{2}{p} + \frac{d_w}{p} = 1 - \frac{1}{p} + \frac{d_h}{p}$.

The Korevaar-Schoen type $p$-variation $\text{Var}_{A,p}(f)$ is defined by
\[
\liminf_{r \to 0^+} \frac{1}{r^{\alpha_p}} \left( \int_A \int_{B(x,r) \cap A} \frac{|f(y) - f(x)|^p}{\mu(B(x,r))} \, d\mu(x) \, d\mu(y) \right)^{1/p}.
\]

The case $p = 1$ defines the BV space $BV(X)$ and the variation $\text{Var}_A(f)$.

There is an equivalent characterization of $\text{Var}_{A,p}(f)$, in form of the sub-Gaussian $p$-variation $\text{Var}^*_A,f(p)$. 
Poincaré inequality for $1 \leq p \leq 2$

**Theorem (Baudoin-C. 2020)**

On the Vicsek set, for any $f \in W^{1,p}(X)$ if $1 < p \leq 2$, or $f \in BV(X)$ if $p = 1$, there holds

$$
\| f - f_{B(x_0,R)} \|_{L^p(B(x_0,R),\mu)} \leq CR^{1-\frac{2}{p}} + \frac{d_w}{p} \text{Var}_{B(x_0,cR),p}(f).
$$

When $p = 1$, the power of $R$ is $d_h$; when $p = 2$, the power is $\frac{d_w}{2}$. 
Outline of the proof

Step 1: Pseudo-Poincaré inequality

\[ \| f - P^K_t f \|_{L^p(K, \mu)} \leq C \text{Var}^*_K, p(f). \]

Key words: symmetry of the heat kernel, spectral theory and weak Bakry-Émery condition. In fact, we write

\[ \int_K (f - P^K_t f)gd\mu = \lim_{\tau \to 0^+} \int_0^t \mathcal{E}_\tau(P^K_s f, g)ds, \]

where

\[ \mathcal{E}^K_\tau(u, v) := \frac{1}{2\tau} \int_K \int_K p^K_\tau(x, y)(u(x) - u(y))(v(x) - v(y))d\mu(x)d\mu(y). \]

Step 2: Poincaré inequalities on simplices

\[ \left\| f - \int_{K_w} f d\mu \right\|_{L^p(K_w, \mu)} \leq C r(K_w)^{1-\frac{2}{p} + \frac{d_w}{p}} \text{Var}^*_K, p(f). \]

Key words: convergence to equilibrium, i.e., \( P^K_t f \to \int_K f d\mu \), and scaling.
Step 3: Comparing $\text{Var}_K^*,p(f)$ and $\text{Var}_K,p(f)$

It is straightforward to get $\text{Var}_K,p(f) \leq C\text{Var}_K^*,p(f)$. However, the other direction requires more precise metric analysis.

Step 4: From simplices to metric balls

- The case $1 < p \leq 2$.
  Morrey type estimates on simplices: for $x, y$ in an $n$-simplex $K_w$
  \[
  |f(x) - f(y)| \leq CL^{n(d_w - d_h)(1 - \frac{1}{p})}\text{Var}_K^*,p(f).
  \]

  Covering: the structure of $K$ is “regular” in the sense that two disjoint $n$-simplices are “not close”. Hence we can cover a metric ball with some neighboring simplices whose diameter are comparable to the radius. (idea from Pietruska-Pałuba & Stós 2013).

- The case $p = 1$. Cutoff argument which relies on the topology of $X$ and requires that $X$ has a treelike structure (e.g. Vicsek set).
Poincaré inequality for $p > 2$

**Theorem (Baudoin-C. 2022+)**

Let $p > 2$. There exist constants $c, C > 0$ such that for any $x_0 \in K$ and $R > 0$ with $B(x_0, cR) \subset K$ we have

$$\int_{B(x_0,R)} |f(x) - f_{B(x_0,R)}|^p d\mu(x) \leq CR^{p-1+d_h} E^{KS}_{B(x_0,cR),p}(f),$$

where $E^{KS}_{A,p}(f)$ is the $L^p$ Korevaar-Schoen energy defined by

$$E^{KS}_{A,p}(f) := \limsup_{r \to 0^+} \frac{1}{r^{p-1+d_h}} \int_A \int_{B(x,r) \cap A} \frac{|f(y) - f(x)|^p}{\mu(B(x,r))} d\mu(y) d\mu(x).$$

The power of $R$ is consistent with the case $1 \leq p \leq 2$. However, the $E^{KS}_{A,p}(f)$ is defined from $\limsup$, instead of $\liminf$ in $\textbf{Var}_{K,p}(f)$. 
Idea for the proof

**Step 1:** $L^p$ Poincaré inequality on $V_n$ in terms of the discrete $p$-energy

\[ E_{A,p}^n(f) := 3^{(p-1)n} \sum_{x,y \in A, e(x,y) \in E_n} |f(x) - f(y)|^p. \]

(for discrete $p$-energies on other fractals, see Herman & Peirone & Strichartz 2004, and recent work of J. Kigami, R. Shimizu, Cao & Gu & Qiu in 2021)

**Step 2:** $L^p$ Poincaré inequality on $K$ in terms of the $p$-energy from limit approximation

\[ E_{A,p}(f) := \lim_{m \to \infty} E_{A,p}^m(f). \]

**Step 3:** Comparison of $p$-energies $E^{KS}_{A,p}(f)$ and $E_{A,p}(f)$.

Important ingredient: approximation by piecewise affine functions on the Vicsek set.
Applications: Sobolev inequalities on balls

Apply $L^p$ Poincaré inequalities and the general theory developed by Bakry & Coulhon & Ledoux & Saloff-Coste in the paper “Sobolev inequalities in disguise”, then
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**Theorem (Baudoin-C. 2020)**

Let $p \geq 1$. Then for $f \in W^{1,p}(X)$ if $1 < p \leq 2$, or $f \in BV(X)$ if $p = 1$,

$$
\|f\|_{L^\infty(B(x,r))} \leq C \left( r^{-\frac{d_h}{p}} \|f\|_{L^p(B(x,r))} + r^{1-\frac{1}{p}} \text{Var}_{B(x,cr),p}(f) \right).
$$
Let \(1 \leq p \leq 2\), define the fractal version of maximal function by

\[
g(x) := \sup_{r>0} \frac{1}{\mu(B(x, r))^{1/p}} \text{Var}_B(x, r, p)(f).
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Let $1 \leq p \leq 2$. Then for $f \in W^{1, p}(X)$ if $1 < p \leq 2$, or $f \in BV(X)$ if $p = 1$,

$$|f(x) - f(y)| \leq Cd(x, y)^{1 - \frac{2}{p} + \frac{d_w}{p}} (g(x) + g(y)).$$
Applications: Maximal function

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In this direction, we expect more applications on Hajłasz-Sobolev spaces, which have been extensively studied in the work by J. Heinonen, P. Koskela, N. Shanmugalingam, and J. Tyson et al.