L^p Poincaré inequalities on some fractals

The 7th Cornell Conference on Analysis, Probability, and Mathematical Physics on Fractals

Li Chen, Louisiana State University
Joint with Fabrice Baudoin (UCONN)

June 5, 2022

Introduction

Poincaré inequalities: In many situations a (q, p)-Poincaré inequality

$$\inf_a \int_{B(x,r)} |f - a|^q d\mu \leq C(r) \int_{B(x,\lambda r)} |\nabla f|^p d\mu$$

gives useful information about the space.

Example: On Euclidean space, Riemannian manifolds with non-negative Ricci curvature,

$$\int_{B(x,r)} |f - f_{B(x,r)}|^2 d\mu \leq Cr^2 \int_{B(x,r)} |\nabla f|^2 d\mu.$$

In this talk, we will discuss Poincaré inequalities on some tree-like fractals for which an illustrative example is the Vicsek set.

Vicsek set

Let $V_0 = \{q_1, q_2, q_3, q_4, q_5\}$ be the 4 corners of a unit square and the center. Define

$$\psi_i(z) = \frac{1}{3}(z - q_i) + q_i, \quad 1 \le i \le 5.$$

The Vicsek set is the unique non-empty compact set such that

$$K = \bigcup_{i=1}^{5} \psi_i(K) =: \Psi(K).$$

Denote $W_n = \{1, 2, \cdots, 5\}^n$. For any $w = \{i_1, \cdots, i_n\} \in W_n$, the set $K_w := \Psi_w(K) = \psi_{i_1} \circ \cdots \circ \psi_{i_n}(K)$ is an n-simplex.

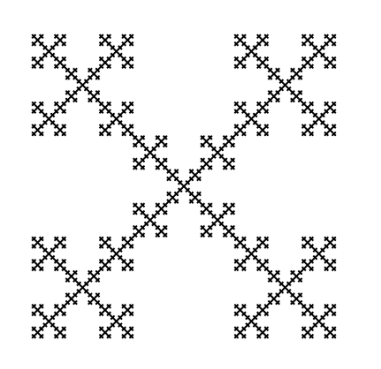

Figure 1: Vicsek set

Vicsek (metric) graph

Vicsek graphs Define the set of vertices inductively by

$$V_{n+1} = \Psi(V_n), \quad n \geq 0.$$

The Vicsek graph G_n is equipped with the set of vertices V_n and the set of edges E_n for which each edge has length 3^{-n}.

Vicsek metric graphs Let $\bar{V}_0$ be the metric graph with vertices V_0 which consists of the union of the two diagonals of the unit square. Define Vicsek metric graphs inductively by

$$\bar{V}_{n+1} = \Psi(\bar{V}_n), \quad n \geq 0.$$

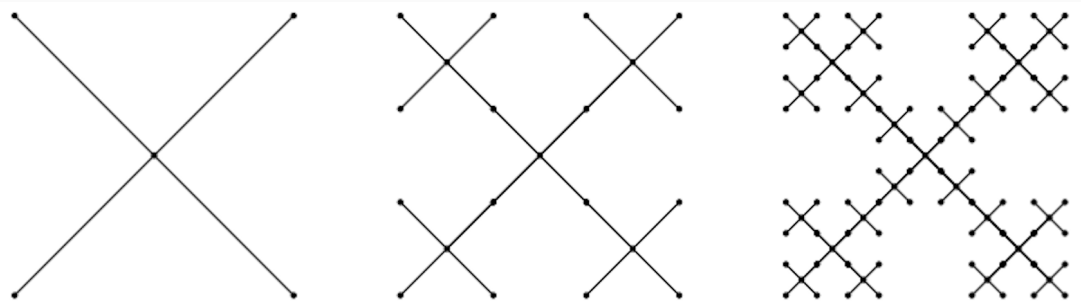

Figure 2: Vicsek graphes V_0, V_1, V_2

Unbounded Vicsek set

By unbounded Vicsek set we mean blow-ups of compact Vicsek set. For instance, assume that $\psi_1(x) = x/3$ without loss of generality, then the unbounded nested fractal X defined by

$$X = \bigcup_{n=1}^{\infty} K^{\langle n \rangle} := \bigcup_{n=1}^{\infty} 3^n K.$$

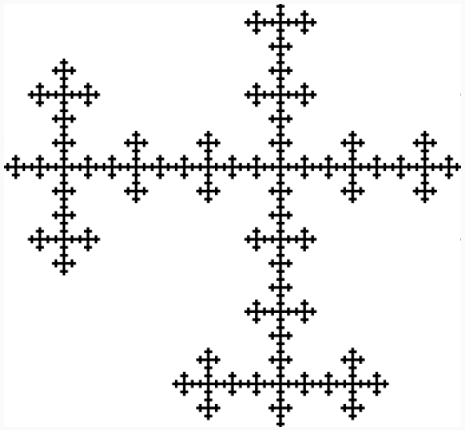

Figure 3: Infinite Vicsek set

Dirichlet form on the Vicsek set

μ: normalized Hausdorff measure; d: Euclidean distance.

Dirichlet form on $L^2(K,\mu)$:

$$\mathcal{E}_K(f,f) = \lim_{n\to\infty} 3^n \sum_{w\in W_n} \sum_{\mathbf{e}(x,y)\in E_0} (f\circ\psi_w(x) - f\circ\psi_w(y))^2.$$

Then $\mathcal{E}_K$ can be extended to X through dilation and limiting procedures. (see J. Kigami, M. Barlow, Fitzsimmons & Hambly & Kumagai, R. Strichartz et al).

Energy measure: for any $f,g\in\mathcal{F}$, there exists a measure $d\Gamma(f,g)$ in the sense of Beurling-Deny:

$$\mathcal{E}(f,g) = \int_X d\Gamma(f,g).$$

$d\Gamma$ satisfies Leibniz type rule: $d\Gamma(fg,h) = fd\Gamma(g,h) + gd\Gamma(f,h)$.

$d\Gamma(f,g)$ is not absolutely continuous with respect to μ.

Volume and heat kernel bounds on the Vicsek set

Associated with the Dirichlet form there are a Laplacian type operator $\mathcal{L}$ and a Markov semigroup $P_t = \exp(t\mathcal{L})$ satisfying

$$\mathcal{E}(f,f) = -\langle \mathcal{L}f, f\rangle = \lim_{t\to 0^+} \frac{1}{t}\langle (I-P_t)f, f\rangle.$$

The heat semigroup P_t admits a density $p_t(x,y)$ w.r.t. μ.

- Ahlfors d_h-regularity: $\mu(B(x,r)) \simeq r^{d_h}$, where $d_h = \frac{\log 5}{\log 3}$ is the Hausdorff dimension.
- Sub-Gaussian heat kernel (Neumann) estimate:

$$p_t^K(x,y) \simeq Ct^{-d_h/d_w} \exp\left(-c\left(d(x,y)^{d_w}/t\right)^{1/(d_w-1)}\right), \quad \forall t \in (0,1),$$

  where $d_w = \frac{\log 15}{\log 3} = d_h + 1$ is the walk dimension.

Our goal

The L^2 Poincaré inequality $\mathrm{PI}(d_w)$ holds on X (or a local version on K):

$$\int_{B(x,r)} |u - u_{B(x,r)}|^2 \, d\mu \leq C r^{d_w} \int_{B(x,cr)} d\Gamma(u,u), \quad \forall x \in X, r > 0.$$

In fact, the sub-Gaussian heat kernel bound implies the volume doubling property and $\mathrm{PI}(d_w)$ (see the work of Barlow & Bass & Kumagai, Hebisch & Saloff-Coste, Grigor'yan & Telcs in 2000s).

Question

Can we extend the $\mathrm{PI}(d_w)$ to L^p type Poincaré inequalities on tree-like fractals such as Vicsek set?

Our goal

The L^2 Poincaré inequality $\mathrm{PI}(d_w)$ holds on X (or a local version on K):

$$\int_{B(x,r)} |u - u_{B(x,r)}|^2 \, d\mu \leq C r^{d_w} \int_{B(x,cr)} d\Gamma(u,u), \quad \forall x \in X, r > 0.$$

In fact, the sub-Gaussian heat kernel bound implies the volume doubling property and $\mathrm{PI}(d_w)$ (see the work of Barlow & Bass & Kumagai, Hebisch & Saloff-Coste, Grigor'yan & Telcs in 2000s).

Question

Can we extend the $\mathrm{PI}(d_w)$ to L^p type Poincaré inequalities on tree-like fractals such as Vicsek set?

Difficulty: no analogue of curvature; no differential structure and L^p version of energy measure.

Positive evidences

- On infinite Vicsek graph, there holds for $p \geq 1$ ([C. 2019])

$$\|f - f_{B(x,n)}\|_{\ell^p(B(x,n))} \leq Cn^{1-\frac{1}{p}+\frac{d_h}{p}} \| \, |\nabla f| \, \|_{\ell^p(B(x,2n))}.$$

- Weak Bakry-Émery type condition on Vicsek set proved in [ABCRST III] (following Barlow):

$$|P_t f(x) - P_t f(y)| \leq C \frac{d(x,y)}{t^{1/d_w}} \|f\|_\infty.$$

- Recent development of L^p Korevaar-Schoen and BV spaces on fractals using heat kernel methods in [ABCRST I, III, Alonso-Ruiz & Baudoin].

Korevaar-Schoen-Sobolev and BV spaces on Vicsek set

Definition

For any $p > 1$, the Korevaar-Schoen-Sobolev space $W^{1,p}(X)$ is the space of functions $f \in L^p(X, \mu)$ such that

$$\limsup_{r \to 0^+} \frac{1}{r^{\alpha_p}} \Big(\int_X \int_{B(x,r)} \frac{|f(y) - f(x)|^p}{\mu(B(x,r))} \, d\mu(y) \, d\mu(x) \Big)^{1/p} < +\infty,$$

where $\alpha_p = 1 - \frac{2}{p} + \frac{d_w}{p} = 1 - \frac{1}{p} + \frac{d_h}{p}$.

The Korevaar-Schoen type p-variation $\underline{\mathbf{Var}}_{A,p}(f)$ is defined by

$$\liminf_{r \to 0^+} \frac{1}{r^{\alpha_p}} \left(\int_A \int_{B(x,r) \cap A} \frac{|f(y) - f(x)|^p}{\mu(B(x,r))} d\mu(x) d\mu(y) \right)^{1/p}.$$

The case $p = 1$ defines the BV space $BV(X)$ and the variation $\underline{\mathbf{Var}}_A(f)$.

There is an equivalent characterization of $\underline{\mathbf{Var}}_{A,p}(f)$, in form of the sub-Gaussian p-variation $\underline{\mathbf{Var}}^*_{A,p}(f)$.

Poincaré inequality for $1 \leq p \leq 2$

Theorem (Baudoin-C. 2020)

On the Vicsek set, for any $f \in W^{1,p}(X)$ if $1 < p \leq 2$, or $f \in BV(X)$ if $p = 1$, there holds

$$\left\| f - f_{B(x_0,R)} \right\|_{L^p(B(x_0,R),\mu)} \leq C R^{1-\frac{2}{p}+\frac{d_w}{p}} \underline{\mathbf{Var}}_{B(x_0,cR),p}(f).$$

When $p = 1$, the power of R is d_h; when $p = 2$, the power is $\frac{d_w}{2}$.

Outline of the proof

Step 1: Pseudo-Poincaré inequality

$$\|f - P_t^K f\|_{L^p(K,\mu)} \leq C\, \underline{\mathbf{Var}}^*_{K,p}(f).$$

Key words: symmetry of the heat kernel, spectral theory and weak Bakry-Émery condition. In fact, we write

$$\int_K (f - P_t^K f) g d\mu = \lim_{\tau \to 0^+} \int_0^t \mathcal{E}_\tau(P_s^K f, g) ds,$$

where

$$\mathcal{E}_\tau^K(u,v) := \frac{1}{2\tau} \int_K \int_K p_\tau^K(x,y)(u(x) - u(y))(v(x) - v(y)) d\mu(x) d\mu(y).$$

Step 2: Poincaré inequalities on simplices

$$\left\| f - \fint_{K_w} f d\mu \right\|_{L^p(K_w,\mu)} \leq C r(K_w)^{1 - \frac{2}{p} + \frac{d_w}{p}} \underline{\mathbf{Var}}^*_{K_w,p}(f).$$

Key words: convergence to equilibrium, i.e., $P_t^K f \to \fint_K f d\mu$, and scaling.

Step 3: Comparing $\underline{\mathbf{Var}}^*_{K,p}(f)$ and $\underline{\mathbf{Var}}_{K,p}(f)$

It is straightforward to get $\underline{\mathbf{Var}}_{K,p}(f) \leq C\underline{\mathbf{Var}}^*_{K,p}(f)$. However, the other direction requires more precise metric analysis.

Step 4: From simplices to metric balls

- The case $1 < p \leq 2$.
  Morrey type estimates on simplices: for x, y in an n-simplex K_w

  $$|f(x) - f(y)| \leq CL^{-n(d_w - d_h)(1-\frac{1}{p})}\underline{\mathbf{Var}}^*_{K_w,p}(f).$$

  Covering: the structure of K is "regular" in the sense that two disjoint n-simplices are "not close". Hence we can cover a metric ball with some neighboring simplices whose diameter are comparable to the radius. (idea from Pietruska-Pałuba & Stós 2013).
- The case $p = 1$. Cutoff argument which relies on the topology of X and requires that X has a treelike structure (e.g. Vicsek set).

Poincaré inequality for $p > 2$

Theorem (Baudoin-C. 2022+)

Let $p > 2$. There exist constants $c, C > 0$ such that for any $x_0 \in K$ and $R > 0$ with $B(x_0, cR) \subset K$ we have

$$\int_{B(x_0,R)} |f(x) - f_{B(x_0,R)}|^p d\mu(x) \leq CR^{p-1+d_h} \mathcal{E}^{KS}_{B(x_0,cR),p}(f),$$

where $\mathcal{E}^{KS}_{A,p}(f)$ is the L^p Korevaar-Schoen energy defined by

$$\mathcal{E}^{KS}_{A,p}(f) := \limsup_{r \to 0^+} \frac{1}{r^{p-1+d_h}} \int_A \int_{B(x,r)\cap A} \frac{|f(y) - f(x)|^p}{\mu(B(x,r))} \, d\mu(y) \, d\mu(x).$$

The power of R is consistent with the case $1 \leq p \leq 2$. However, the $\mathcal{E}^{KS}_{A,p}(f)$ is defined from lim sup, instead of lim inf in $\underline{\mathbf{Var}}_{K,p}(f)$.

Idea for the proof

Step 1: L^p Poincaré inequality on V_n in terms of the discrete p-energy

$$\mathcal{E}^n_{A,p}(f) := 3^{(p-1)n} \sum_{x,y \in A, \mathbf{e}(x,y) \in E_n} |f(x) - f(y)|^p.$$

(for discrete p-energies on other fractals, see Herman & Peirone & Strichaatz 2004, and recent work of J. Kigami, R. Shimizu, Cao & Gu & Qiu in 2021)

Step 2: L^p Poincaré inequality on K in terms of the p-energy from limit approximation

$$\mathcal{E}_{A,p}(f) := \lim_{m \to \infty} \mathcal{E}^m_{A,p}(f).$$

Step 3: Comparison of p-energies $\mathcal{E}^{KS}_{A,p}(f)$ and $\mathcal{E}_{A,p}(f)$.

Important ingredient: approximation by piecewise affine functions on the Vicsek set.

Applications: Sobolev inequalities on balls

Apply L^p Poincaré inequalities and the general theory developed by Bakry & Coulhon & Ledoux & Saloff-Coste in the paper "Sobolev inequalities in disguise", then

Applications: Sobolev inequalities on balls

Apply L^p Poincaré inequalities and the general theory developed by Bakry & Coulhon & Ledoux & Saloff-Coste in the paper "Sobolev inequalities in disguise", then

Theorem (Baudoin-C. 2020)

Let $p \geq 1$. Then for $f \in W^{1,p}(X)$ if $1 < p \leq 2$, or $f \in BV(X)$ if $p = 1$,

$$\|f\|_{L^\infty(B(x,r))} \leq C \left(r^{-\frac{d_h}{p}} \|f\|_{L^p(B(x,r))} + r^{1-\frac{1}{p}} \underline{\mathbf{Var}}_{B(x,cr),p}(f) \right).$$

Applications: Maximal function

Let $1 \leq p \leq 2$, define the fractal version of maximal function by

$$g(x) := \sup_{r>0} \frac{1}{\mu(B(x,r))^{1/p}} \underline{\mathbf{Var}}_{B(x,r),p}(f).$$

Applications: Maximal function

Let $1 \leq p \leq 2$, define the fractal version of maximal function by

$$g(x) := \sup_{r>0} \frac{1}{\mu(B(x,r))^{1/p}} \underline{\mathbf{Var}}_{B(x,r),p}(f).$$

Theorem (Baudoin-C. 2020)

Let $1 \leq p \leq 2$. Then for $f \in W^{1,p}(X)$ if $1 < p \leq 2$, or $f \in BV(X)$ if $p = 1$,

$$|f(x) - f(y)| \leq Cd(x,y)^{1-\frac{2}{p}+\frac{d_W}{p}}(g(x) + g(y)).$$

Applications: Maximal function

Let $1 \leq p \leq 2$, define the fractal version of maximal function by

$$g(x) := \sup_{r>0} \frac{1}{\mu(B(x,r))^{1/p}} \underline{\mathbf{Var}}_{B(x,r),p}(f).$$

Theorem (Baudoin-C. 2020)

Let $1 \leq p \leq 2$. Then for $f \in W^{1,p}(X)$ if $1 < p \leq 2$, or $f \in BV(X)$ if $p = 1$,

$$|f(x) - f(y)| \leq Cd(x,y)^{1-\frac{2}{p}+\frac{d_W}{p}}(g(x) + g(y)).$$

In this direction, we expect more applications on Hajłasz-Sobolev spaces, which have been extensively studied in the work by J. Heinonen, P. Koskela, N. Shanmugalingam, and J. Tyson et al.