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On singularity of energy measures for symmetric diffusions with full off-diagonal heat kernel estimates

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Joint work with Mathav Murugan (UBC)

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@Cornell University

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$\Gamma(u) =$ "$|\nabla u|^2\,dx$"

$\beta = 2$ (Gaussian) 09:35–10:05, $\Rightarrow \Gamma(u,u) \ll \mu$.

$\beta > 2$ (sub-Gaussian) 13:15–13:45@207 $\Rightarrow \Gamma(u,u) \perp \mu$!

$$p_t(x,y) \asymp c\mu\bigl(B(x,t^{\frac{1}{\beta}})\bigr)^{-1}\exp\Bigl(-\tilde{c}\bigl(d(x,y)^{\beta}/t\bigr)^{\frac{1}{\beta-1}}\Bigr)$$

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▷ $(K, d, \mu, \mathcal{E}, \mathcal{F})$: strongly local reg. symmet. Dir. sp.

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$$\int_K f\,d^{\exists 1}\Gamma(u,u)=\lim_{t\downarrow 0} t^{-1}\mathbb{E}_\mu\left[\int_0^t f(X_s)\,d\langle M^{[u]}\rangle_s\right]$$

$$=\lim_{t\downarrow 0}\frac{1}{2t}\int_K f(x)\mathbb{E}_x\left[|\widetilde{u}(X_t)-\widetilde{u}(X_0)|^2\right]d\mu(x)$$

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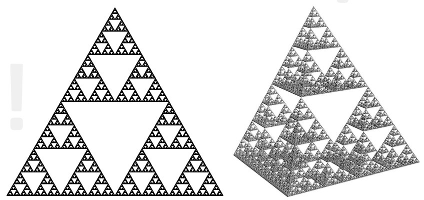

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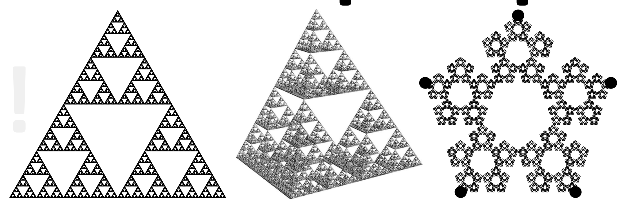

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$$p_t(x,y) \asymp c\mu(B(x,t^{\frac{1}{\beta}}))^{-1}\exp(-\tilde{c}(d(x,y)^{\beta}/t)^{\frac{1}{\beta-1}})$$

▷ $^{\exists 1}\Gamma(u,u)$: $\int_K f\,d\Gamma(u,u) = \mathcal{E}(fu,u) - \mathcal{E}(f,u^2)/2$

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Preceding $\perp$ results (ONLY for self-similar sets!)

- (Kusuoka '89, '93) p.c.f. s.-s. sets $\supset$ d-dim SG $(d \geq 2)$
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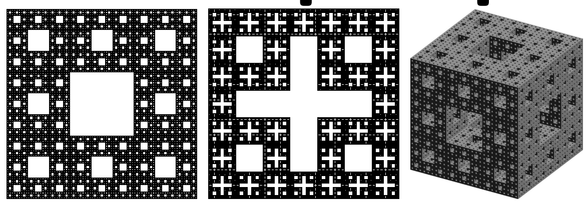
- (Hino '05) General s.-s. sets $\supset {}^{\forall}$GSCs! 

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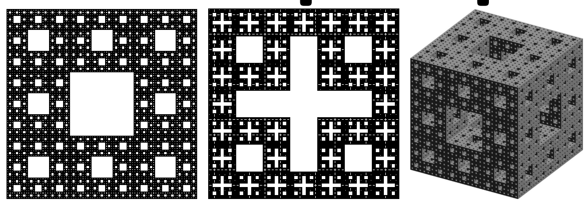

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2 Main result: Almost dichotomy $\Gamma \perp \mu$ v.s. $\Gamma \sim \mu$

▷ Ψ: homeo. on $[0,\infty)$ with $1 < {}^{\exists}\beta_0 \le {}^{\exists}\beta_1$, ${}^{\exists}c \ge 1$, $0 < {}^{\forall}r \le {}^{\forall}R$, $c^{-1}(R/r)^{\beta_0} \le \Psi(R)/\Psi(r) \le c(R/r)^{\beta_1}$.

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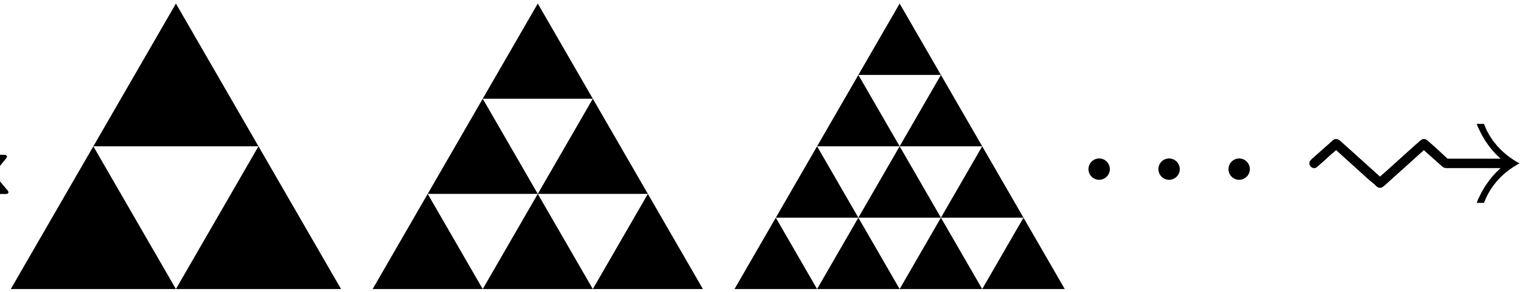
Example: Scale irregular SGs (Hambly '92, Barlow–Hambly '97)

Mix 

$\ell = 2$ $\ell = 3$ $\ell = 4$

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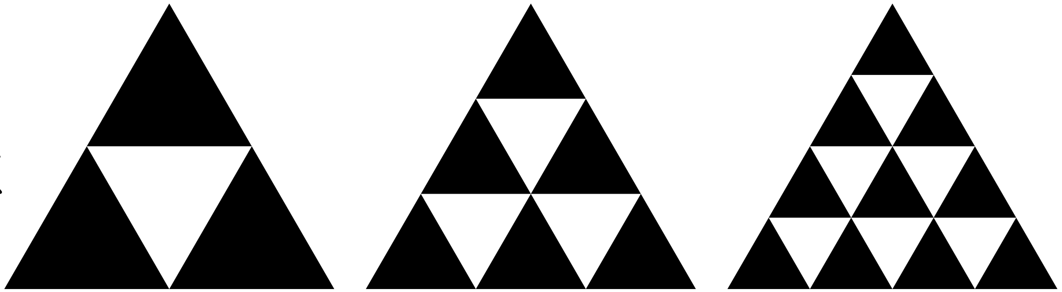
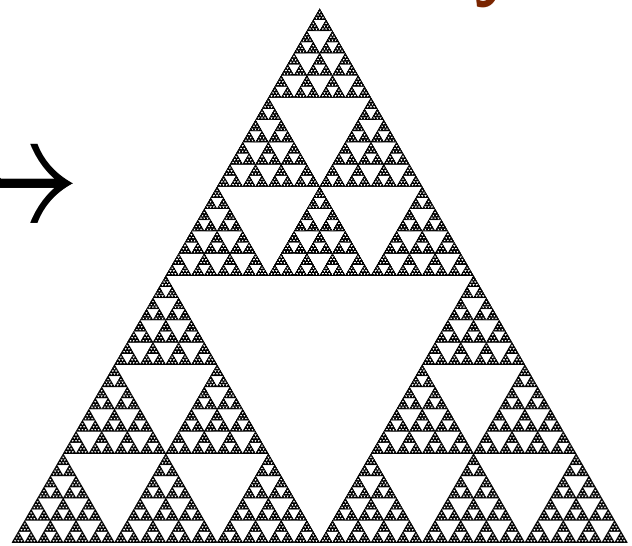
Mix  $\cdots \rightsquigarrow$ 

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$K^{(2,3,4,\ell_4,\ell_5,\ldots)}$

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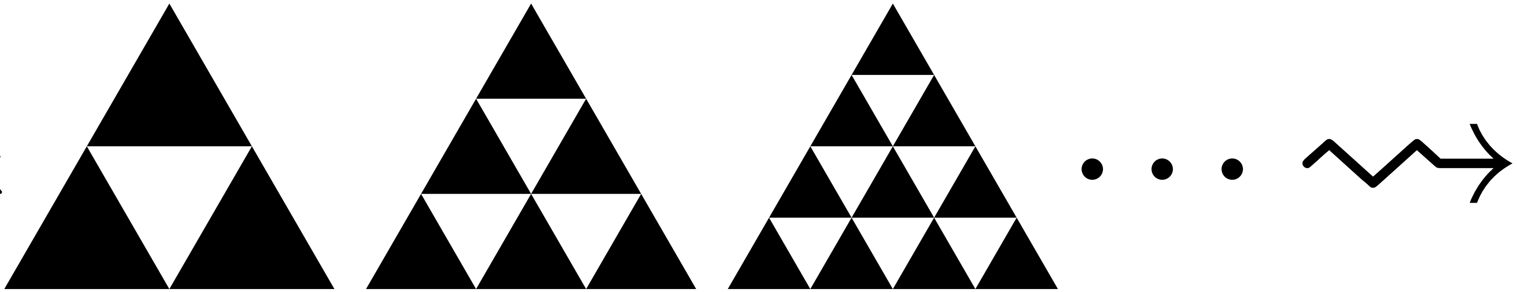
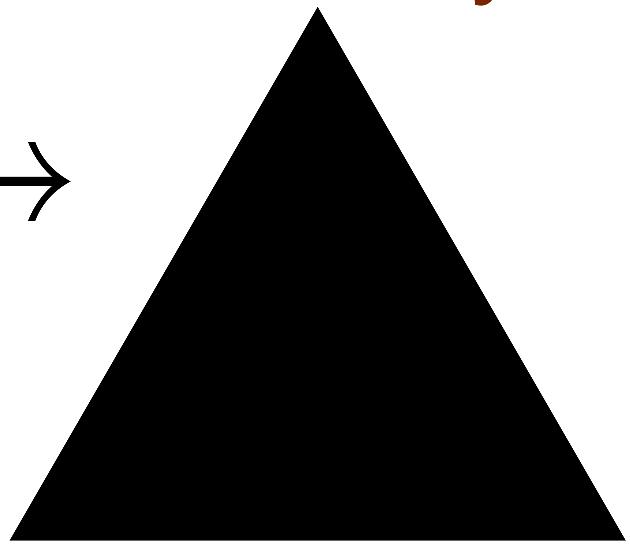


$\ell=2$ $\ell=3$ $\ell=4$



$K^{(}$

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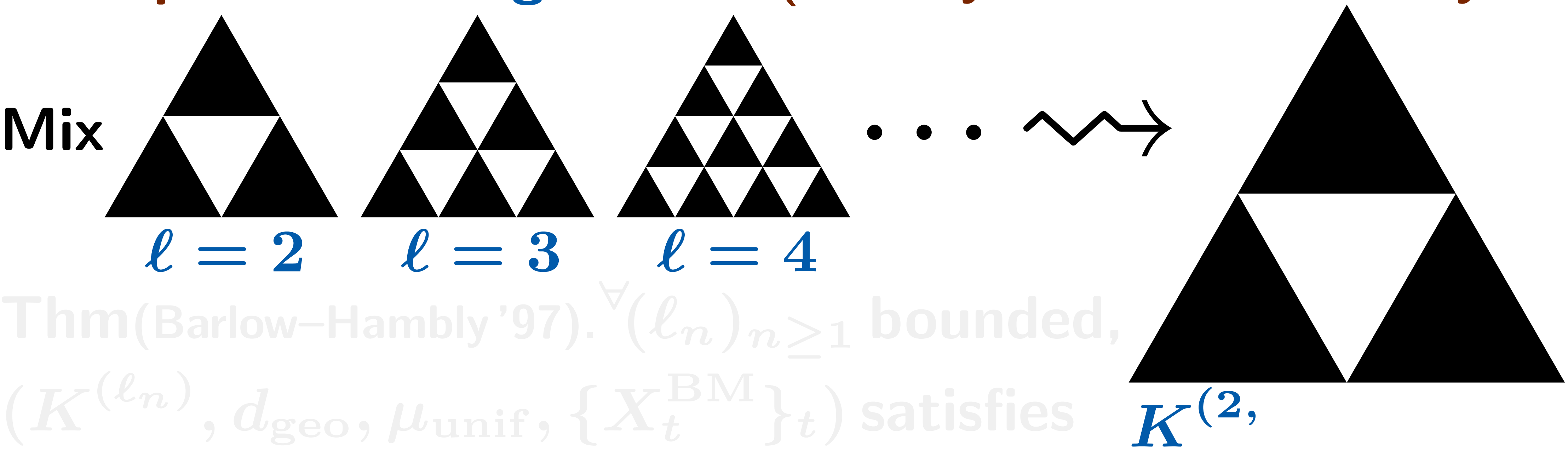

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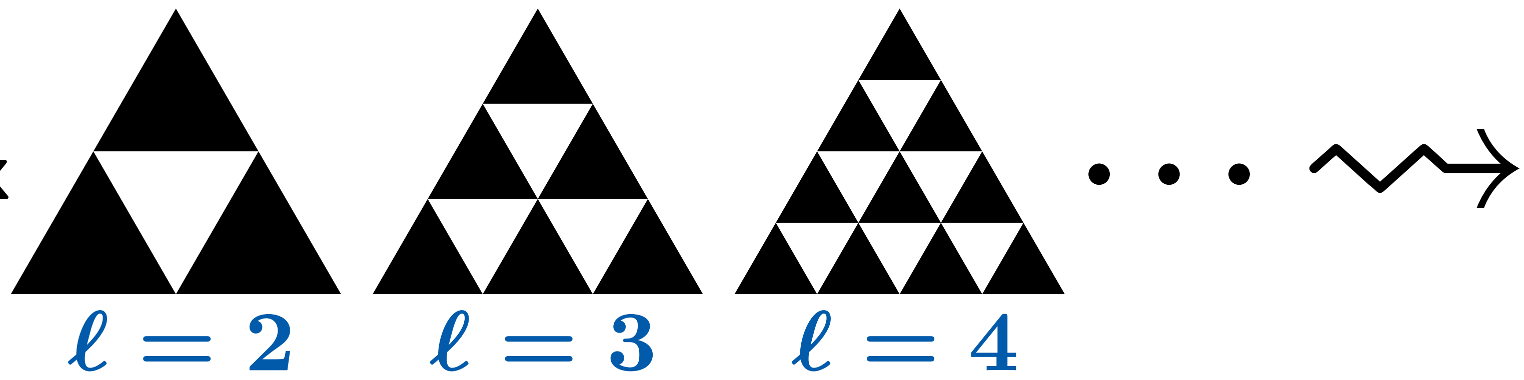

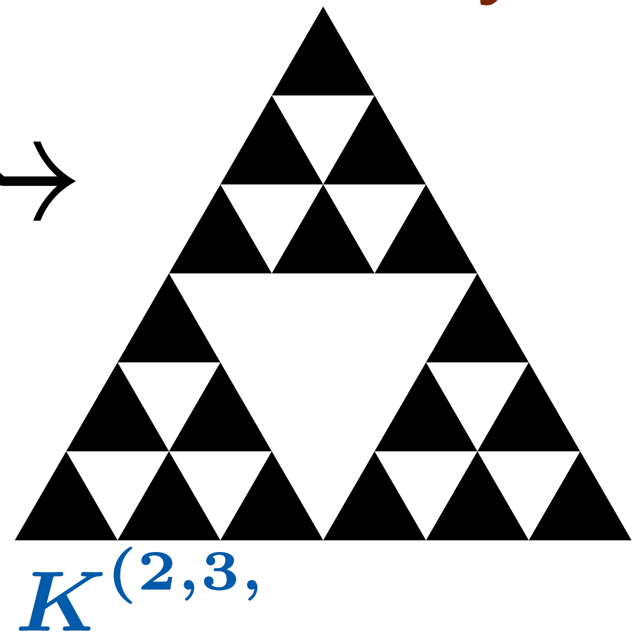

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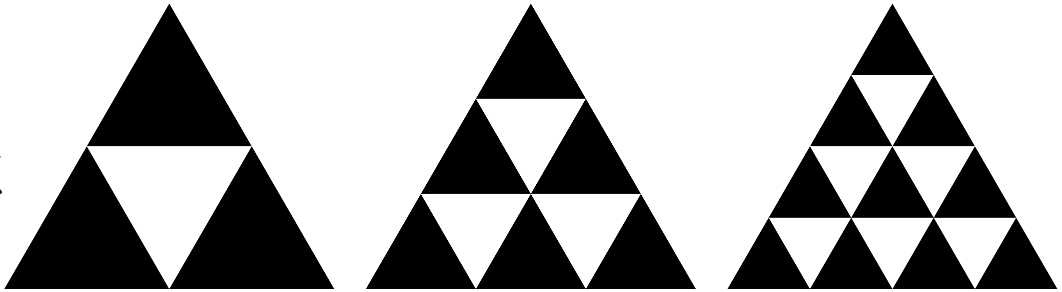

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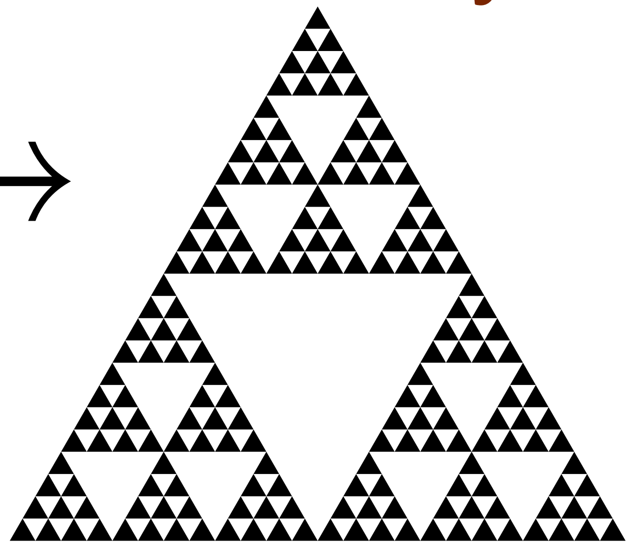

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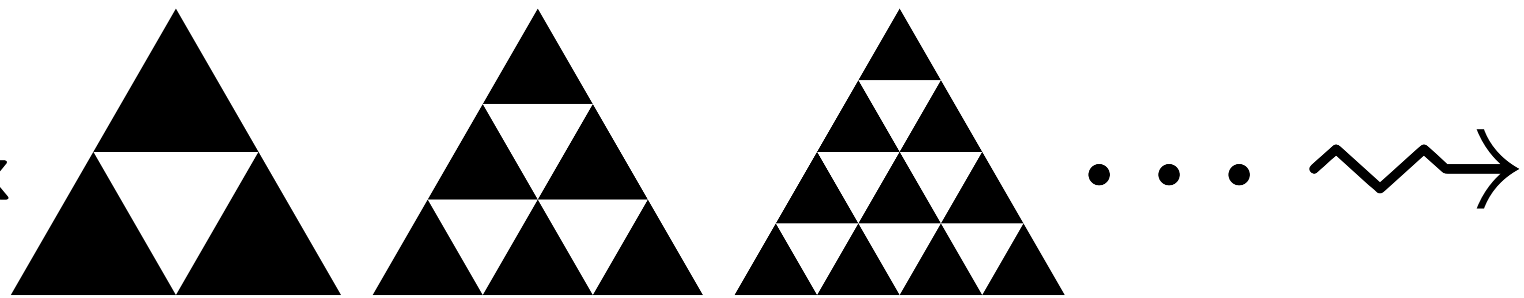

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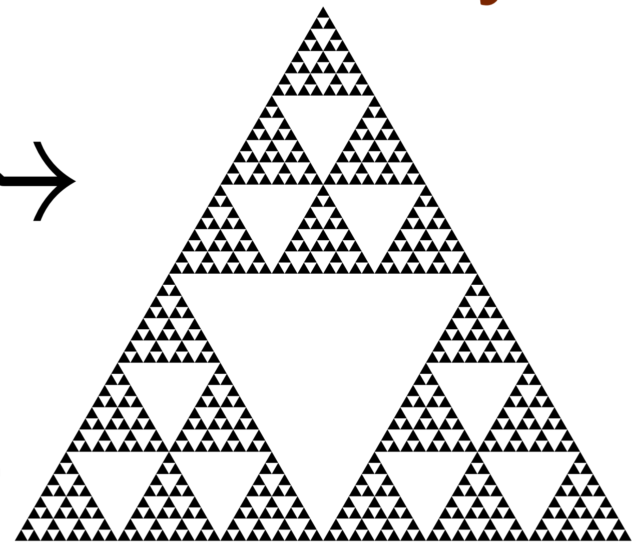

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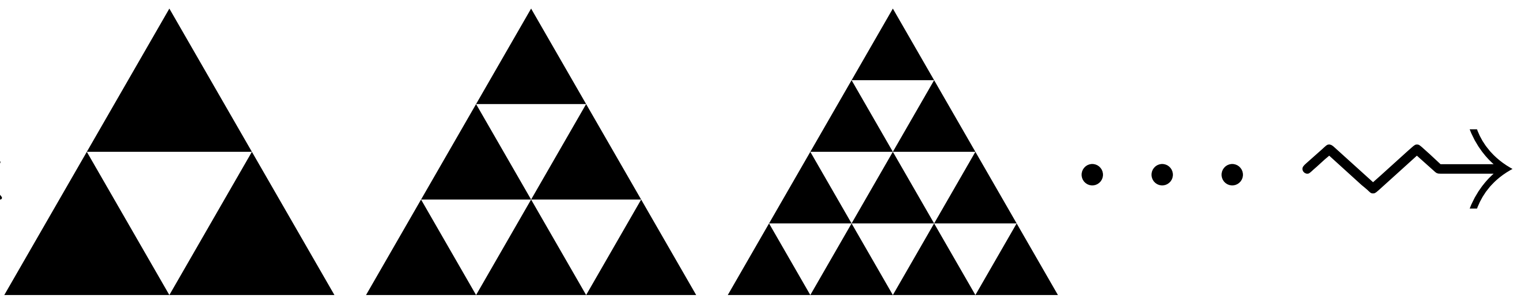
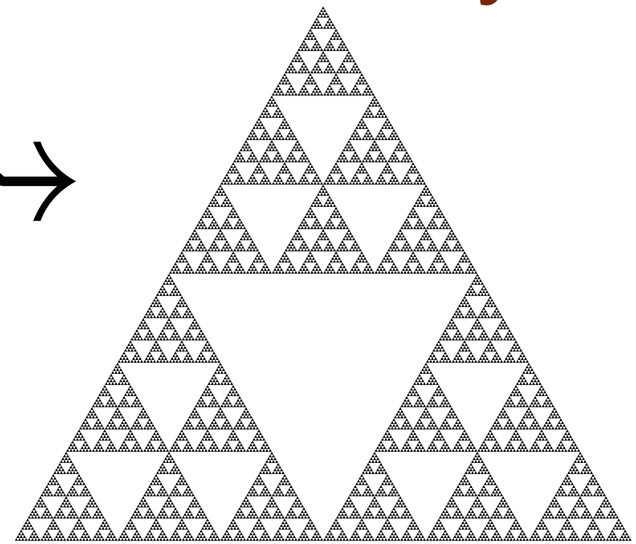
Mix 

$\ell = 2$ $\ell = 3$ $\ell = 4$



$K^{(2,3,4,\ell_4,\ell_5,}$

Thm (Barlow–Hambly '97). ${}^\forall(\ell_n)_{n\geq 1}$ bounded, $(K^{(\ell_n)}, d_{\mathrm{geo}}, \mu_{\mathrm{unif}}, \{X_t^{\mathrm{BM}}\}_t)$ satisfies

Prop. $0 < {}^\forall r \leq {}^\forall R \leq 1$, $\frac{\Psi_{(\ell_n)}(R)}{\Psi_{(\ell_n)}(r)} \geq c\left(\frac{R}{r}\right)^{\beta^{\min}_{(\ell_n)} > 2} \rightsquigarrow {}^\forall u \in \mathcal{F}, \Gamma(u,u) \perp \mu$.

Rmk (Hino–Yasui '22). $\Gamma(u,u) \perp \mu$ for wider class of random SGs and μ

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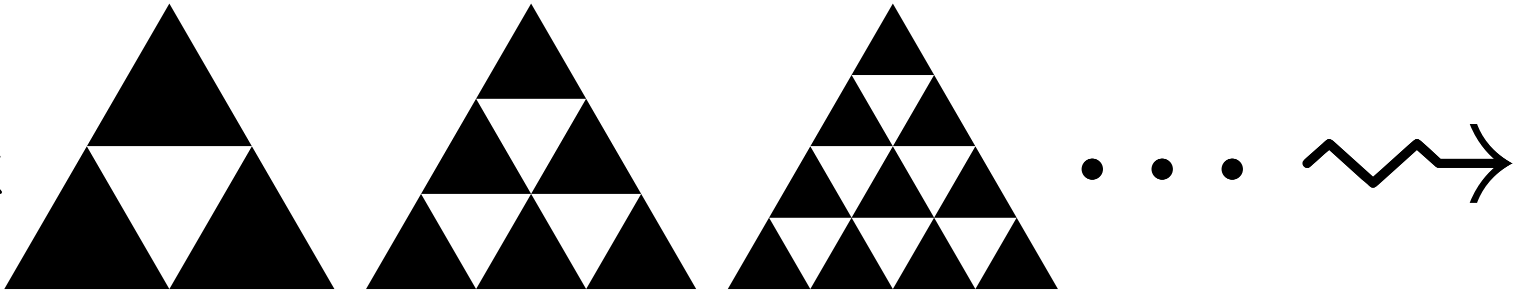
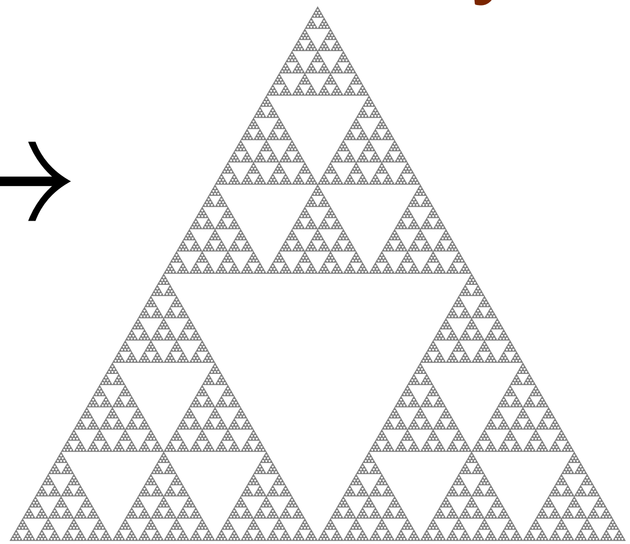


$\ell = 2$ $\ell = 3$ $\ell = 4$



$K^{(2,3,4,\ell_4,\ell_5,\ldots}$

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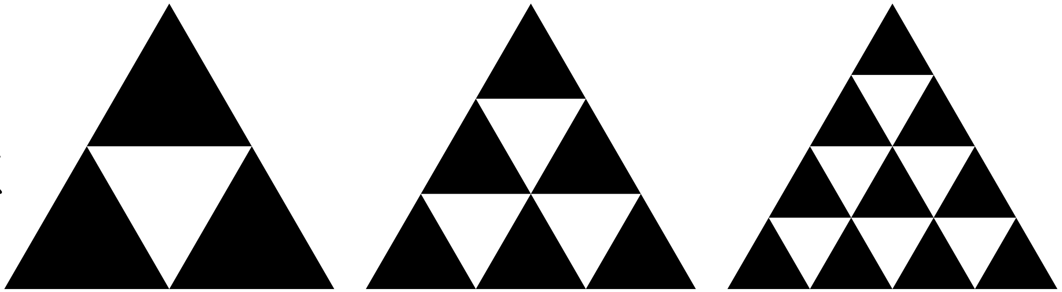
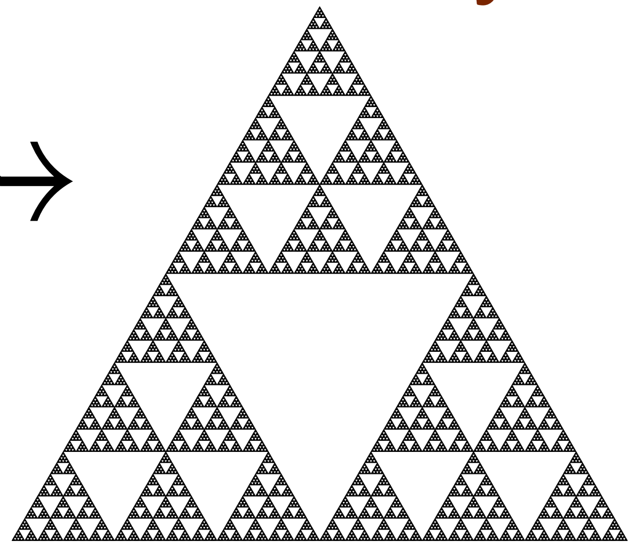
Mix



$\ell=2$ $\ell=3$ $\ell=4$ $\cdots \rightsquigarrow$



$K^{(2,3,4,\ell_4,\ell_5,\dots)}$

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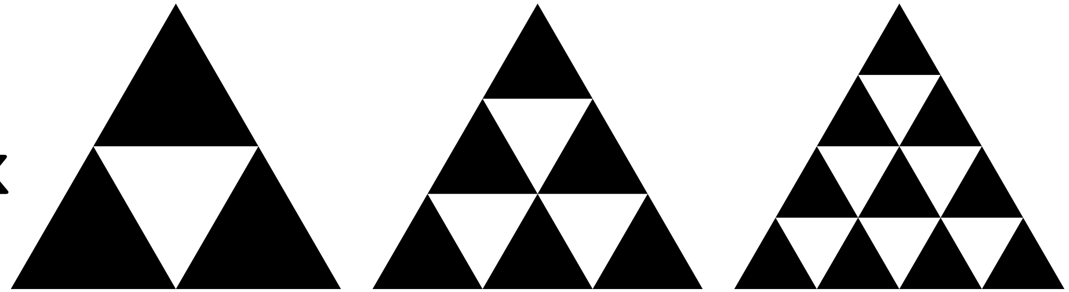
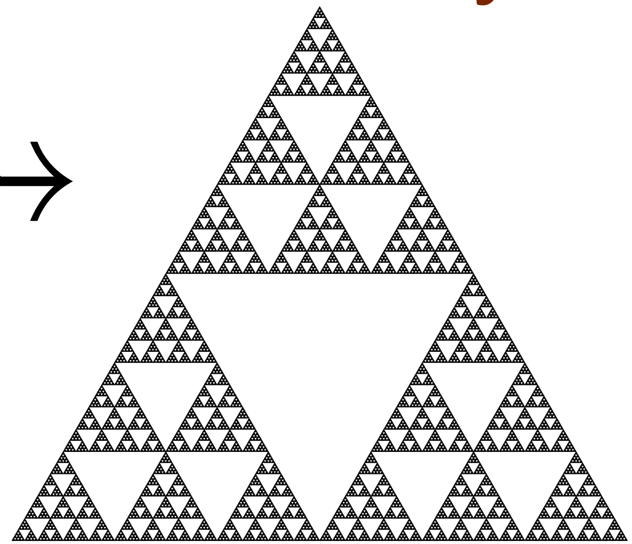
Mix



$\ell=2$ $\ell=3$ $\ell=4$ $\cdots \rightsquigarrow$



$K^{(2,3,4,\ell_4,\ell_5,\dots)}$

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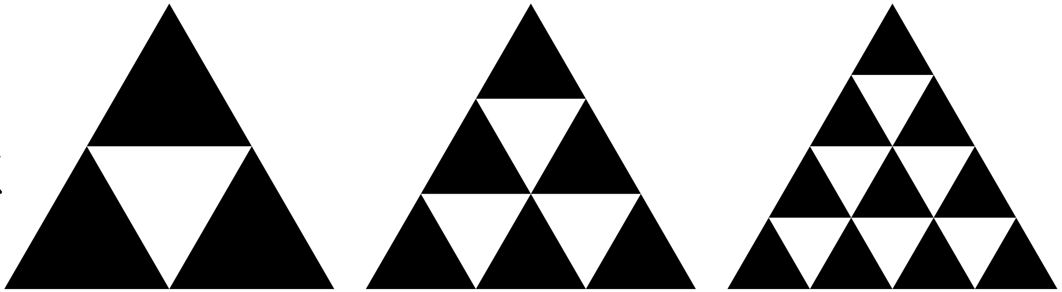
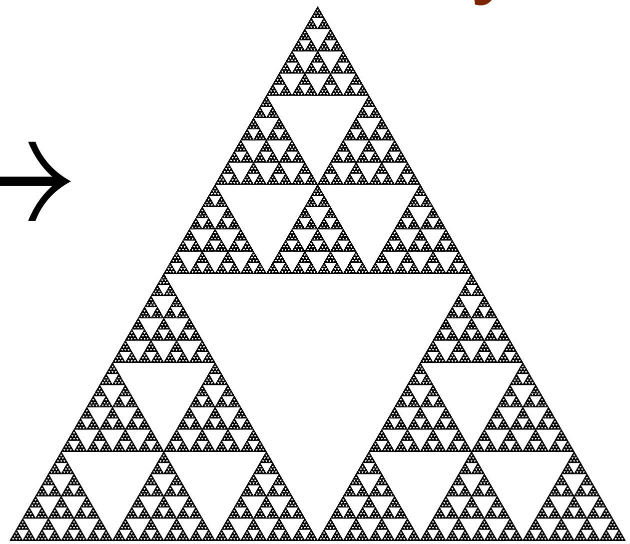
Mix



$\ell = 2$ $\qquad \ell = 3$ $\qquad \ell = 4$

$\cdots \rightsquigarrow$



$K^{(2,3,4,\ell_4,\ell_5,\ldots)}$

Thm (Barlow–Hambly '97). ${}^\forall(\ell_n)_{n\geq 1}$ **bounded**, $(K^{(\ell_n)}, d_{\mathrm{geo}}, \mu_{\mathrm{unif}}, \{X_t^{\mathrm{BM}}\}_t)$ satisfies HKE($\Psi_{(\ell_n)}$) with $\Psi_{(\ell_n)}(\ell_1^{-1}\cdots\ell_k^{-1}) := \tau_{\ell_1}^{-1}\cdots\tau_{\ell_k}^{-1}$.

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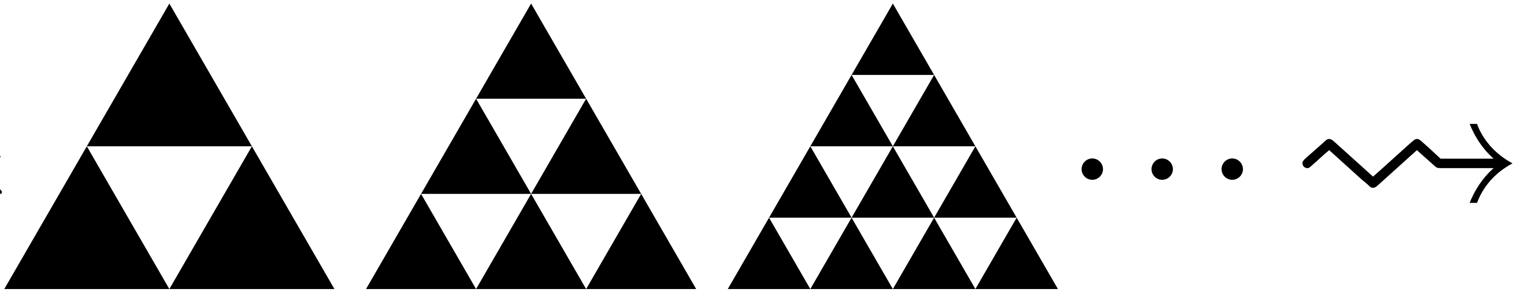
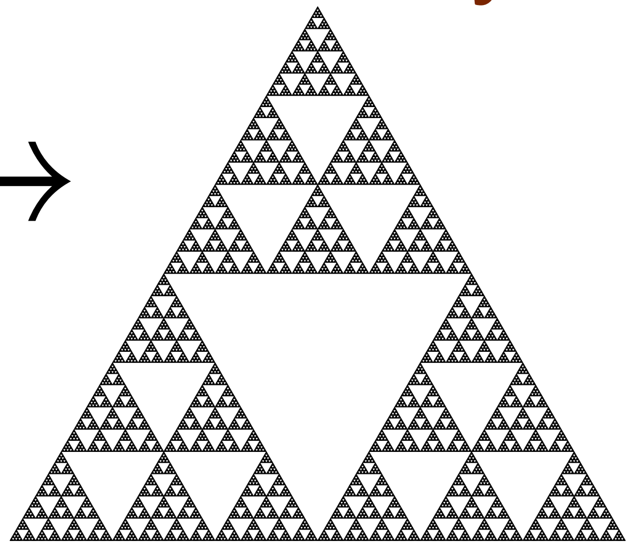
Mix



$\ell=2$ $\ell=3$ $\ell=4$



$K^{(2,3,4,\ell_4,\ell_5,\dots)}$

Thm (Barlow–Hambly '97). ${}^{\forall}(\ell_n)_{n\geq 1}$ bounded, $(K^{(\ell_n)},d_{\mathrm{geo}},\mu_{\mathrm{unif}},\{X_t^{\mathrm{BM}}\}_t)$ satisfies HKE($\Psi_{(\ell_n)}$) with $\Psi_{(\ell_n)}(\ell_1^{-1}\cdots\ell_k^{-1}):=\ell_1^{-\beta_{\ell_1}}\cdots\ell_k^{-\beta_{\ell_k}}$.

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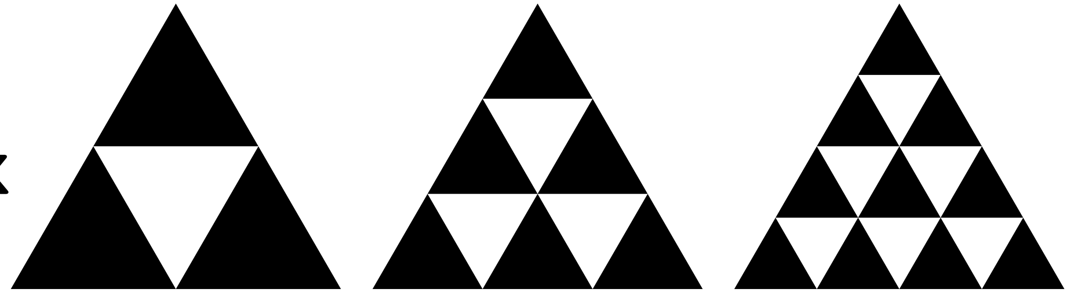
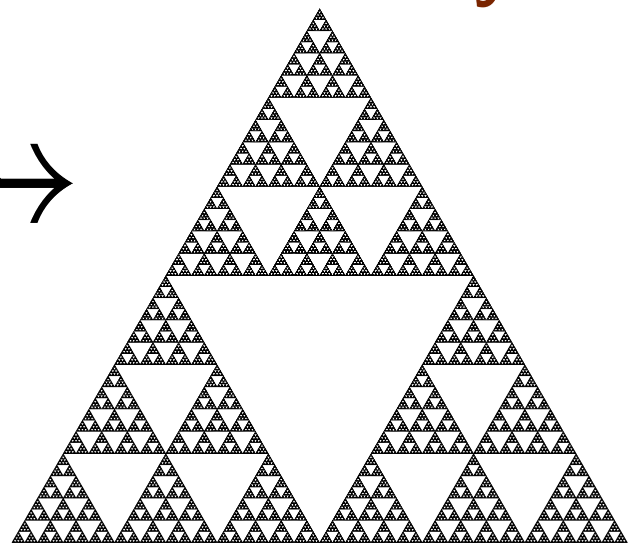
Mix



$\ell=2$ $\quad\ell=3$ $\quad\ell=4$ $\quad\cdots\rightsquigarrow$



$K^{(2,3,4,\ell_4,\ell_5,\ldots)}$

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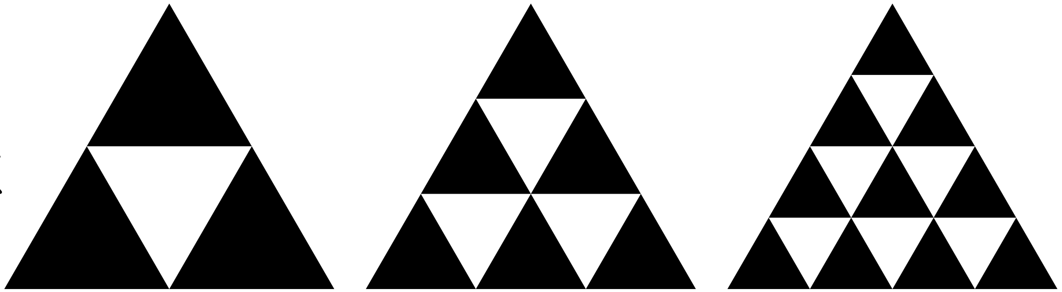
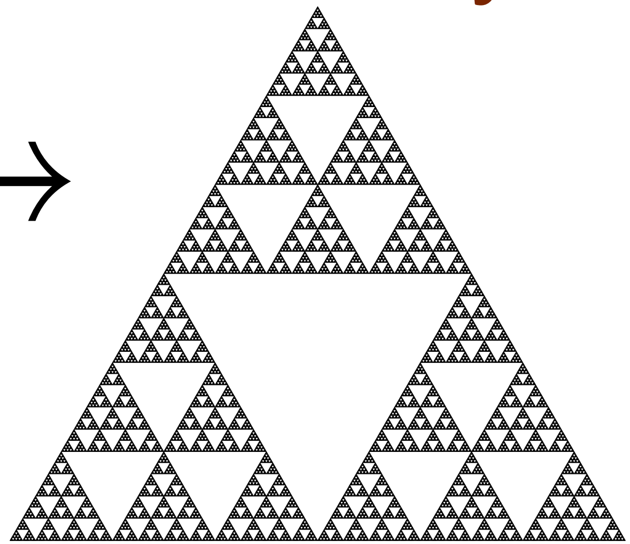
Mix



$\ell=2 \qquad \ell=3 \qquad \ell=4$

$\cdots \rightsquigarrow$



$K^{(2,3,4,\ell_4,\ell_5,\dots)}$

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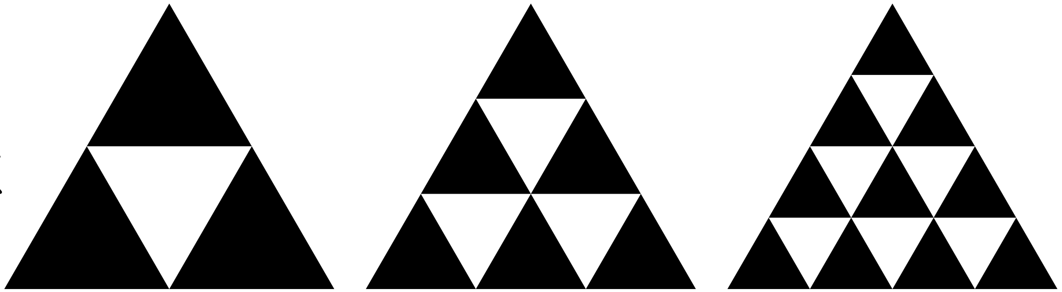
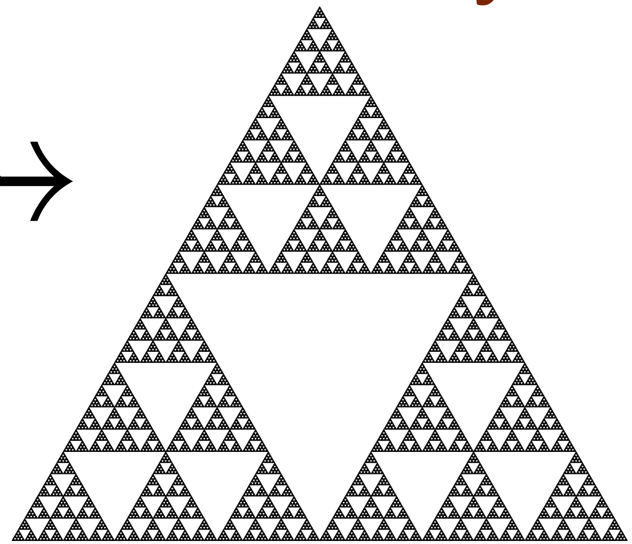
Mix



$\ell=2$ $\quad\ell=3$ $\quad\ell=4$ $\quad\cdots\rightsquigarrow$



$K^{(2,3,4,\ell_4,\ell_5,\ldots)}$

Thm (Barlow–Hambly '97). ${}^{\forall}(\ell_n)_{n\geq 1}$ bounded, $(K^{(\ell_n)}, d_{\mathrm{geo}}, \mu_{\mathrm{unif}}, \{X_t^{\mathrm{BM}}\}_t)$ satisfies HKE$(\Psi_{(\ell_n)})$ with $\Psi_{(\ell_n)}(\ell_1^{-1}\cdots\ell_k^{-1}):=\ell_1^{-\beta_{\ell_1}}\cdots\ell_k^{-\beta_{\ell_k}}$.

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Conj 2 (K.–Murugan AOP '20). **Assume that (K, d) is complete and that HKE(Ψ) holds. Then:**

(b)^c $\lim_{r\downarrow 0} r^{-2}\Psi(r) = 0 \Rightarrow {}^\forall u \in \mathcal{F}$, $\Gamma(u,u) \perp \mu$.

Thm (K. '22). For *any* $(\ell_n)_{n\geq 1}$, $(K^{(\ell_n)}_{\text{thin}}, d_{\text{geo}}, \mu_{\text{unif}}, \{X^{\text{BM}}_t\}_t)$ satisfies HKE($\Psi^{\text{M}}_{(\ell_n)}\Psi^{\text{R}}_{(\ell_n)}$) with $\Psi^{\text{M}}_{(\ell_n)}(\ell_1^{-1}\cdots\ell_k^{-1}) := \mu_{\ell_1}\cdots\mu_{\ell_k}$, $\Psi^{\text{R}}_{(\ell_n)}(\ell_1^{-1}\cdots\ell_k^{-1}) := r_{\ell_1}\cdots r_{\ell_k}$ & *linear* interpolation.

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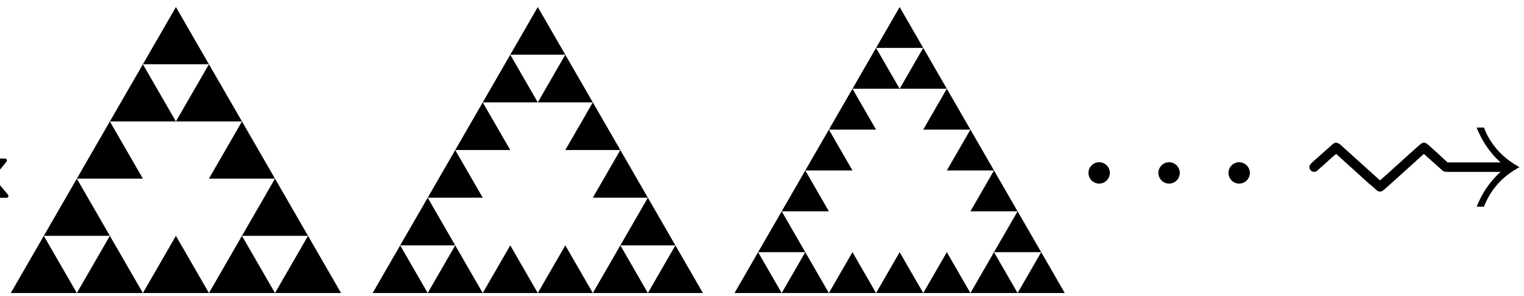
Example: *Thin* scale irregular SGs (suggested by Barlow, '19)

Mix 

$\ell = 5$ $\ell = 6$ $\ell = 7$

Thm (K. '22). For *any* $(\ell_n)_{n\geq 1}$, $(K^{(\ell_n)}_{\text{thin}}, d_{\text{geo}}, \mu_{\text{unif}}, \{X^{\text{BM}}_t\}_t)$ satisfies HKE$(\Psi^{\text{M}}_{(\ell_n)}\Psi^{\text{R}}_{(\ell_n)})$ with $\Psi^{\text{M}}_{(\ell_n)}(\ell_1^{-1}\cdots\ell_k^{-1}) := \mu_{\ell_1}\cdots\mu_{\ell_k}$, $\Psi^{\text{R}}_{(\ell_n)}(\ell_1^{-1}\cdots\ell_k^{-1}) := r_{\ell_1}\cdots r_{\ell_k}$ & *linear* interpolation.

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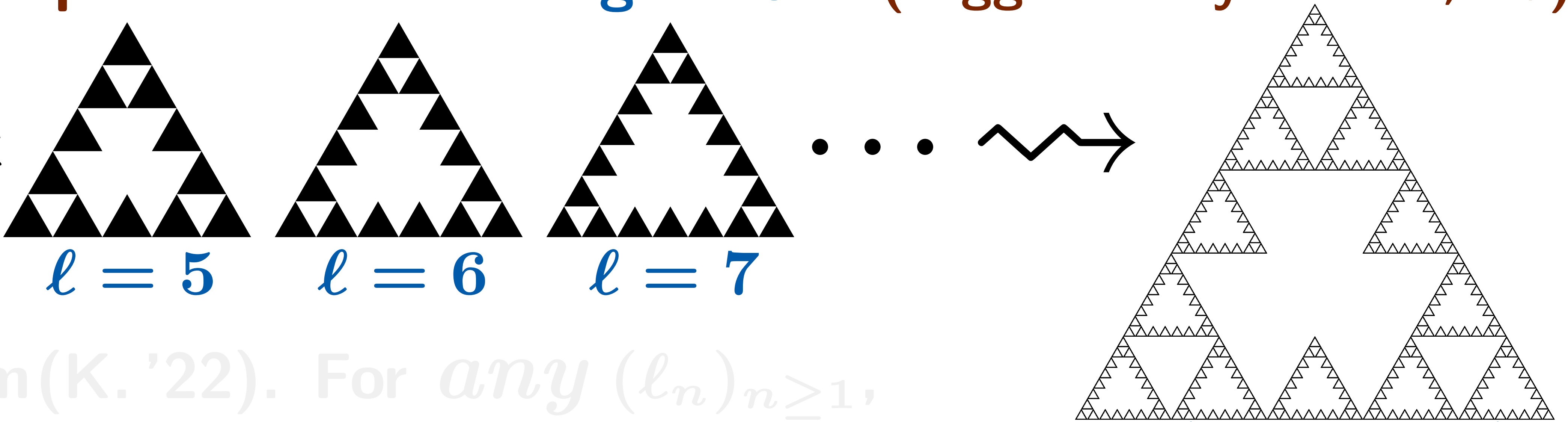
Example: *Thin* scale irregular SGs (suggested by Barlow, '19)

Mix



$\ell = 5$ $\ell = 6$ $\ell = 7$ $\cdots \rightsquigarrow$ $K_{\mathrm{thin}}^{(5,8,12,\ell_4,\ldots)}$

Thm(K. '22). For *any* $(\ell_n)_{n\geq 1}$, $(K_{\mathrm{thin}}^{(\ell_n)}, d_{\mathrm{geo}}, \mu_{\mathrm{unif}}, \{X_t^{\mathrm{BM}}\}_t)$ satisfies HKE$(\Psi_{(\ell_n)}^{\mathrm{M}} \Psi_{(\ell_n)}^{\mathrm{R}})$ with $\Psi_{(\ell_n)}^{\mathrm{M}}(\ell_1^{-1}\cdots\ell_k^{-1}) := \mu_{\ell_1}\cdots\mu_{\ell_k}$, $\Psi_{(\ell_n)}^{\mathrm{R}}(\ell_1^{-1}\cdots\ell_k^{-1}) := r_{\ell_1}\cdots r_{\ell_k}$ & *linear* interpolation.

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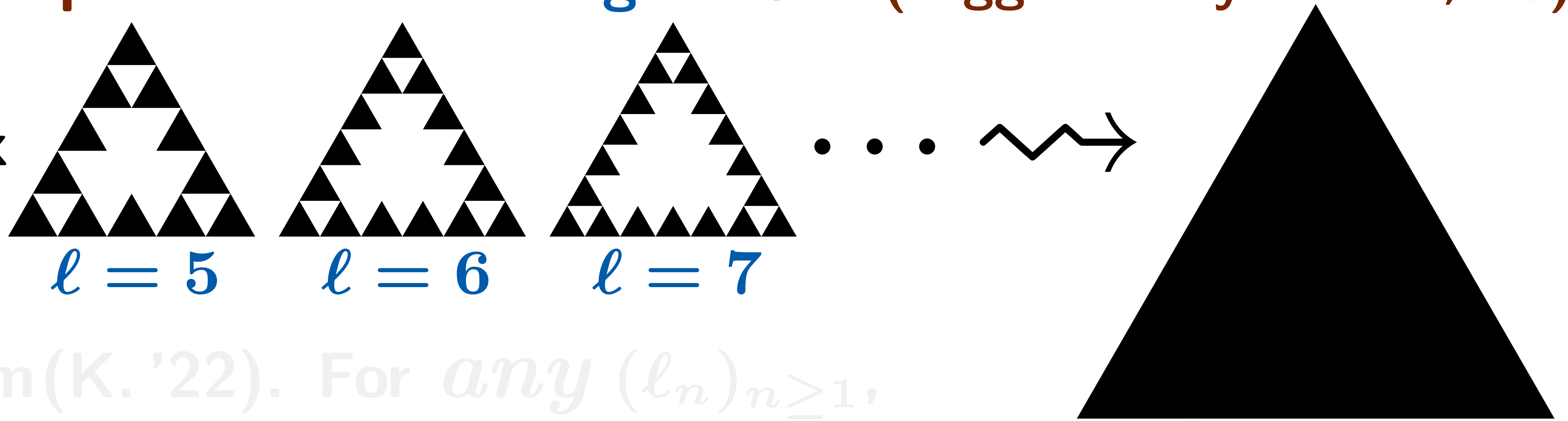
Example: *Thin* scale irregular SGs (suggested by Barlow, '19)

Mix



Thm (K. '22). For *any* $(\ell_n)_{n\ge1}$, $(K_{\mathrm{thin}}^{(\ell_n)}, d_{\mathrm{geo}}, \mu_{\mathrm{unif}}, \{X_t^{\mathrm{BM}}\}_t)$ satisfies HKE$(\Psi^{\mathrm{M}}_{(\ell_n)}, \Psi^{\mathrm{R}}_{(\ell_n)})$ with $\Psi^{\mathrm{M}}_{(\ell_n)}(\ell_1^{-1}\cdots\ell_k^{-1}) := \mu_{\ell_1}\cdots\mu_{\ell_k}$, $\Psi^{\mathrm{R}}_{(\ell_n)}(\ell_1^{-1}\cdots\ell_k^{-1}) := r_{\ell_1}\cdots r_{\ell_k}$ & *linear* interpolation.

Thm (K. '22). For *any* $(\ell_n)_{n\ge1}$, ${}^{\forall}u\in\mathcal{F}$, $\Gamma(u,u)\perp\mu$.

Conj 2 (K.–Murugan AOP '20). Assume that (K,d) is **complete** and that $\mathrm{HKE}(\Psi)$ holds. Then:

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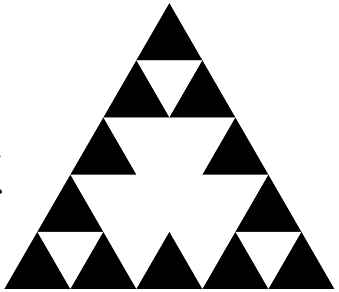
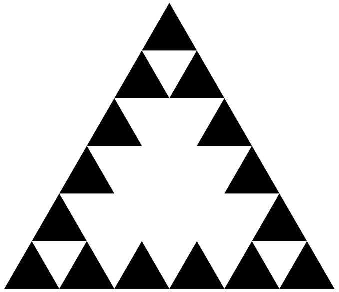
Example: *Thin* scale irregular SGs (suggested by Barlow, '19)

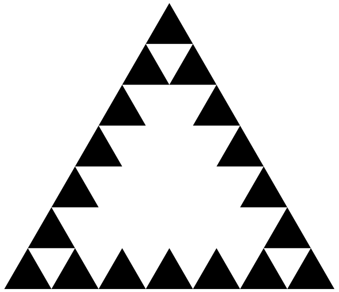
Mix



$\ell=5$

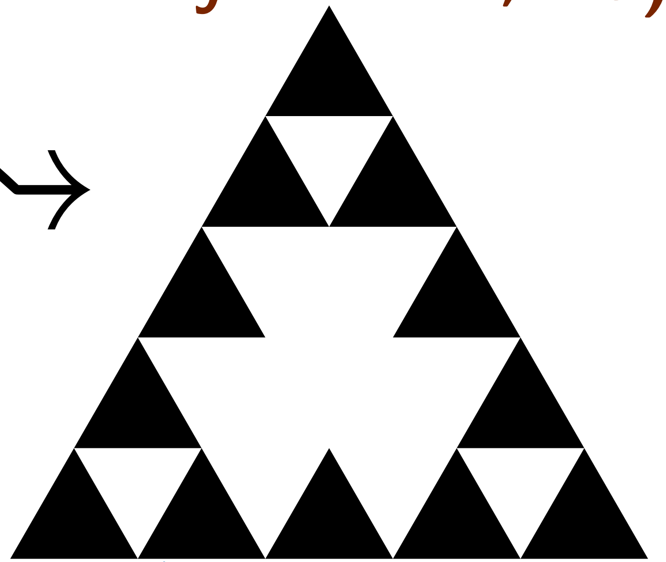


$\ell=6$



$\ell=7$

$\cdots \rightsquigarrow$



$K^{(5,}_{\text{thin}}$

Thm(K. '22). For *any* $(\ell_n)_{n\geq 1}$, $(K^{(\ell_n)}_{\text{thin}}, d_{\text{geo}}, \mu_{\text{unif}}, \{X^{\text{BM}}_t\}_t)$ satisfies $\mathrm{HKE}(\Psi^{\mathrm{M}}_{(\ell_n)}\Psi^{\mathrm{R}}_{(\ell_n)})$ with $\Psi^{\mathrm{M}}_{(\ell_n)}(\ell_1^{-1}\cdots\ell_k^{-1}):=\mu_{\ell_1}\cdots\mu_{\ell_k}$, $\Psi^{\mathrm{R}}_{(\ell_n)}(\ell_1^{-1}\cdots\ell_k^{-1}):=r_{\ell_1}\cdots r_{\ell_k}$ & *linear* interpolation.

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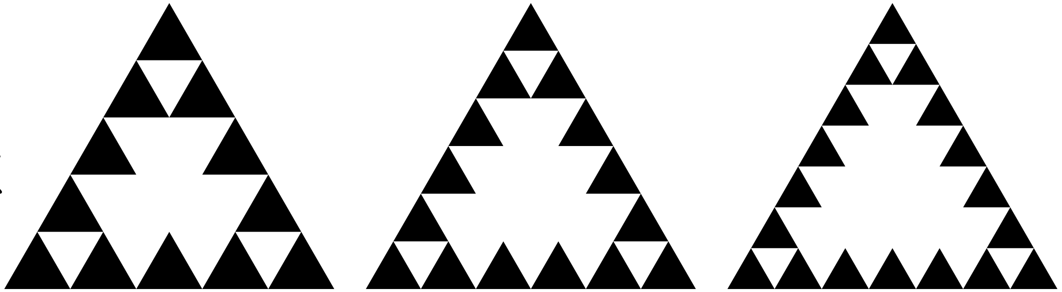
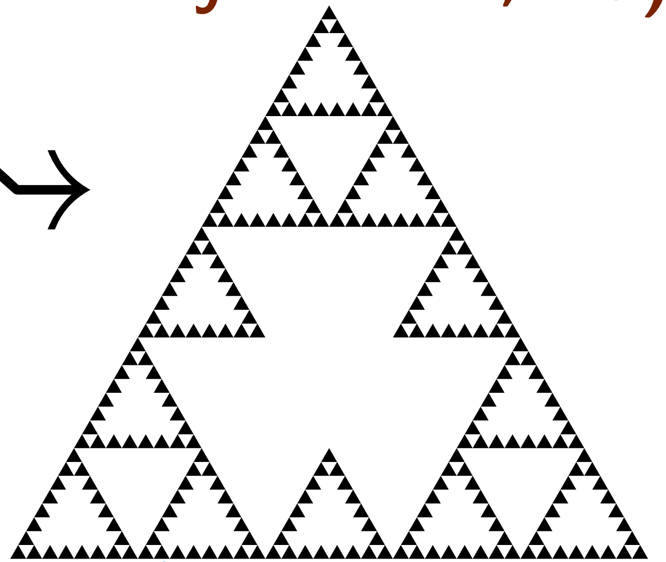
(b)^c $\lim_{r\downarrow 0} r^{-2}\Psi(r)=0 \Rightarrow {}^\forall u\in\mathcal{F}$, $\Gamma(u,u)\perp\mu$.

Example: *Thin* scale irregular SGs (suggested by Barlow, '19)

Mix  $\cdots \rightsquigarrow$ 

$\ell=5$ $\quad \ell=6$ $\quad \ell=7$

$K_{\mathrm{thin}}^{(5,8,}$

Thm(K. '22). For *any* $(\ell_n)_{n\geq 1}$, $(K_{\mathrm{thin}}^{(\ell_n)}, d_{\mathrm{geo}}, \mu_{\mathrm{unif}}, \{X_t^{\mathrm{BM}}\}_t)$ satisfies HKE($\Psi^{\mathrm{M}}_{(\ell_n)}\Psi^{\mathrm{R}}_{(\ell_n)}$) with $\Psi^{\mathrm{M}}_{(\ell_n)}(\ell_1^{-1}\cdots\ell_k^{-1}) := \mu_{\ell_1}\cdots\mu_{\ell_k}$, $\Psi^{\mathrm{R}}_{(\ell_n)}(\ell_1^{-1}\cdots\ell_k^{-1}) := r_{\ell_1}\cdots r_{\ell_k}$ & *linear* interpolation.

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Conj 2 (K.–Murugan AOP '20). Assume that (K,d) **is complete and that HKE(Ψ) holds. Then:**

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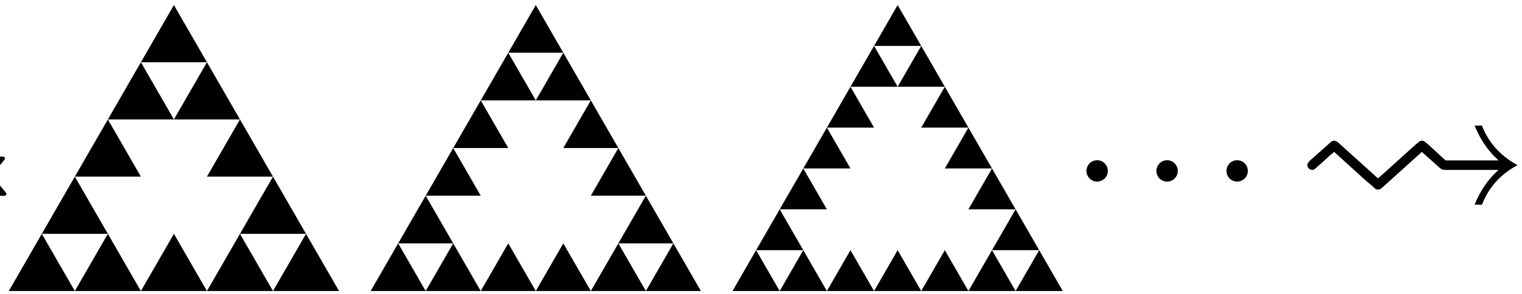
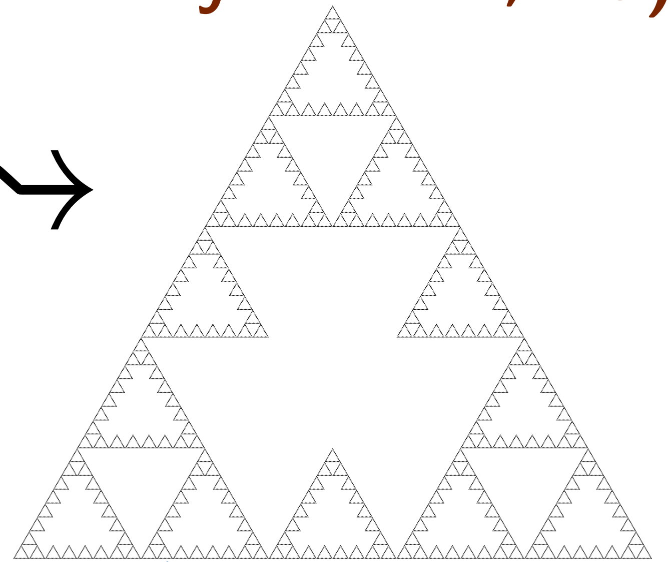


$\ell=5$ $\ell=6$ $\ell=7$



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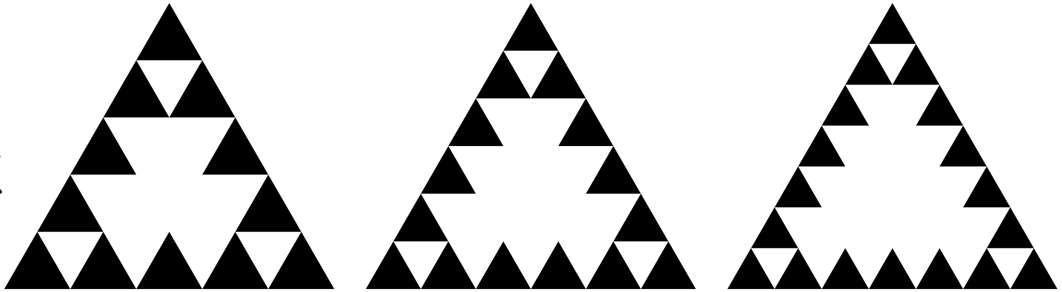

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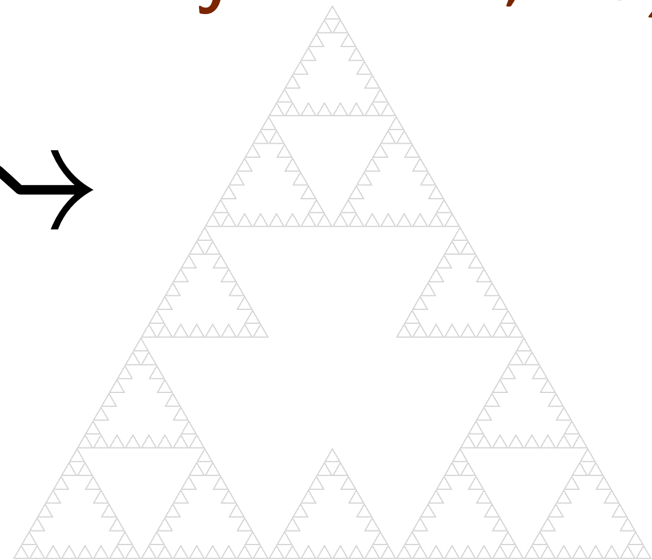

$K_{\mathrm{thin}}^{(5,8,12,\ell_4,}$

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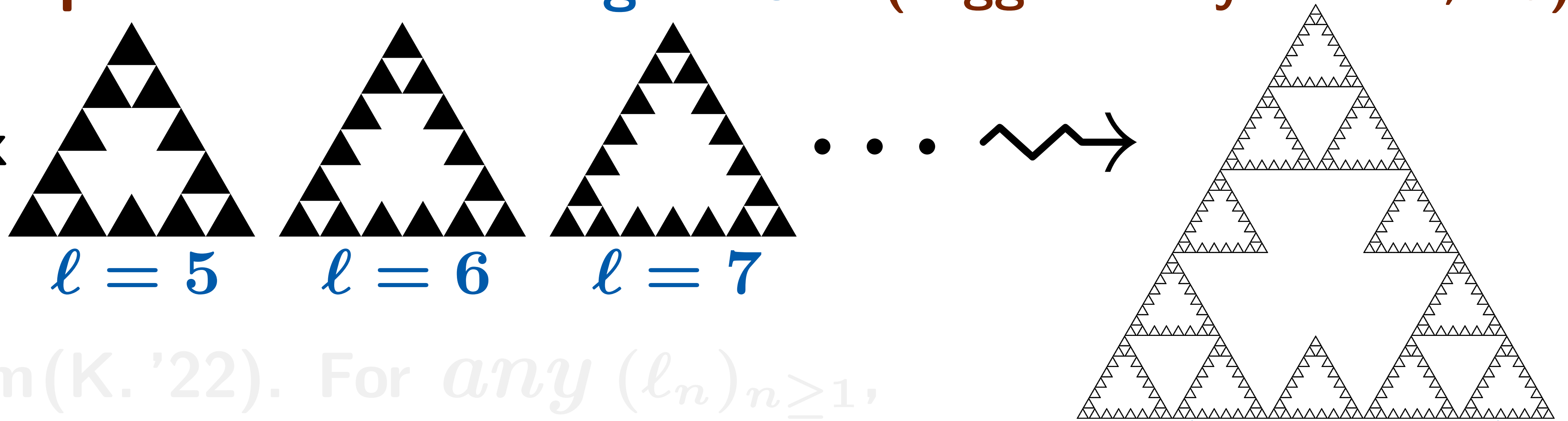
Mix 

$\ell = 5$ $\ell = 6$ $\ell = 7$ $K_{\mathrm{thin}}^{(5,8,12,\ell_4,\ldots)}$

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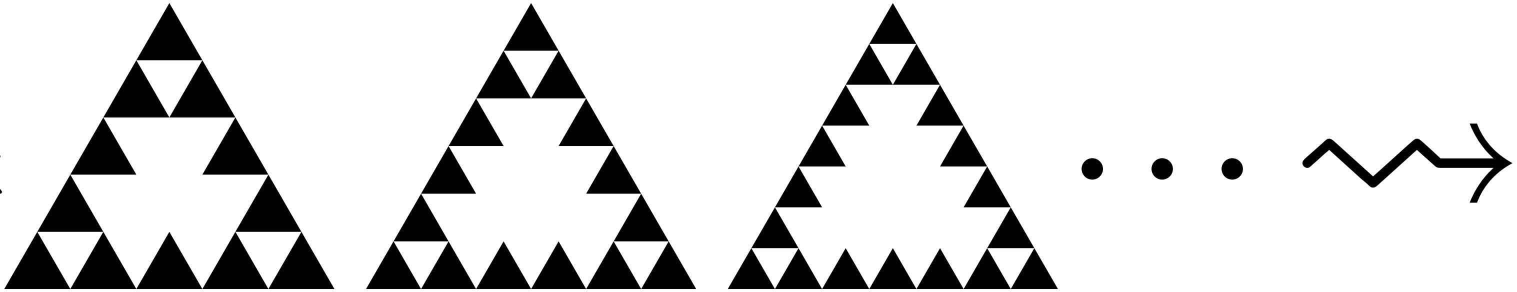

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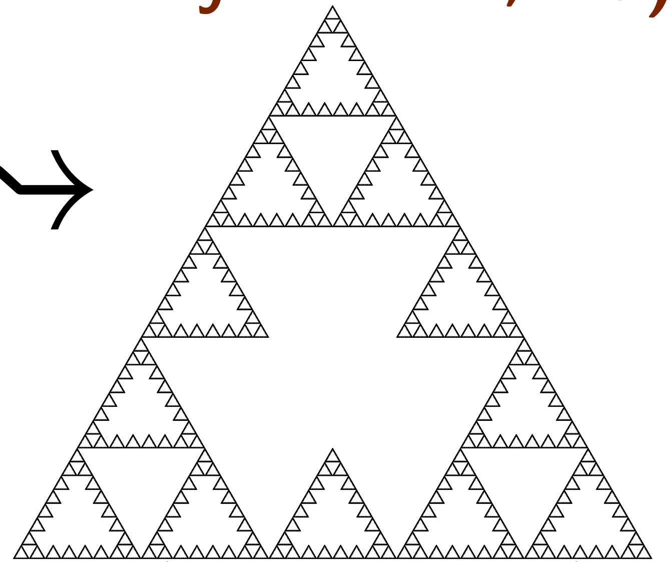

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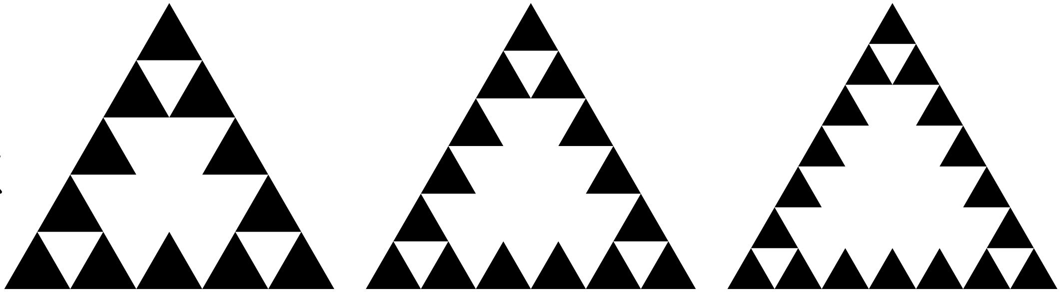

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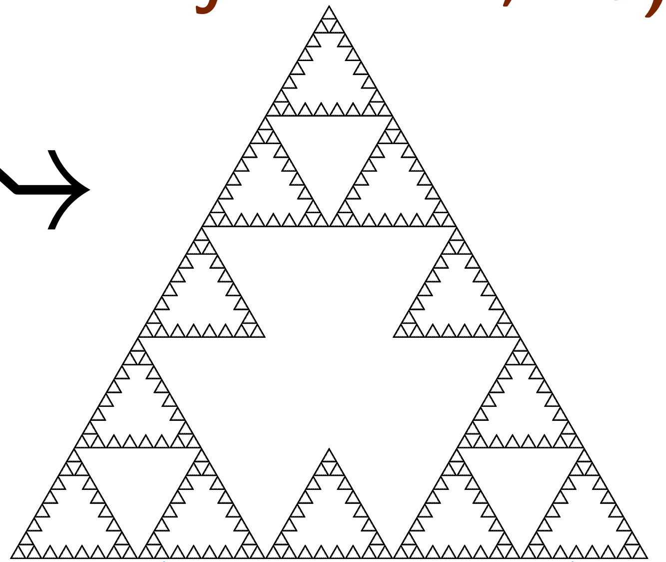

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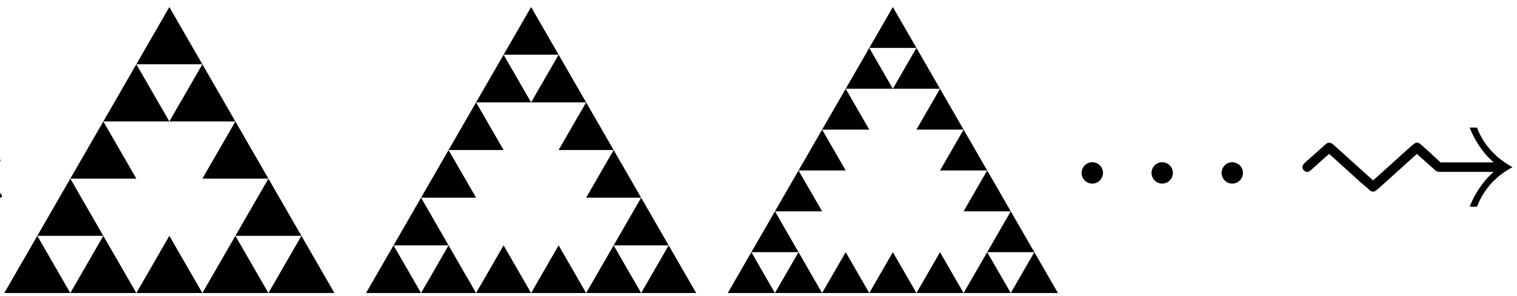
Mix 

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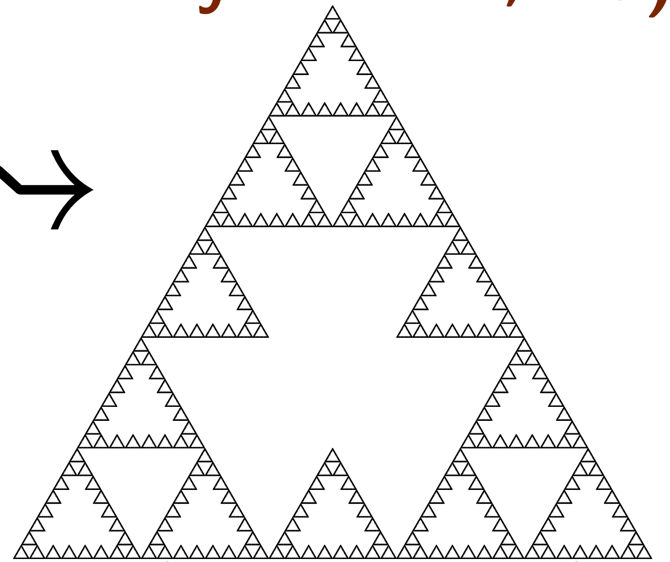

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3 General setting & conditions for HK estimates

▷ $(K, d, \mu, \mathcal{E}, \mathcal{F})$: strongly local reg. symmet. Dir. sp.
$\leftrightarrow (\{X_t\}_{t\geq 0}, \{\mathbb{P}_x\}_{x\in K_\partial})$: μ-sym. diffusion, no killing

▷ Ψ: homeo. on $[0,\infty)$ with $1 < {}^{\exists}\beta_0 \leq {}^{\exists}\beta_1$, ${}^{\exists}c \geq 1$,
$0 < {}^{\forall}r \leq {}^{\forall}R$, $c^{-1}(R/r)^{\beta_0} \leq \Psi(R)/\Psi(r) \leq c(R/r)^{\beta_1}$.

▷ $\Phi(R,t) := \Phi_\Psi(R,t) := \sup_{r>0}\bigl(R/r - t/\Psi(r)\bigr)$.

HKE(Ψ): ${}^{\exists}p_t(x,y)$, ${}^{\forall}t > 0$, (μ-almost)${}^{\forall}x, y \in K$,
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Thm(Grigor'yan–Hu–Lau '15, Andres–Barlow '15, "$\Rightarrow$qG": Murugan JFA '20, *cf.* Barlow–Bass–Kumagai '06, Lierl '15)

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Assume that (K, d) is complete. Then

↓crucial for us!

HKE$_{\mathrm{cont}}$(Ψ) $\Leftrightarrow$ VD + PI(Ψ) + CS(Ψ) + quasiGeodesic(d).

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(a) $\liminf_{\lambda\to\infty}\liminf_{r\downarrow 0}\frac{\lambda^2\Psi(r/\lambda)}{\Psi(r)}=0 \Rightarrow {}^{\forall}u\in\mathcal{F}$, $\Gamma(u,u)\perp\mu$.

● Suffices to show $\Gamma(u,u)|_V\perp\mu|_V$ when $u|_V$ is harmonic:

Lem. Let $u\in\mathcal{F}\cap\mathcal{C}_c(K)$, $u\ge 0$, $F_n:=u^{-1}(2^{-n}\mathbb{Z})$ and

Then $\Gamma(u_n)(F_n)=0$ & $\|u-u_n\|_{\mathrm{qsup}}+\mathcal{E}(u-u_n)\to 0$.

Prop (Reverse PI). If $h\in\mathcal{F}\cap L^\infty$ & $h|_{B(x,2r)}$ is harmonic,

CS(Ψ) $\Rightarrow \int_{B(x,r)}d\Gamma(h,h)\le 8c_{\mathrm{S}}\Psi(r)^{-1}\int_{B(x,2r)}h^2\,d\mu$.

cf. PI(Ψ): ${}^{\exists}c_{\mathrm{P}}>0$, ${}^{\exists}A\ge 1$, ${}^{\forall}x\in K$, ${}^{\forall}r>0$, ${}^{\forall}u\in\mathcal{F}$,

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