Harmonic and Analytic Properties of Second Generation IFS

Theory and Numerical Experiments

Dedicated to the memory of Bob Strichartz

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**IFS Convolution of measures**

\[ \nu, \mu \in \mathcal{B}([-1, 1]) \]

\[ (\nu \ast \mu)(f) = \iint f(x + \beta) \, d\mu(x) \, d\nu(\beta) \quad f \in C([-1, 1]) \]

\[ (\nu \ast_\delta \mu)(f) = \iint f(\delta x + (1 - \delta) \beta) \, d\mu(x) \, d\nu(\beta) \quad \delta \in [0, 1] \]

\[ \nu \ast_\delta \mu = \mu \ast_{\bar{\delta}} \nu, \; \nu \ast_0 \mu = \nu, \; \nu \ast_1 \mu = \mu \]

\[ \nu \ast_\delta : \mu \rightarrow \nu \ast_\delta \mu \quad \Leftarrow \text{IFS Convolution operator} \]

\[ \mathcal{T}_{\delta, \nu}(\mu) = \nu \ast_\delta \mu \]

For any \( \delta \in [0, 1) \) the above defines a transformation \( \mathcal{T}_{\delta, \nu} \) from the space \( \mathcal{M}([-1, 1]) \) of probability measures to itself. The transformation \( \mathcal{T}_{\delta, \nu} \) is a contraction in \( \mathcal{M}([-1, 1]) \) w.r.t. the weak \( \ast \) topology — i.e the Hutchinson’s metric

\[ \mathcal{T}_{\delta, \nu}(\mu_0) \rightarrow \mu^*, \; \mathcal{T}_{\delta, \nu}(\mu^*) = \mu^* \]

IFS Convolution

\[ T_{\delta, \nu}(\mu)(f) = (\nu *_{\delta} \mu)(f) = \int \int f(\delta x + (1 - \delta) \beta) \, d\mu(x) d\nu(\beta) \]

\[ T_{\delta, \nu}(\mu_0) \rightarrow \mu^*, \quad T_{\delta, \nu}(\mu^*) = \mu^* \]

\[ \nu = \sum_{j=1}^{M} \pi_j \Delta_{\beta_j} \quad \phi_{\delta}(\beta, x) = \delta x + \bar{\delta} \beta, \quad \bar{\delta} = 1 - \delta \]

\[ T_{\delta, \nu}(\mu^*)(f) = \sum_{j=1}^{M} \pi_j \int f(\delta x + \bar{\delta} \beta_j) \, d\mu^*(x) = \int f(x) \, d\mu^*(x) \]

\( \mu^* \) is the balanced measure of an \( M \)-maps homogeneous IFS with contraction ratio \( \delta \) and fixed points \( \beta_j \)

If \( \text{Supp}(\nu) \) has infinite cardinality, \( \mu^* \) is the balanced measure of an IFS with infinitely many maps.

I am convinced that this is a very natural class of measures to consider, and thinking about the usual ifs measures in this context is very illuminating. My guess is that there may be a lot more that can be said in the future about this class of measures. R.S.S.
IFS lineage

\[
T_{\delta, \nu}(\mu^*) = \mu^* \implies \mu^* = \Phi_{\delta}(\nu)
\]

For any \( \delta \in [0, 1) \) the above defines a transformation \( \Phi_{\delta} \) from the space \( \mathcal{M}([-1, 1]) \) of probability measures to itself.

\[
\nu_0 \rightarrow \Phi_{\delta}(\nu_0) = \nu_1 \rightarrow \Phi_{\delta}(\nu_1) = \Phi_{\delta}^2(\nu_0) = \nu_2 \rightarrow \ldots
\]

Initial measure  First generation IFS  Second generation IFS

\[
\nu_0 = \frac{1}{2}(\Delta_{-1} + \Delta_{-1}) \rightarrow \Phi_{\frac{1}{3}}(\nu_0) = \nu_1 \rightarrow \Phi_{\delta}(\nu_1) = \nu_2
\]

Bernoulli measure  Devil’s staircase  can’t find a good name
The first IFS in history

Phocea: Ancient Greek port in Asia Minor, the northernmost city of Ionia, established c. 1000 BC. The city was one of the first to colonize the Western Mediterranean (founded Massilia, modern Marseille). After being besieged in 545, most of its citizens left for Elea (Velia) in Italy.

Zeno of Elea (488 BC – 425 BC)
Zeno’s Cat: a one-map IFS

\[ T_{\delta,\nu}^{n} (\mu_0) \to \mu^*, \quad T_{\delta,\nu} (\mu^*) = \mu^* \]

\[ T_{\delta,\nu} (\mu^*) = \mu^* \implies \mu^* = \Phi_\delta (\nu) \]

\[ \delta = \frac{1}{2}, \quad \nu = \Delta_0, \quad \mu_0 = \Delta_x \]

\[ \mu_2 = T_{\delta,\nu} (\mu_1) = \Delta_{x/4} \]

\[ \mu_1 = T_{\delta,\nu} (\mu_0) = \Delta_{x/2} \]

\[ \mu_0 = \Delta_x \]

\[ \mu_n \to \mu^* = \Phi_{\frac{1}{2}} (\nu) = \Delta_0 = \nu \]

Zeno of Elea (488 BC – 425 BC)
Alice’s Cat(s)

\begin{align*}
\beta_0 & \\
\beta_1 & \\
x_0 & \\
x_1 & \\
x_2 & \\
x_i & \rightarrow \delta(x_i - \beta_i) + \beta_i = x_{i+1} \\
\beta_i & \text{ is sampled from the distribution } \nu_0(\beta) \\
\text{The Bernoulli cat: } \nu_0 = \frac{1}{2}(\Delta_{-1} + \Delta_1) \\
\frac{1}{N} \sum_{i=0}^{N-1} \Delta x_i & \rightarrow \mu^* = \Phi_\delta(\nu_0) = \nu_1 \\
\nu_0 = \frac{1}{2}(\Delta_{-1} + \Delta_1) & \rightarrow \Phi_{\frac{1}{3}}(\nu_0) = \nu_1 \rightarrow \Phi_\delta(\nu_1) = \nu_2
\end{align*}
Alice's Cat

\[ \beta_1 x x x 2 0 x 1 \beta_0 z 1 \]

\[ x_i \rightarrow \delta (x_i - \beta_i) + \beta_i = x_{i+1} \]

\( \beta_i \) is sampled from the distribution \( \nu_0(\beta) \)

The Bernoulli cat: \( \nu_0 = \frac{1}{2}(\Delta_{-1} + \Delta_1) \)

\[ \frac{1}{N} \sum_{i=0}^{N-1} \Delta x_i \rightarrow \mu^* = \Phi_\delta(\nu_0) = \nu_1 \]

\[ \nu_0 = \frac{1}{2}(\Delta_{-1} + \Delta_1) \rightarrow \Phi_{\frac{1}{3}}(\nu_0) = \nu_1 \rightarrow \Phi_\delta(\nu_1) = \nu_2 \]

\[ z_i \rightarrow \delta (z_i - x_i) + x_i = z_{i+1} \]

\[ \frac{1}{N} \sum_{i=0}^{N-1} \Delta z_i \rightarrow \Phi_\delta(\nu_1) = \nu_2 \]
\[ m_1(x) = \int^x dv_1(s) \]
\[ m_2(x) = \int^x dv_2(s) \]

\( \nu_0 \) is a pure-point measure, \\
\( \nu_1 \) is s.c. supported on the Cantor set \( K \).
What are the properties of \( \nu_2 \) ?
The support of the measure

\[ m_1(x) = \int_0^x d\nu_1(s) \]

\[ m_2(x) = \int_0^x d\nu_2(s) \]

Thm. Let \( \text{Supp}(\nu_i), \text{Supp}(\nu_{i+1}) \) be the support of \( \nu_i \) and \( \nu_{i+1} \). For any \( \delta > 0 \), \( \text{Supp}(\nu_i) \subset \text{Supp}(\nu_{i+1}) \subset B_{L\delta}(\text{Supp}(\nu_i)) \).
**The support of the measure**

\[ A \oplus B = \{ a + b, \ a \in A, b \in B \} \]
\[ \delta A = \{ \delta a, \ a \in A \} \]
\[ \text{Supp}(\nu_{i+1}) = \bigoplus_{j=0}^{\infty} \delta^j \text{Supp}(\nu_i) \]
\[ \text{Supp}(\nu_1) = \bigoplus_{j=0}^{\infty} \frac{1}{3}^j \{ -1, 1 \} = K \]
\[ \text{Supp}(\nu_2) = \bigoplus_{j=0}^{\infty} \delta^j K = \bigcup_{l=1}^{N} I_l \]

Thm. M. - Peirone (2017) Let \( \sum_{j=0}^{\infty} \alpha_j \) be a convergent series of real positive entries and let \( K \) be a non–empty compact set admitting a construction of uniformly lower-bounded dissection. Then:

The series \( \bigoplus_{j=0}^{\infty} \alpha_j K \) is convergent.

Any permutation of its terms yields the same value for the sum of the series, which is a finite union of closed intervals.
The nature of the measure

\[ \nu_0 \rightarrow \Phi_\delta(\nu_0) = \nu_1 \rightarrow \Phi_\delta(\nu_1) = \Phi^2_\delta(\nu_0) = \nu_2 \rightarrow \ldots \]

Thm. The measures \( \nu_i, \ i \geq 1 \) are of pure type.

Thm. If \( \nu_i \) is a.c. with bounded density, so is \( \nu_{i+1} \).

Example. When \( \nu_0 \) is the Lebesgue measure on \([-1, 1]\), \( \nu_1 \) is a.c. and its density \( \rho(\nu_1) \) is infinitely differentiable.
The P-F-R operator

\[ \nu_0 = \frac{1}{2} (\Delta_{-1} + \Delta_{-1}) \rightarrow \Phi_2(\nu_0) = \nu_1 \rightarrow \Phi_\delta(\nu_1) = \nu_2 \]

\[ \mathcal{P}_{\delta,\nu_1}(\rho)(x) = \frac{1}{\delta} \int d\nu_1(\beta) \rho\left(\frac{x - \bar{\delta}_\beta}{\delta}\right) \]

\[ d_n = \| \mathcal{P}^{n+1}_{\delta,\nu_1}(\rho) - \mathcal{P}^n_{\delta,\nu_1}(\rho) \| \sim \| \cdot \|_{\text{Var}}, \| \cdot \|_1 \]

\[ \delta = \frac{2}{10} \]

\[ \delta = \frac{1}{10} \]

Nonlinear Self–similar Measures and their Fourier Transforms

David Glickenstein & Robert S. Strichartz
Numerical Experiments in Fourier Asymptotics of Cantor Measures and Wavelets

Prem Janardhan, David Rosenblum and Robert S. Strichartz

\( \hat{\mu}(t) = \int d\mu(s)e^{-its} \)

\[(\nu *_{\delta} \mu)(f) = \int \int f(\delta x + (1-\delta)\beta) \, d\mu(x) \, d\nu(\beta) \]

\[(\nu \hat{*}_{\delta} \mu)(t) = \hat{\nu}(\delta t) \hat{\mu}(\delta t) \quad \text{Elton and Yan (1989)} \]

\[\hat{\mu}(t) = \hat{\nu}(\delta t) \hat{\mu}(\delta t) = \hat{\nu}(\delta t) \hat{\nu}(\delta \delta t) \hat{\mu}(\delta^2 t) = \prod_{j=0}^{\infty} \hat{\nu}(\delta \delta^j t) \]

\[M_1(|\hat{\mu}|^2; z) := \int_1^\infty t^{z-1}|\hat{\mu}(t)|^2 \, dt \]

\[d_S(\mu) := \sup\{s \in \mathbb{R} \text{ s.t. } M_1(|\hat{\mu}|^2; s) < \infty\} \]

\[d_S(\mu) \leq 1 \Rightarrow d_S(\mu) = D_2(\mu) \]

\[d_S(\mu) > 1 \Rightarrow \mu \text{ a.c. with density in } L^2, \]

\[d_S(\mu) > 2 \Rightarrow \mu \text{ a.c. with a continuous density} \]
Numerical Experiments

\[ M_1(|\hat{\mu}|^2; z) := \int_1^\infty t^{z-1} |\hat{\mu}(t)|^2 \, dt \quad \tau = \frac{\log(t)}{h} \]

\[ M_1(|\hat{\mu}|^2; z) = h \int_0^\infty e^{zh\tau} |\hat{\mu}(e^{h\tau})|^2 \, d\tau \]

\[ = h \int_0^\infty e^{(zh+1)\tau} |\hat{\mu}(e^{h\tau})|^2 \, e^{-\tau} \, d\tau \quad \text{Laguerre integration} \]

\[ = h \sum_{i=1}^n e^{(zh+1)\theta_i^n} |\hat{\mu}(e^{h\theta_i^n})|^2 w_i^n \quad \text{Laguerre points and weights} \]

\[ M_1(z) \]

\[ h \]

\[ z \]
Pade' extrapolation

\[ M_1(\vert \hat{\mu} \vert^2; z) \equiv \int_{1}^{\infty} t^{z-1} \vert \hat{\mu}(t) \vert^2 \, dt \]

\[ M_1(\vert \hat{\mu} \vert^2; z_l) = \frac{P_{m-1}(z_l)}{Q_m(z_l)}, \quad l = 1, \ldots, 2m \]

\[ d_S(\mu) \equiv \sup\{s \in \mathbb{R} \text{ s.t. } M_1(\vert \hat{\mu} \vert^2; s) < \infty\} \]

\[ d_S(\mu) \sim \text{smallest real zero of } Q_m(z) \]

\[ \nu_0 = \frac{1}{2}(\Delta_{-1} + \Delta_{-1}) \rightarrow \Phi_{1/2}(\nu_0) = \nu_1 \]

\[ \hat{\nu}_1(t) = \frac{\sin(t)}{t} \Rightarrow d_S(\nu_1) = 2 \]

Multipoint Pade' Lebesgue measure
\[ M_1(|\hat{\mu}|^2; z) := \int_1^\infty t^{z-1} |\hat{\mu}(t)|^2 \, dt \]
\[ \nu(t) = \frac{\sin(t)}{t} \Rightarrow d_S(\nu_1) = 2 \]
Pade' extrapolation

\[ M_1(|\hat{\mu}|^2; z) := \int_1^\infty t^{z-1}|\hat{\mu}(t)|^2 \, dt \]

\[ \nu_1(t) = \frac{\sin(t)}{t} \Rightarrow d_S(\nu_1) = 2 \quad M_1(|\hat{\mu}|^2; z) \sim \frac{P_{m-1}(z)}{Q_m(z)} \]

Zeros of \( Q_4 \):

\[ z = 1.9979.. \]

Zeros of \( P_3 \)
Pade’ extrapolation

\[ M_1(|\hat{\mu}|^2; z) := \int_1^\infty t^{z-1}|\hat{\mu}(t)|^2 \, dt \]

\[ M_1(|\hat{\mu}|^2; z_l) = \frac{P_{m-1}(z_l)}{Q_m(z_l)}, \quad l = 1, \ldots, 2m \]

\[ d_S(\mu) := \sup\{s \in \mathbb{R} \text{ s.t. } M_1(|\hat{\mu}|^2; s) < \infty\} \]

\[ d_S(\mu) \sim \text{smallest real zero of } Q_m(z) \]

\[ \nu_0 = \frac{1}{2}(\Delta_{-1} + \Delta_{-1}) \quad \rightarrow \quad \Phi_{1/2}(\nu_0) = \nu_1 \]

\[ \nu_1(t) = \frac{\sin(t)}{t} \quad \Rightarrow \quad d_S(\nu_1) = 2 \]

\[ \nu_0 = \frac{1}{2}(\Delta_{-1} + \Delta_{-1}) \quad \rightarrow \quad \Phi_{1/3}(\nu_0) = \nu_1 \]

\[ \nu_1(t) = \prod_{j=0}^{\infty} \cos\left(\frac{2}{3j+1}t\right) \quad \Rightarrow \quad d_S(\nu_1) = \frac{\log 2}{\log 3} \]
\[ M_1(\vert \hat{\mu} \vert^2; z) := \int_1^\infty t^{z-1} \vert \hat{\mu}(t) \vert^2 \, dt \]

\[ \tilde{\nu}_1(t) = \prod_{j=0}^{\infty} \cos\left( \frac{2}{3j+1} t \right) \Rightarrow d_S(\nu_1) = \frac{\log 2}{\log 3} \]
\[ M_1(\vert \hat{\mu} \vert^2; z) := \int_1^\infty t^{z-1} \vert \hat{\mu}(t) \vert^2 \, dt \]

\[ \tilde{\nu}_1(t) = \prod_{j=0}^\infty \cos\left( \frac{2}{3j+1} t \right) \Rightarrow d_S(\nu_1) = \frac{\log 2}{\log 3} \quad \Rightarrow M_1(\vert \hat{\mu} \vert^2; z) \approx \frac{P_{m-1}(z)}{Q_m(z)} \]
\[ M_1(|\hat{\mu}|^2; z) := \int_1^\infty t^{z-1}|\hat{\mu}(t)|^2 \, dt \]

\[ M_1(|\hat{\mu}|^2; z_l) = \frac{P_{m-1}(z_l)}{Q_m(z_l)}, \quad l = 1, \ldots, 2m \]

\[ d_S(\mu) := \sup\{s \in \mathbb{R} \text{ s.t. } M_1(|\hat{\mu}|^2; s) < \infty\} \]

\[ d_S(\mu) \sim \text{smallest real zero of } Q_m(z) \]

\[ \nu_0 = \frac{1}{2}(\Delta_{-1} + \Delta_{-1}) \quad \rightarrow \quad \Phi_{1/2}(\nu_0) = \nu_1 \]

\[ \nu_0 = \frac{1}{2}(\Delta_{-1} + \Delta_{-1}) \quad \rightarrow \quad \Phi_{1/3}(\nu_0) = \nu_1 \]

\[ \nu_0 \quad \rightarrow \quad \Phi_\delta(\nu_0) = \nu_1 \quad \rightarrow \quad \Phi_\delta(\nu_1) = \nu_2 \]
\[ \nu_0 \rightarrow \Phi_\delta(\nu_0) = \nu_1 \rightarrow \Phi_\delta(\nu_1) = \nu_2 \]
\[ \nu_0 = \frac{1}{2}(\Delta_{-1} + \Delta_{-1}) \]

1st gen IFS
Devil’s staircase

Supp (\nu_1) \subset \text{Supp}(\nu_2) \subset B_{2\delta}(\text{Supp}(\nu_1)).

\[ \tilde{\nu}_1(t) = \prod_{j=0}^{\infty} \cos(\bar{\delta}^j t) \Rightarrow d_S(\nu_1) = -\frac{\log 2}{\log \delta} \]
\[ \tilde{\nu}_2(t) = \prod_{j,k=0} \cos(\bar{\delta}^2 \delta^j + kt) \Rightarrow d_S(\nu_2) =? \]

\[ d_S(\nu_2) = D_2(\nu_2) < 1 \Rightarrow \nu_2 \ s.c.
\]
\[ d_S(\nu_2) > 1 \Rightarrow \nu_2 \ a.c. \ with \ density \ in \ L^2, \]
\[ d_S(\nu_2) > 2 \Rightarrow \nu_2 \ a.c. \ with \ a \ continuous \ density \]
\[ M_1(|\hat{\mu}|^2; z) := \int_1^\infty t^{z-1}|\hat{\mu}(t)|^2 \, dt \]

\[ M_1(z) = \int_1^\infty t^{z-1} M_1(t) \, dt \]

\[ \Delta(z) \]

\[ z \]

\[ \delta = 1/10 \]
\[ d_S \simeq .61 \]

\[ \delta = 2/10 \]
\[ d_S \simeq 1.08 \]

\[ \delta = 4/10 \]
\[ d_S \geq 2 \]

Conjecture: there is a transition value of \( \delta \) from s.c. to a.c.
Conclusions

\[ T_{\delta, \nu}(\mu^*) = \mu^* \implies \mu^* = \Phi_\delta(\nu) \]

For any \( \delta \in [0, 1) \) the above defines a transformation \( \Phi_\delta \) from the space \( \mathcal{M}([-1, 1]) \) of probability measures to itself

\[ \nu_0 \rightarrow \Phi_\delta(\nu_0) = \nu_1 \rightarrow \Phi_\delta(\nu_1) = \Phi_\delta^2(\nu_0) = \nu_2 \rightarrow \ldots \]

Initial measure  First generation IFS  Second generation IFS

\[ \nu_0 = \frac{1}{2} (\Delta_{-1} + \Delta_{-1}) \rightarrow \Phi_{\frac{1}{3}}(\nu_0) = \nu_1 \rightarrow \Phi_{\frac{1}{3}}(\nu_1) = \nu_2 \]

Atomic measure  singular continuous  absolutely continuous

To be continued...


