Center for Nonlinear and Complex Systems

Dipartimento di Fisica e Matematica

Harmonic and Analytic Properties of Second Generation IFS

Theory and Numerical Experiments

Dedicated to the memory of Bob Strichartz

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IFS Convolution of measures

$$\nu, \mu \in \mathcal{B}([-1,1])$$

$$(\nu * \mu)(f) = \iint f(x+\beta)d\mu(x)d\nu(\beta) \quad f \in C([-1,1])$$

$$(\nu *_\delta \mu)(f) = \iint f(\delta x + (1-\delta)\beta)\, d\mu(x)d\nu(\beta) \qquad \delta \in [0,1]$$

$$\nu *_\delta \mu = \mu *_{\bar{\delta}} \nu, \;\; \nu *_0 \mu = \nu, \;\; \nu *_1 \mu = \mu \qquad \bar{\delta} = 1 - \delta$$

$$\nu *_\delta : \mu \longrightarrow \nu *_\delta \mu \quad \Longleftarrow \text{IFS Convolution operator}$$

$$\mathcal{T}_{\delta,\nu}(\mu) = \nu *_\delta \mu$$

For any $\delta \in [0,1)$ the above defines a transformation $\mathcal{T}_{\delta,\nu}$ from the space $\mathcal{M}([-1,1])$ of probability measures to itself
The transformation $\mathcal{T}_{\delta,\nu}$ is a contraction in $\mathcal{M}([-1,1])$ w.r.t. the weak $*$ topology $-$ *i.e* the Hutchinson's metric

$$\mathcal{T}^n_{\delta,\nu}(\mu_0) \to \mu^*, \;\; \mathcal{T}_{\delta,\nu}(\mu^*) = \mu^*$$

Hutchinson 1981, Barnsley Demko 1985, Elton 1989, Mendivil 2000

IFS Convolution

$$\mathcal{T}_{\delta,\nu}(\mu)(f) = (\nu *_\delta \mu)(f) = \iint f(\delta x + (1-\delta)\beta)\, d\mu(x) d\nu(\beta)$$

$$\mathcal{T}^n_{\delta,\nu}(\mu_0) \to \mu^*, \quad \mathcal{T}_{\delta,\nu}(\mu^*) = \mu^*$$

$$\nu = \sum_{j=1}^{M} \pi_j \Delta_{\beta_j} \qquad \phi_\delta(\beta, x) = \delta x + \bar{\delta}\beta, \quad \bar{\delta} = 1 - \delta$$

$$\mathcal{T}_{\delta,\nu}(\mu^*)(f) = \sum_{j=1}^{M} \pi_j \int f(\delta x + \bar{\delta}\beta_j)\, d\mu^*(x) = \int f(x)\, d\mu^*(x)$$

μ^* is the balanced measure of an M-maps homogeneous IFS with contraction ratio δ and fixed points β_j

If Supp(ν) has infinite cardinality, μ^* is the balanced measure of an IFS with infinitely many maps.

I am convinced that this is a very natural class of measures to consider, and thinking about the usual ifs measures in this context is very illuminating. My guess is that there may be a lot more that can be said in the future about this class of measures. R.S.S.

IFS lineage

$$\mathcal{T}_{\delta,\nu}(\mu^*) = \mu^* \Longrightarrow \mu^* = \Phi_\delta(\nu)$$

For any $\delta \in [0,1)$ the above defines a transformation Φ_δ from the space $\mathcal{M}([-1,1])$ of probability measures to itself

$$\nu_0 \longrightarrow \Phi_\delta(\nu_0) = \nu_1 \longrightarrow \Phi_\delta(\nu_1) = \Phi_\delta^2(\nu_0) = \nu_2 \longrightarrow \dots$$

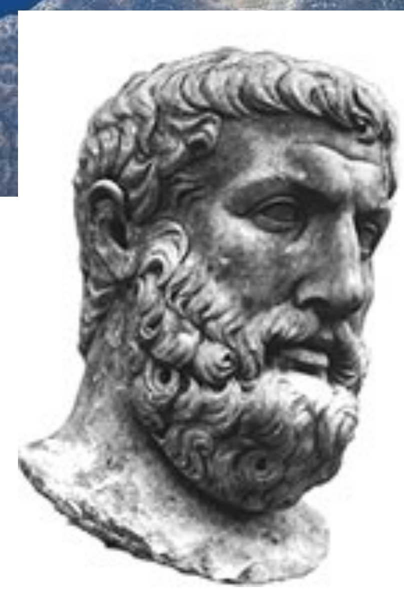
↑ Initial measure (for ν_0) ↑ First generation IFS (for ν_1) ↑ Second generation IFS (for ν_2)

$$\nu_0 = \frac{1}{2}(\Delta_{-1} + \Delta_{-1}) \longrightarrow \Phi_{\frac{1}{3}}(\nu_0) = \nu_1 \longrightarrow \Phi_\delta(\nu_1) = \nu_2$$

↑ Bernoulli measure (for ν_0) ↑ Devil's staircase (for ν_1) ↑ can't find a good name (for ν_2)

The first IFS in history

Phocea: Ancient Greek port in Asia Minor, the northernmost city of Ionia, established c. 1000BC. The city was one of the first to colonize the Western Mediterranean (founded Massilia, modern Marseille). After being besieged in 545, most of its citizens left for Elea (Velia) in Italy.



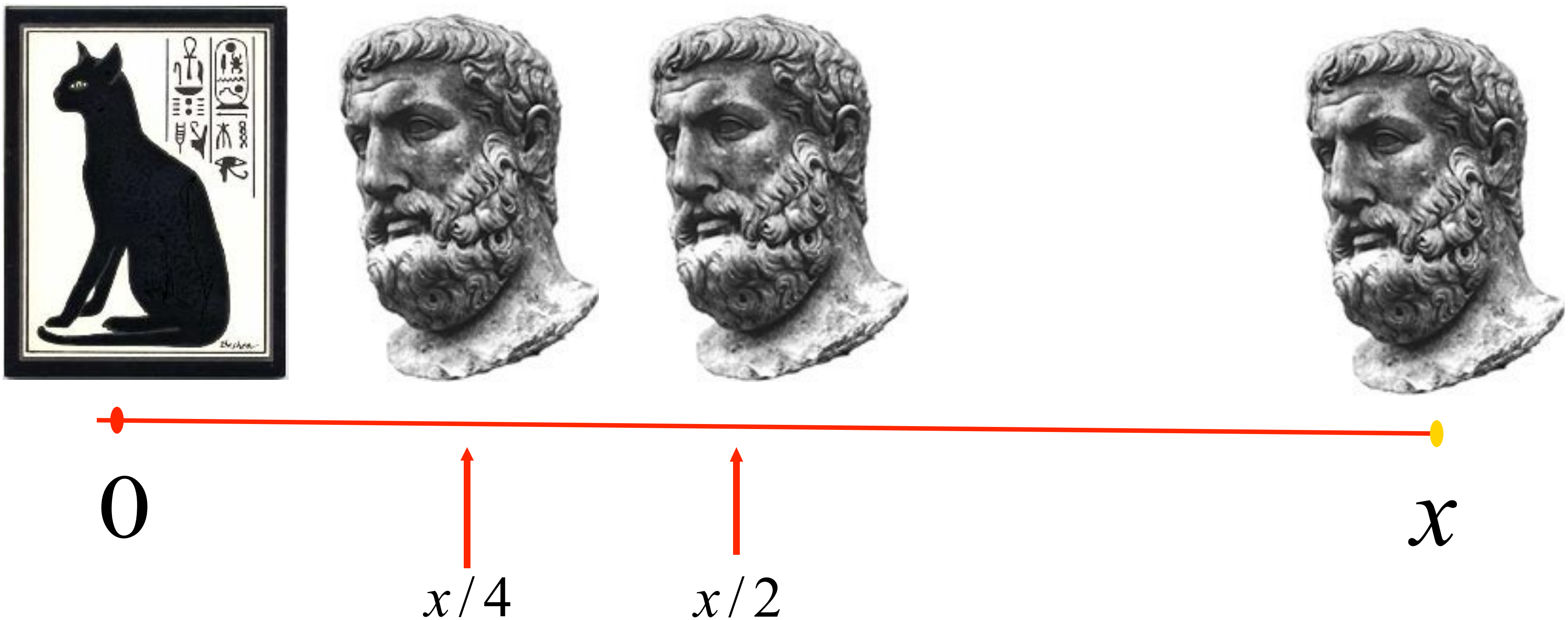
Zeno of Elea (488 BC – 425 BC)

Zeno's Cat: a one-map IFS

$$\mathcal{T}^n_{\delta,\nu}(\mu_0) \to \mu^*, \quad \mathcal{T}_{\delta,\nu}(\mu^*) = \mu^*$$

$$\mathcal{T}_{\delta,\nu}(\mu^*) = \mu^* \Longrightarrow \mu^* = \Phi_\delta(\nu)$$

$$\delta = \frac{1}{2}, \quad \nu = \Delta_0, \quad \mu_0 = \Delta_x$$



$$\mu_2 = \mathcal{T}_{\delta,\nu}(\mu_1) = \Delta_{x/4} \qquad \mu_1 = \mathcal{T}_{\delta,\nu}(\mu_0) = \Delta_{x/2} \qquad \mu_0 = \Delta_x$$

$$\mu_n \to \mu^* = \Phi_{\frac{1}{2}}(\nu) = \Delta_0 = \nu$$

Zeno of Elea (488 BC – 425 BC)

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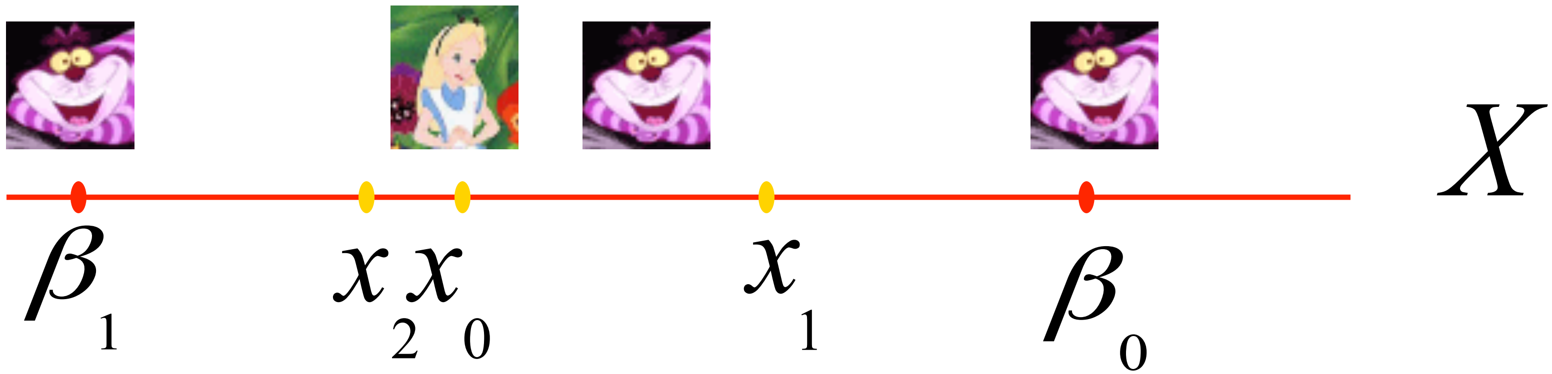
Alice's Cat(s)



$$x_i \to \delta(x_i - \beta_i) + \beta_i = x_{i+1}$$

β_i is sampled from the distribution $\nu_0(\beta)$

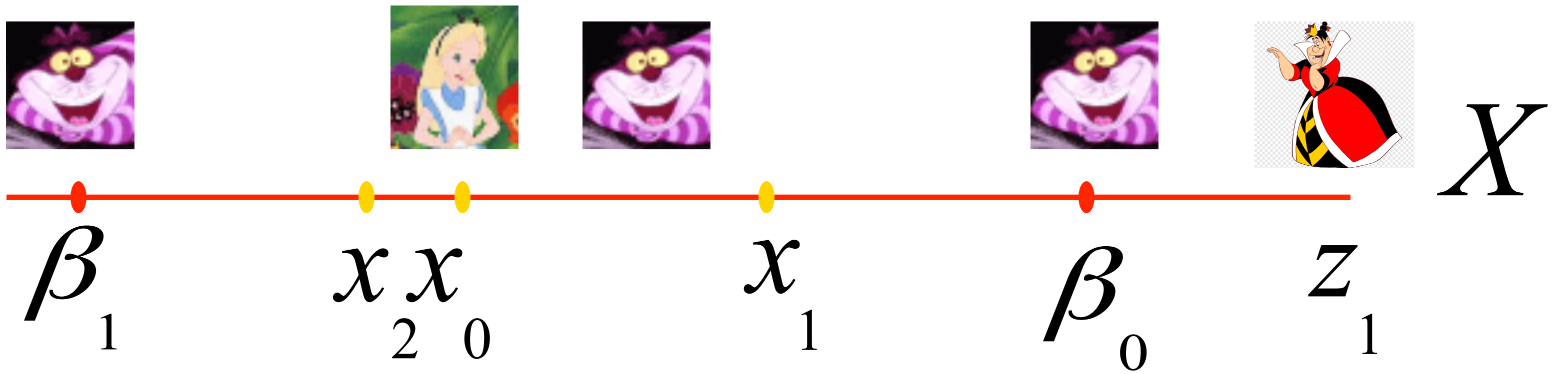
The Bernoulli cat: $\nu_0 = \frac{1}{2}(\Delta_{-1} + \Delta_1)$

$$\frac{1}{N}\sum_{i=0}^{N-1} \Delta_{x_i} \to \mu^* = \Phi_\delta(\nu_0) = \nu_1$$

$$\nu_0 = \frac{1}{2}(\Delta_{-1} + \Delta_1) \longrightarrow \Phi_{\frac{1}{3}}(\nu_0) = \nu_1 \longrightarrow \Phi_\delta(\nu_1) = \nu_2$$

Cheshire cat | Alice | who is this ?

Alice's Cat



$$x_i \to \delta(x_i - \beta_i) + \beta_i = x_{i+1}$$

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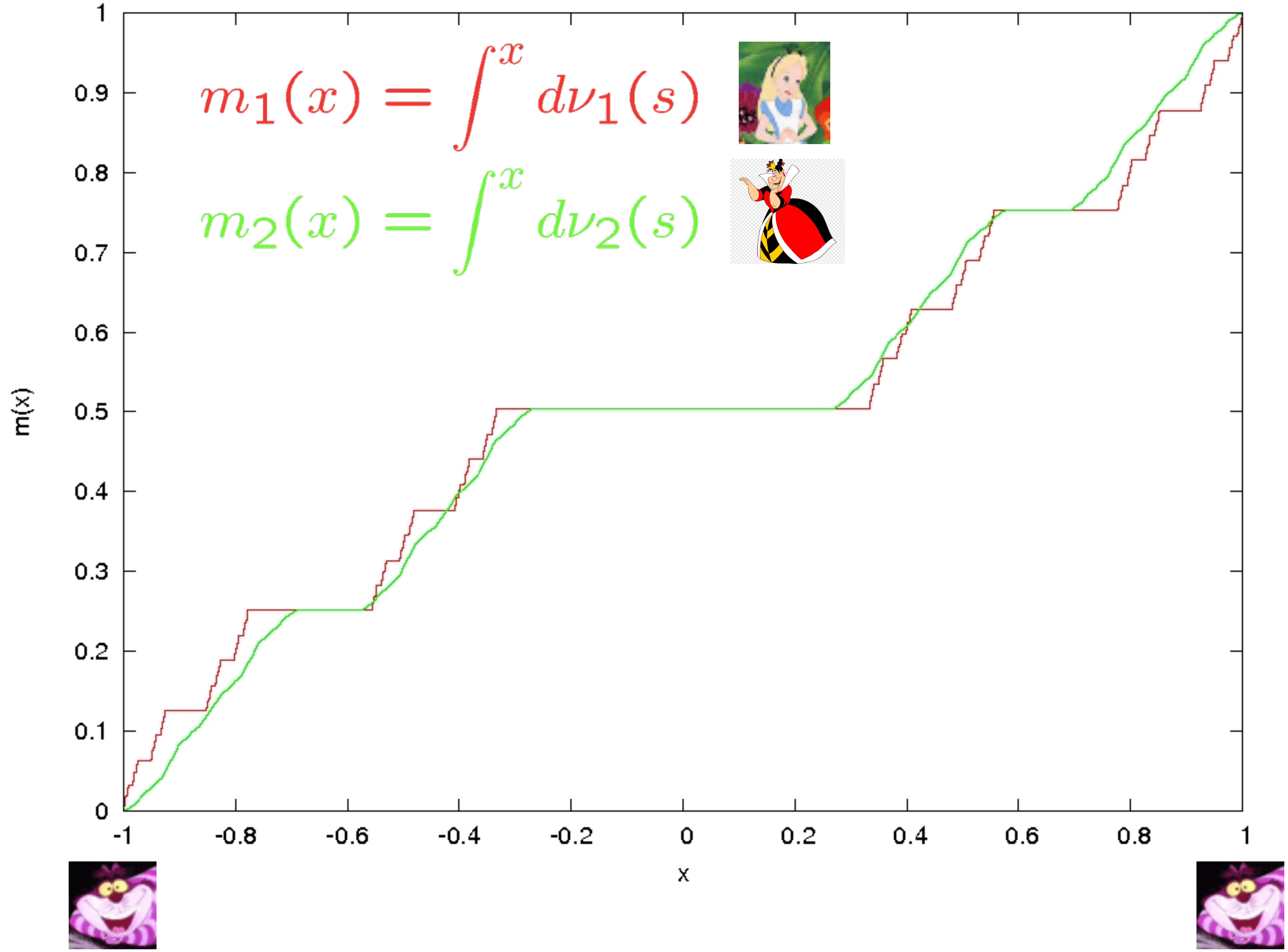
The Bernoulli cat: $\nu_0 = \frac{1}{2}(\Delta_{-1} + \Delta_1)$

$$\frac{1}{N}\sum_{i=0}^{N-1} \Delta_{x_i} \to \mu^* = \Phi_\delta(\nu_0) = \nu_1$$

$$\nu_0 = \frac{1}{2}(\Delta_{-1} + \Delta_1) \longrightarrow \Phi_{\frac{1}{3}}(\nu_0) = \nu_1 \longrightarrow \Phi_\delta(\nu_1) = \nu_2$$

$$z_i \to \delta(z_i - x_i) + x_i = z_{i+1}$$

$$\frac{1}{N}\sum_{i=0}^{N-1} \Delta_{z_i} \to \Phi_\delta(\nu_1) = \nu_2$$

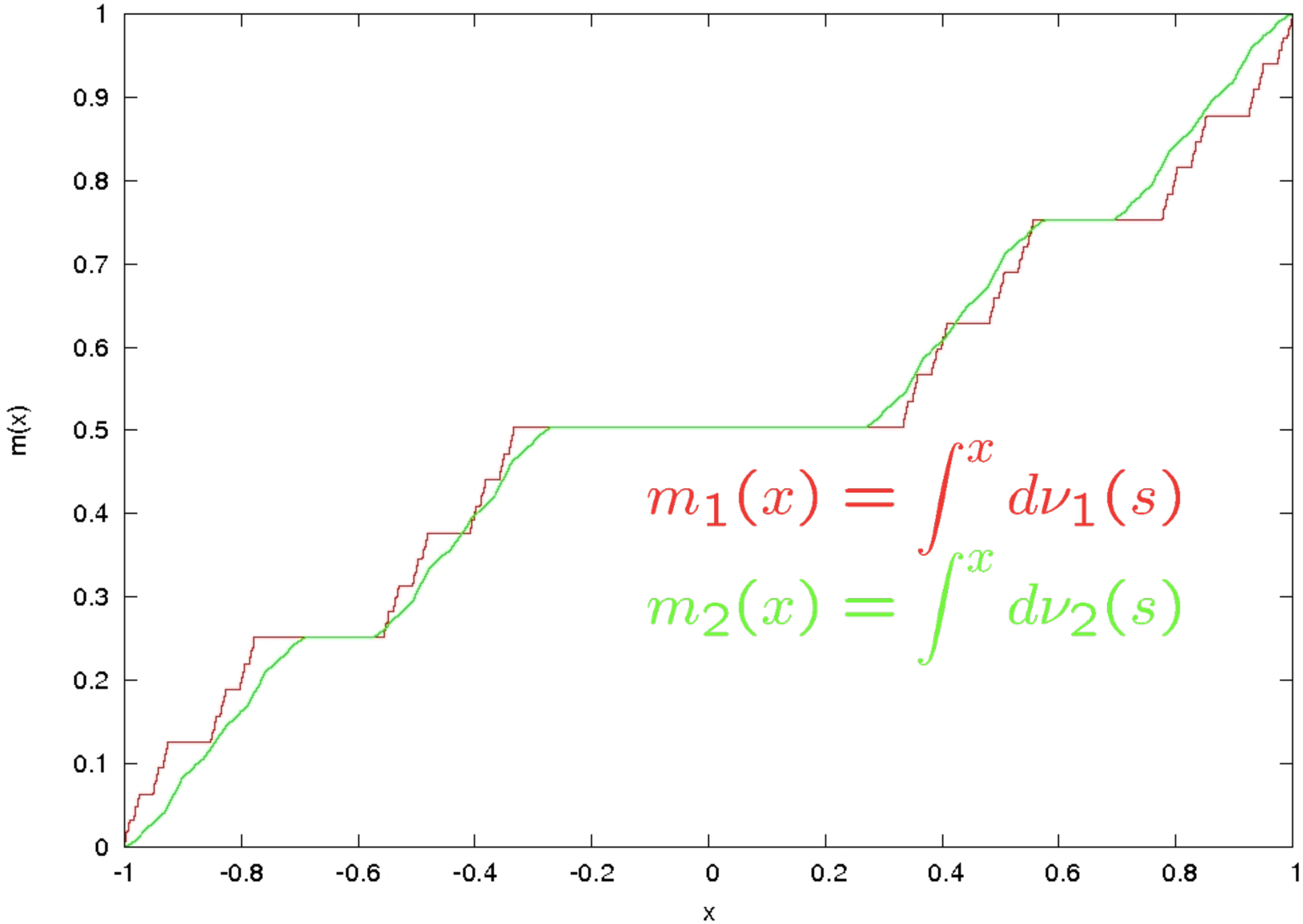


ν_0 is a pure-point measure,

ν_1 is s.c. supported on the Cantor set K.

What are the properties of ν_2 ?

The support of the measure



Thm. Let Supp(ν_i), Supp(ν_{i+1}) be the support of ν_i and ν_{i+1}.
For any $\delta > 0$, Supp (ν_i) $\subset$ Supp(ν_{i+1}) $\subset B_{L\delta}$(Supp(ν_i)).

The support of the measure

$A \oplus B = \{a + b,\ a \in A, b \in B\}$

$\delta A = \{\delta a,\ a \in A\}$

$\mathrm{Supp}(\nu_{i+1}) = \oplus_{j=0}^{\infty}\, \delta^j\, \mathrm{Supp}(\nu_i)$

$\mathrm{Supp}(\nu_1) = \oplus_{j=0}^{\infty}\, (\frac{1}{3})^j\, \{-1, 1\} = K$

$\mathrm{Supp}(\nu_2) = \oplus_{j=0}^{\infty}\, \delta^j\, K = \bigcup_{l=1}^{N} I_l$

Thm. M. - Peirone (2017) Let $\sum_{j=0}^{\infty} \alpha_j$ be a convergent series of real positive entries and let K be a non–empty compact set admitting a construction of uniformly lower-bounded dissection. Then:

The series $\oplus_{j=0}^{\infty} \alpha_j K$ is convergent.

Any permutation of its terms yields the same value for the sum of the series, which is a finite union of closed intervals.

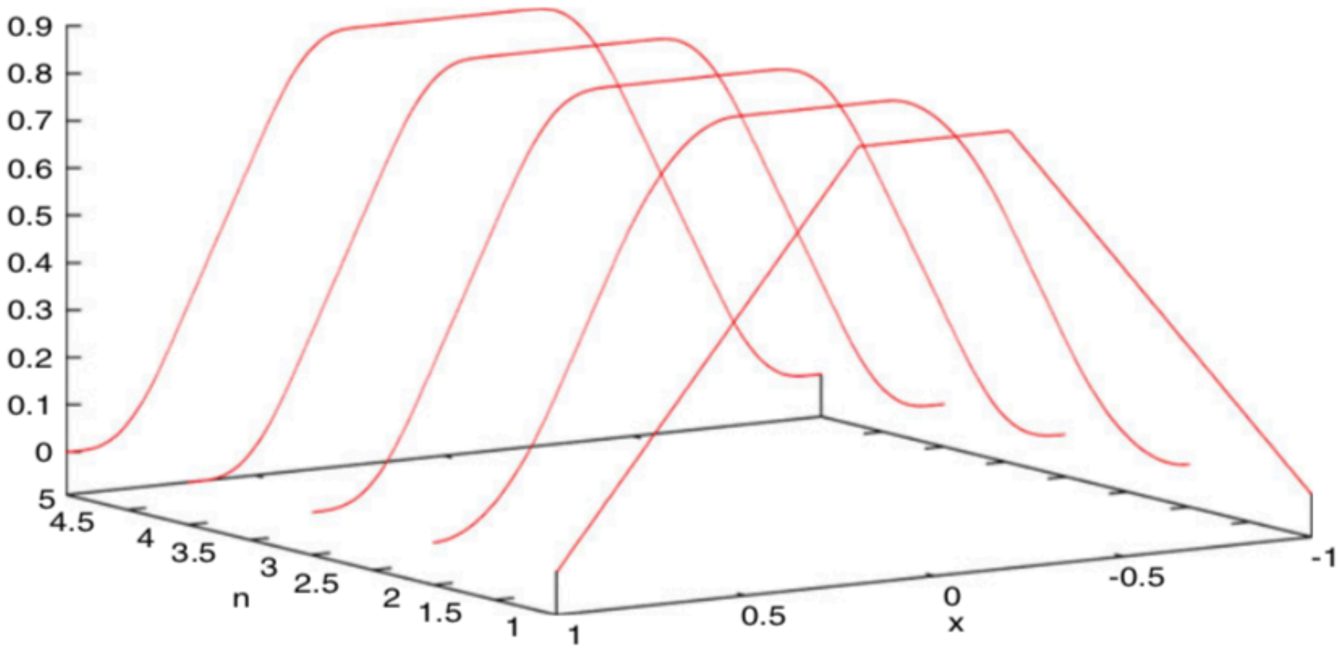
The nature of the measure

$$\nu_0 \longrightarrow \Phi_\delta(\nu_0) = \nu_1 \longrightarrow \Phi_\delta(\nu_1) = \Phi_\delta^2(\nu_0) = \nu_2 \longrightarrow \ldots$$

Thm. The measures ν_i, $i \geq 1$ are of pure type.

Thm. If ν_i is a.c. with bounded density, so is ν_{i+1}.

Example. When ν_0 is the Lebesgue measure on $[-1, 1]$, ν_1 is a.c. and its density $\rho(\nu_1)$ is infinitely differentiable.



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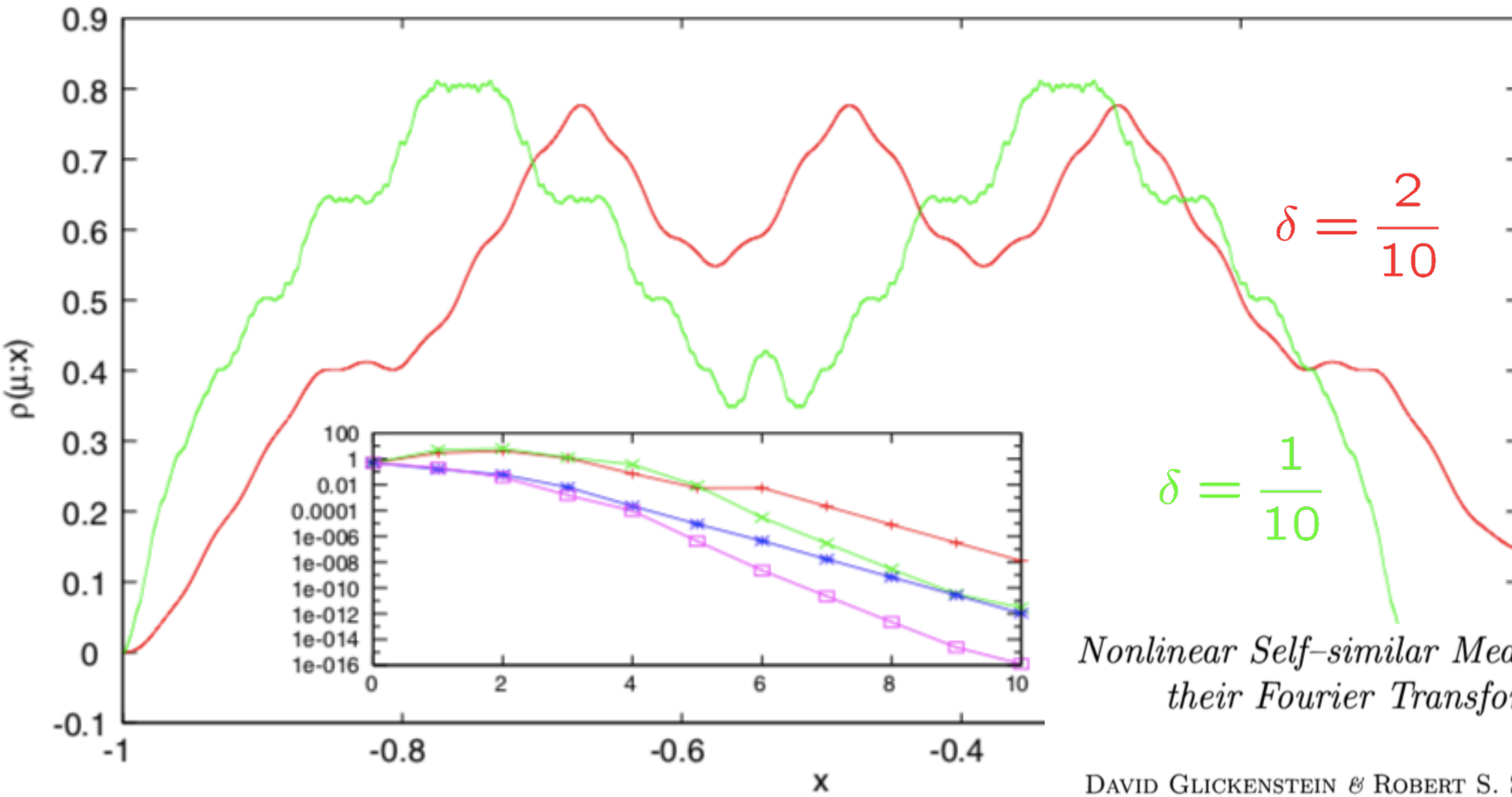
The P-F-R operator

$$\nu_0 = \frac{1}{2}(\Delta_{-1} + \Delta_{-1}) \longrightarrow \Phi_{\frac{2}{5}}(\nu_0) = \nu_1 \longrightarrow \Phi_\delta(\nu_1) = \nu_2$$

$$\mathcal{P}_{\delta,\nu_1}(\rho)(x) = \frac{1}{\delta}\int d\nu_1(\beta)\rho(\frac{x-\bar{\delta}\beta}{\delta})$$

$$d_n = \|\mathcal{P}^{n+1}_{\delta,\nu_1}(\rho) - \mathcal{P}^n_{\delta,\nu_1}(\rho)\|_? \sim \ ? \qquad \|\cdot\|_{Var}, \ \ \|\cdot\|_1$$



Nonlinear Self–similar Measures and their Fourier Transforms

David Glickenstein & Robert S. Strichartz

The Sobolev dimension

Numerical Experiments in Fourier Asymptotics of Cantor Measures and Wavelets

Prem Janardhan, David Rosenblum and Robert S. Strichartz

$$\widehat{\mu}(t) = \int d\mu(s) e^{-its}$$

$$(\nu *_\delta \mu)(f) = \iint f(\delta x + (1-\delta)\beta)\, d\mu(x) d\nu(\beta)$$

$$\widehat{(\nu *_\delta \mu)}(t) = \widehat{\nu}(\bar{\delta} t)\, \widehat{\mu}(\delta t)$$

Elton and Yan (1989)

$$\widehat{\mu}(t) = \widehat{\nu}(\bar{\delta} t)\, \widehat{\mu}(\delta t) = \widehat{\nu}(\bar{\delta} t)\, \widehat{\nu}(\bar{\delta}\delta t)\, \widehat{\mu}(\delta^2 t) = \prod_{j=0}^{\infty} \widehat{\nu}(\bar{\delta}\delta^j t)$$

$$M_1(|\widehat{\mu}|^2; z) := \int_1^\infty t^{z-1} |\widehat{\mu}(t)|^2 \, dt$$

$$d_S(\mu) := \sup\{s \in \mathbf{R} \ s.t. \ M_1(|\widehat{\mu}|^2; s) < \infty\}$$

$$d_S(\mu) \leq 1 \Rightarrow d_S(\mu) = D_2(\mu)$$

$d_S(\mu) > 1 \Rightarrow \mu$ *a.c. with density in* L^2,

$d_S(\mu) > 2 \Rightarrow \mu$ *a.c. with a continuous density*

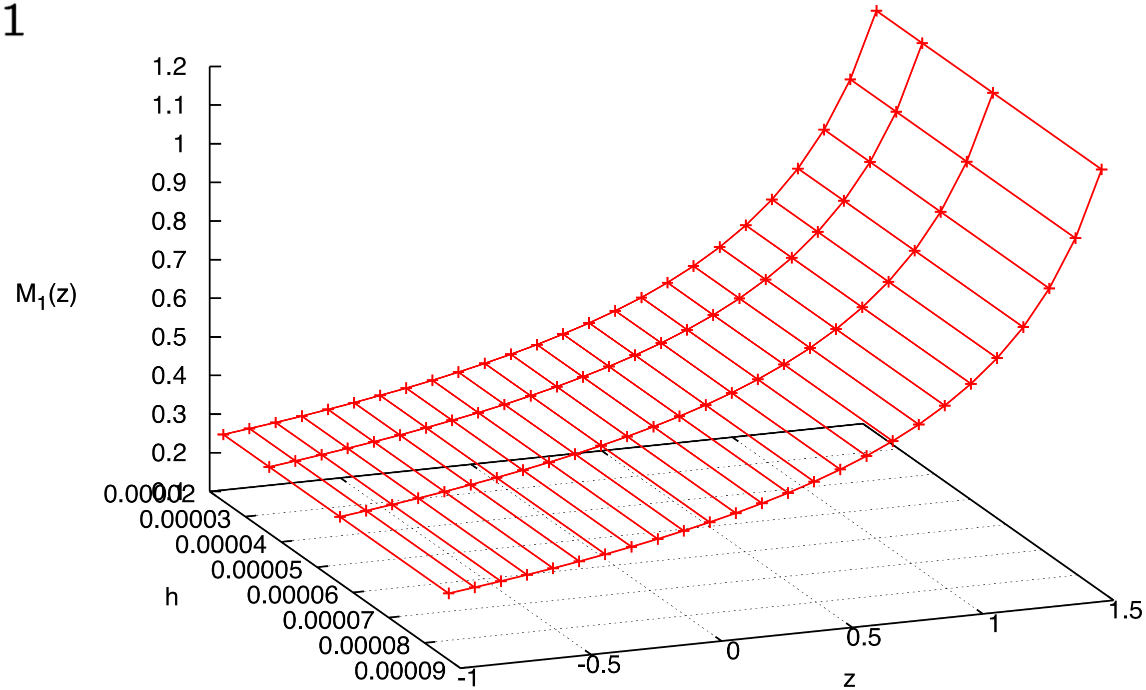
Numerical Experiments

$$M_1(|\hat{\mu}|^2; z) := \int_1^\infty t^{z-1} |\hat{\mu}(t)|^2 \, dt \qquad \tau = \frac{\log(t)}{h}$$

$$M_1(|\hat{\mu}|^2; z) = h \int_0^\infty e^{zh\tau} |\hat{\mu}(e^{h\tau})|^2 \, d\tau$$

$$= h \int_0^\infty e^{(zh+1)\tau} |\hat{\mu}(e^{h\tau})|^2 \; e^{-\tau} d\tau$$ ← Laguerre integration

$$= h \sum_{i=1}^{n} e^{(zh+1)\theta_i^n} |\hat{\mu}(e^{h\theta_i^n})|^2 \; w_i^n$$ ← Laguerre points and weights



Pade' extrapolation

$$M_1(|\widehat{\mu}|^2; z) := \int_1^\infty t^{z-1} |\widehat{\mu}(t)|^2 \, dt$$

$$M_1(|\widehat{\mu}|^2; z_l) = \frac{P_{m-1}(z_l)}{Q_m(z_l)}, \; l = 1, \ldots, 2m$$ ← Multipoint Pade'

$$d_S(\mu) := \sup\{s \in \mathbf{R} \; s.t. \; M_1(|\widehat{\mu}|^2; s) < \infty\}$$

$$d_S(\mu) \sim \text{smallest real zero of } Q_m(z)$$

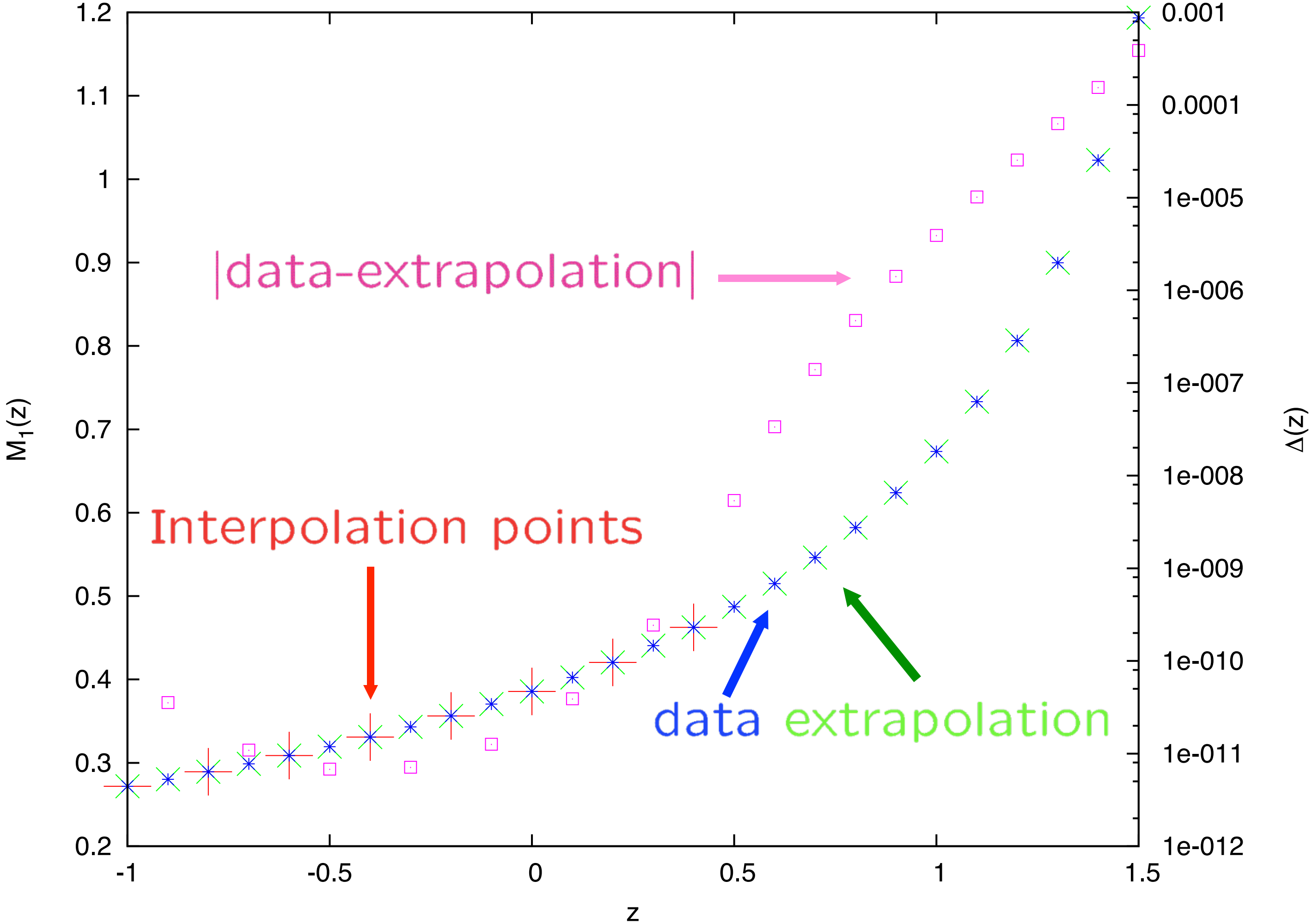
$$\nu_0 = \frac{1}{2}(\Delta_{-1} + \Delta_{-1}) \longrightarrow \Phi_{1/2}(\nu_0) = \nu_1$$ ← Lebesgue measure

$$\widehat{\nu_1}(t) = \frac{\sin(t)}{t} \Rightarrow d_S(\nu_1) = 2$$

Pade' extrapolation

$$M_1(|\widehat{\mu}|^2; z) := \int_1^{\infty} t^{z-1} |\widehat{\mu}(t)|^2 \, dt$$

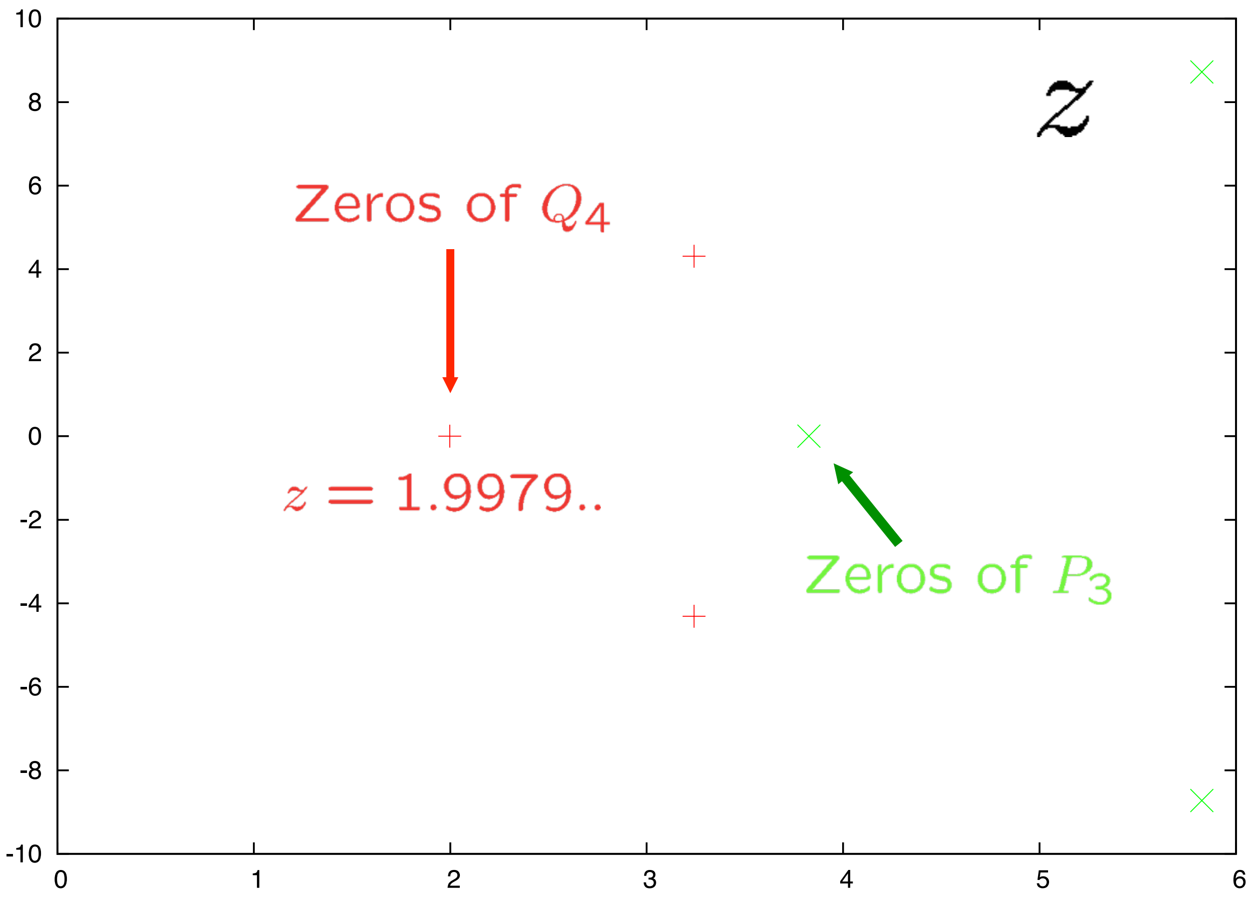
$$\widehat{\nu_1}(t) = \frac{\sin(t)}{t} \Rightarrow d_S(\nu_1) = 2$$



Pade' extrapolation

$$M_1(|\widehat{\mu}|^2; z) := \int_1^\infty t^{z-1} |\widehat{\mu}(t)|^2 \, dt$$

$$\widehat{\nu_1}(t) = \frac{\sin(t)}{t} \Rightarrow d_S(\nu_1) = 2 \qquad M_1(|\widehat{\mu}|^2; z) \simeq \frac{P_{m-1}(z)}{Q_m(z)}$$



Pade' extrapolation

$$M_1(|\widehat{\mu}|^2; z) := \int_1^\infty t^{z-1} |\widehat{\mu}(t)|^2 \, dt$$

$$M_1(|\widehat{\mu}|^2; z_l) = \frac{P_{m-1}(z_l)}{Q_m(z_l)}, \; l = 1, \ldots, 2m$$ ← Multipoint Pade'

$$d_S(\mu) := \sup\{s \in \mathbf{R} \; s.t. \; M_1(|\widehat{\mu}|^2; s) < \infty\}$$

$d_S(\mu) \sim$ smallest real zero of $Q_m(z)$

$$\nu_0 = \frac{1}{2}(\Delta_{-1} + \Delta_{-1}) \longrightarrow \Phi_{1/2}(\nu_0) = \nu_1$$ ← Lebesgue measure

$$\widehat{\nu_1}(t) = \frac{\sin(t)}{t} \Rightarrow d_S(\nu_1) = 2$$

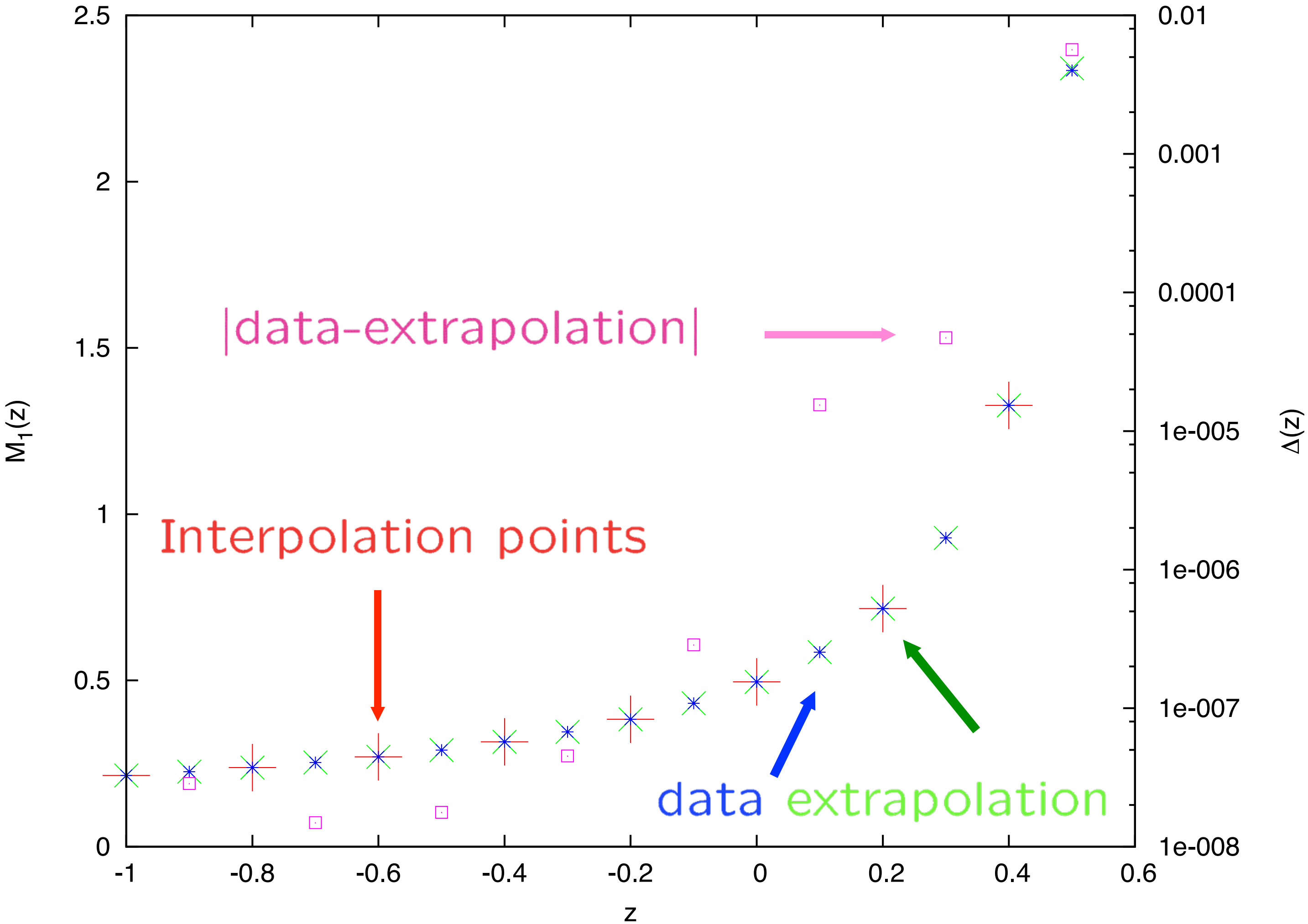
$$\nu_0 = \frac{1}{2}(\Delta_{-1} + \Delta_{-1}) \longrightarrow \Phi_{1/3}(\nu_0) = \nu_1$$ ← Devil's staircase

$$\widehat{\nu_1}(t) = \prod_{j=0}^{\infty} \cos\left(\frac{2}{3^{j+1}} t\right) \Rightarrow d_S(\nu_1) = \frac{\log 2}{\log 3}$$

Pade' extrapolation

$$M_1(|\widehat{\mu}|^2; z) := \int_1^\infty t^{z-1} |\widehat{\mu}(t)|^2 \, dt$$

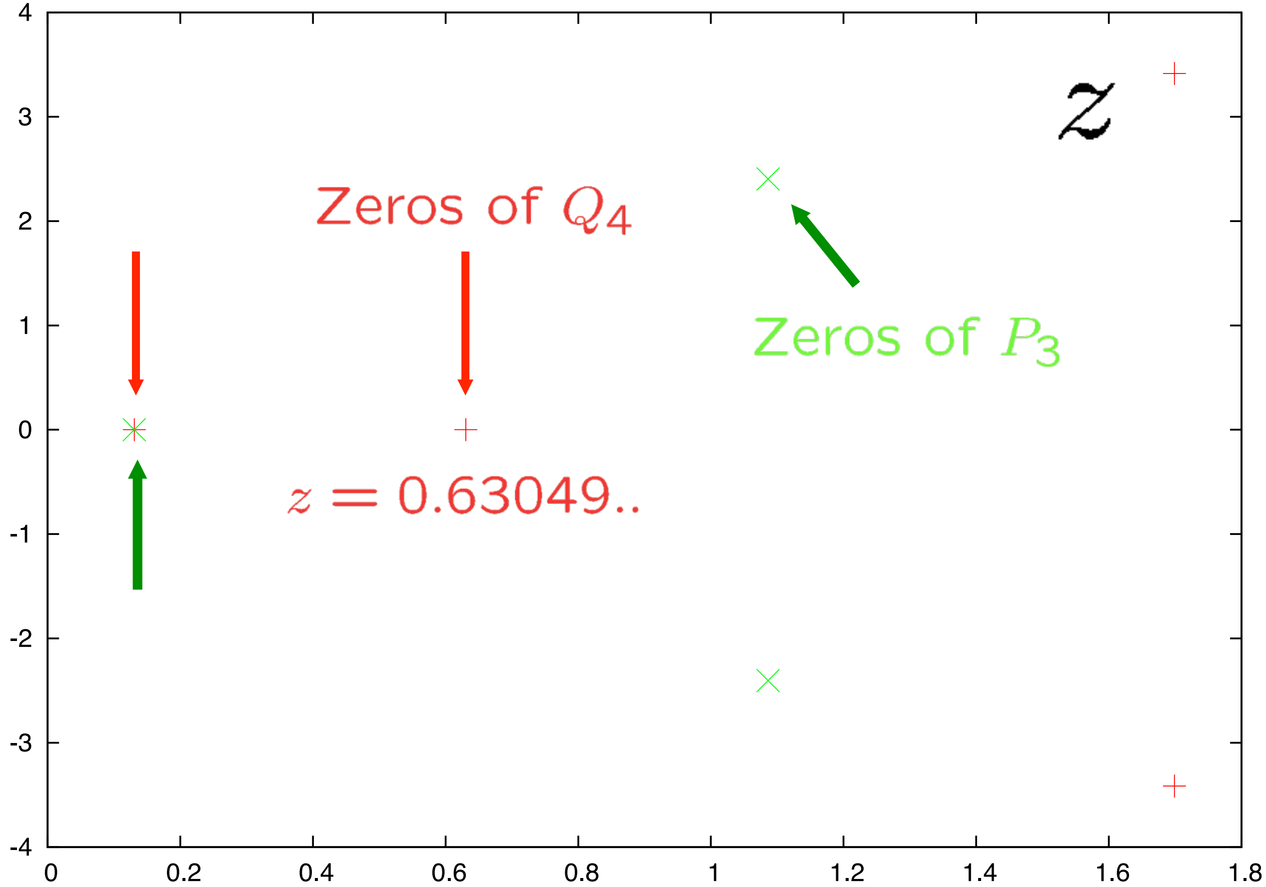
$$\widehat{\nu_1}(t) = \prod_{j=0}^{\infty} \cos(\frac{2}{3^{j+1}} t) \Rightarrow d_S(\nu_1) = \frac{\log 2}{\log 3}$$



Pade' extrapolation

$$M_1(|\widehat{\mu}|^2; z) := \int_1^\infty t^{z-1} |\widehat{\mu}(t)|^2 \, dt$$

$$\widehat{\nu_1}(t) = \prod_{j=0}^{\infty} \cos\left(\frac{2}{3^{j+1}} t\right) \Rightarrow d_S(\nu_1) = \frac{\log 2}{\log 3} \qquad M_1(|\widehat{\mu}|^2; z) \simeq \frac{P_{m-1}(z)}{Q_m(z)}$$



Pade' extrapolation

$$M_1(|\widehat{\mu}|^2; z) := \int_1^\infty t^{z-1} |\widehat{\mu}(t)|^2 \, dt$$

$$M_1(|\widehat{\mu}|^2; z_l) = \frac{P_{m-1}(z_l)}{Q_m(z_l)}, \; l = 1, \ldots, 2m$$ ← Multipoint Pade'

$$d_S(\mu) := \sup\{s \in \mathbf{R} \; s.t. \; M_1(|\widehat{\mu}|^2; s) < \infty\}$$

$$d_S(\mu) \sim \text{smallest real zero of } Q_m(z)$$

$$\nu_0 = \frac{1}{2}(\Delta_{-1} + \Delta_{-1}) \longrightarrow \Phi_{1/2}(\nu_0) = \nu_1$$ ← Lebesgue measure

$$\widehat{\nu_1}(t) = \frac{\sin(t)}{t} \Rightarrow d_S(\nu_1) = 2$$

$$\nu_0 = \frac{1}{2}(\Delta_{-1} + \Delta_{-1}) \longrightarrow \Phi_{1/3}(\nu_0) = \nu_1$$ ← Devil's staircase

$$\widehat{\nu_1}(t) = \prod_{j=0}^{\infty} \cos(\frac{2}{3^{j+1}} t) \Rightarrow d_S(\nu_1) = \frac{\log 2}{\log 3}$$

$$\nu_0 \longrightarrow \Phi_\delta(\nu_0) = \nu_1 \longrightarrow \Phi_\delta(\nu_1) = \nu_2$$ ← 2-nd generation IFS

Spectral transition

$$\nu_0 \longrightarrow \Phi_\delta(\nu_0) = \nu_1 \longrightarrow \Phi_\delta(\nu_1) = \nu_2$$ ← 2-nd generation IFS

$$\nu_0 = \frac{1}{2}(\Delta_{-1} + \Delta_{-1})$$ ← Bernoulli measure

← 1st gen IFS
Devil's staircase

$$\text{Supp }(\nu_1) \subset \text{Supp}(\nu_2) \subset B_{2\delta}(\text{Supp}(\nu_1)).$$

$$\widehat{\nu_1}(t) = \prod_{j=0}^{\infty} \cos(\bar{\delta}\delta^j t) \Rightarrow d_S(\nu_1) = -\frac{\log 2}{\log \delta}$$

$$\widehat{\nu_2}(t) = \prod_{j,k=0}^{\infty} \cos(\bar{\delta}^2 \delta^{j+k} t) \Rightarrow d_S(\nu_2) = ?$$ ← 2nd gen IFS

$d_S(\nu_2) = D_2(\nu_2) < 1 \Rightarrow \nu_2$ *s.c.* ,

$d_S(\nu_2) > 1 \Rightarrow \nu_2$ *a.c. with density in* L^2,

$d_S(\nu_2) > 2 \Rightarrow \nu_2$ *a.c. with a continuous density*

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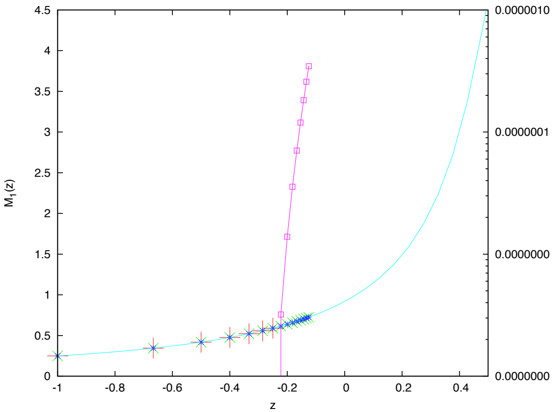
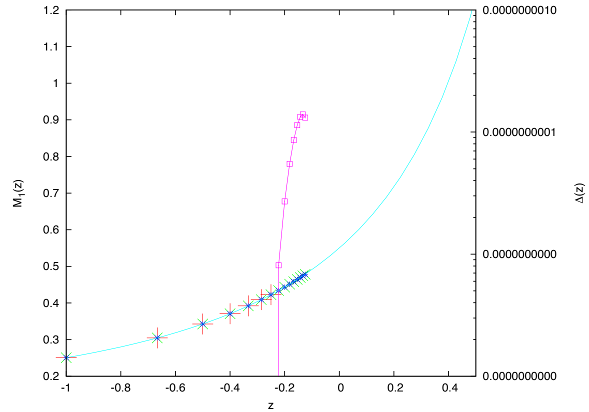
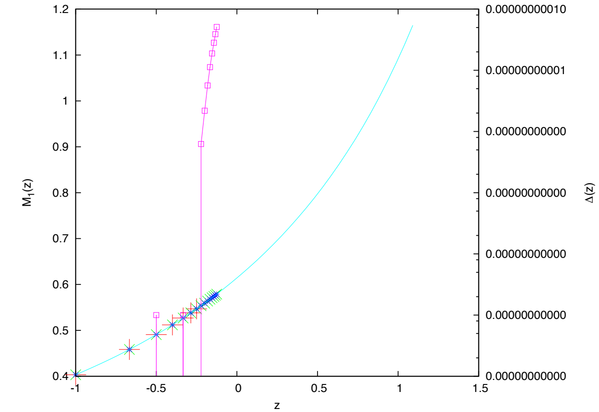
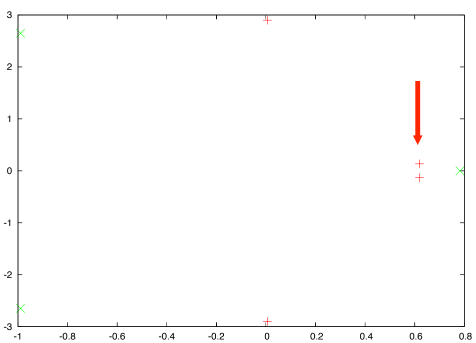
Spectral transition

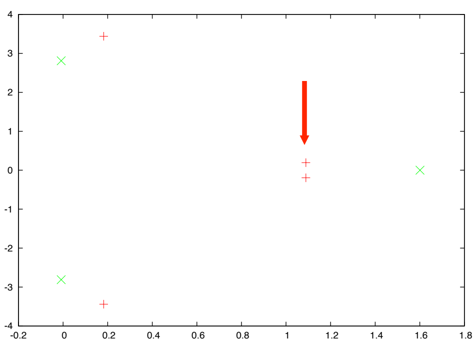
$$M_1(|\widehat{\mu}|^2; z) := \int_1^\infty t^{z-1} |\widehat{\mu}(t)|^2 \, dt$$

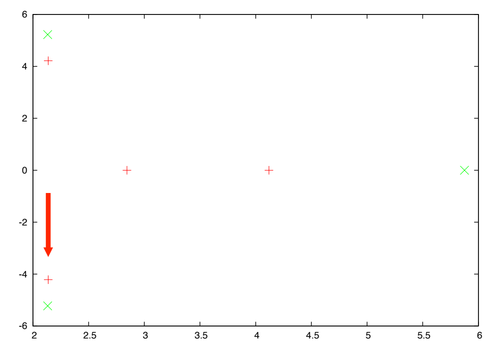












$\delta = 1/10$

$d_S \simeq .61$

$\delta = 2/10$

$d_S \simeq 1.08$

$\delta = 4/10$

$d_S \geq 2$

Conjecture: there is a transition value of δ from s.c. to a.c.

Conclusions

$$\mathcal{T}_{\delta,\nu}(\mu^*) = \mu^* \Longrightarrow \mu^* = \Phi_\delta(\nu)$$

For any $\delta \in [0,1)$ the above defines a transformation Φ_δ from the space $\mathcal{M}([-1,1])$ of probability measures to itself

$$\nu_0 \longrightarrow \Phi_\delta(\nu_0) = \nu_1 \longrightarrow \Phi_\delta(\nu_1) = \Phi_\delta^2(\nu_0) = \nu_2 \longrightarrow \dots$$

Initial measure ↑ First generation IFS ↑ Second generation IFS ↑

$$\nu_0 = \frac{1}{2}(\Delta_{-1} + \Delta_{-1}) \longrightarrow \Phi_{\frac{1}{3}}(\nu_0) = \nu_1 \longrightarrow \Phi_{\frac{1}{3}}(\nu_1) = \nu_2$$

Atomic measure ↑ singular continuous ↑ absolutely continuous ↑

To be continued...

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