Fractal zeta functions generated by orbits of parabolic diffeomorphisms

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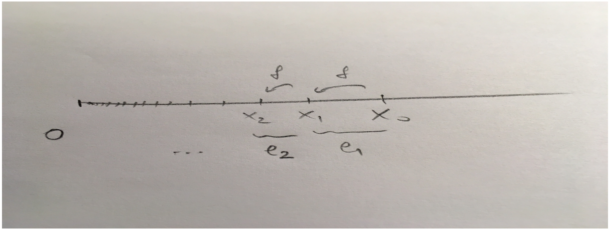
Motivation

- let f be an attracting germ of a diffeo. on $\mathbb{R}$ at a fixed point 0 and let
$$\mathcal{O}_f(x_0) := \{f^{\circ n}(x_0) : \ n \in \mathbb{N}\},$$
be its orbit by f.



- Can one read the formal (or even analytic) class of f from the "fractality" of its one orbit?
- The tube function of the orbit:
$$V_f := V_{f,x_0} : \varepsilon \mapsto |\mathcal{O}_f(x_0)_\varepsilon \cap [0, x_0]|$$

Relative fractal drum (A, Ω)

- $\emptyset \neq A \subset \mathbb{R}^N$, $\Omega \subset \mathbb{R}^N$, Lebesgue measurable, i.e., $|\Omega| < \infty$
- **upper r-dimensional Minkowski content of** (A, Ω):

$$\overline{\mathcal{M}}^r(A, \Omega) := \limsup_{\delta \to 0^+} \frac{|A_\delta \cap \Omega|}{\delta^{N-r}}$$

- **upper Minkowski dimension of** (A, Ω):

  $\overline{\dim}_B(A, \Omega) = \inf\{r \in \mathbb{R} \ : \ \overline{\mathcal{M}}^r(A, \Omega) = 0\}$

- **lower Minkowski content** and **dimension** defined via lim inf
- if $\exists D \in \mathbb{R}$ such that

$$0 < \underline{\mathcal{M}}^D(A, \Omega) = \overline{\mathcal{M}}^D(A, \Omega) < \infty,$$

  we say (A, Ω) is **Minkowski measurable**; in that case $D = \dim_B(A, \Omega)$

The relative distance zeta function LRZ2017

- (A, Ω) RFD in $\mathbb{R}^N$, $s \in \mathbb{C}$ and **fix** $\delta > 0$
- the **distance zeta function of** (A, Ω):

$$\zeta_{A,\Omega}(s; \delta) := \int_{A_\delta \cap \Omega} d(x, A)^{s-N}\, dx$$

- dependence on δ is not essential
- **holomorphic on** $\{\mathrm{Re}\, s > \overline{\dim}_B(A, \Omega)\}$
- the **complex dimensions of** (A, Ω) are defined as the poles of $\zeta_{A,\Omega}$
- in our case $A = \mathcal{O}_f(x_0)$ and $\Omega = [0, x_0]$:

$$\zeta_f(s) = \zeta_{f,x_0}(s) := \int_0^{x_0} d(x, \mathcal{O}_f(x_0))^{s-1}\, dx,$$

Fractal tube formulas for relative fractal drums

- An asymptotic formula for the **tube function** $t \mapsto V_{A,\Omega}(t) := |A_t \cap \Omega|$ as $t \to 0^+$ in terms of $\zeta_{A,\Omega}$.

Theorem (Simplified pointwise formula with error term)

• $\alpha < \overline{\dim}_B(A,\Omega) < N$; $\zeta_{A,\Omega}$ satisfies suitable rational decay conditions (d-**languidity**) on the half-plane $\mathbf{W} := \{\operatorname{Re} s > \alpha\}$, then:

$$V_{A,\Omega}(t) = \sum_{\omega \in \mathcal{P}(\zeta_{A,\Omega}, \mathbf{W})} \operatorname{res}\left(\frac{t^{N-s}}{N-s}\zeta_{A,\Omega}(s), \omega\right) + O(t^{N-\alpha}).$$

- if we allow polynomial growth of $\zeta_{A,\Omega}$, in general, we get a tube formula in the sense of Schwartz distributions

Fractal tube formulas for relative fractal drums

- An asymptotic formula for the **tube function** $t \mapsto V_{A,\Omega}(t) := |A_t \cap \Omega|$ as $t \to 0^+$ in terms of $\zeta_{A,\Omega}$.

Theorem (Case of simple poles)

- In case the the fractal zeta function has only simple poles:

$$V_{A,\Omega}(t) = \sum_{\omega \in \mathcal{P}(\zeta_{A,\Omega}, \mathbf{W})} \frac{t^{N-\omega}}{N-\omega} \operatorname{res}\left(\zeta_{A,\Omega}(s), \omega\right) + O(t^{N-\alpha}).$$

- a pole ω of order m generates terms of type $t^{N-\omega}(-\log t)^{k-1}$ for $k = 1, \ldots, m$
- if $\omega \in \mathbb{C} \setminus \mathbb{R}$ then the term $t^{N-\omega} = t^{N-\operatorname{Re}\omega} \mathrm{e}^{-\mathbb{i} \operatorname{Im}\omega \log t}$ introduces oscillations in the order $t^{N-\operatorname{Re}\omega}$ which are multiplicative periodic with period $T = \mathrm{e}^{2\pi/\operatorname{Im}\omega}$

Parabolic analytic germs

- $$f(x) = x - ax^{k+1} + o(x^{k+1}),\ a > 0,\ x \to 0. \tag{1}$$
- Formal change of variables in the class of formal power series $x + x^2\mathbb{R}[[x]]$ reduces f to a normal form which is a time-one map of a simple vector field:

  $$f_0(x) = \mathrm{Exp}\left(-\frac{x^{k+1}}{1-\rho x^k}\frac{d}{dx}\right).\mathrm{id},\ k \in \mathbb{N},\ \rho \in \mathbb{R}. \tag{2}$$

  Parabolic germs of the type (2) are called *model diffeomorphisms*.
- the pair $(k, \rho) \in \mathbb{N} \times \mathbb{R}$ is called the *formal invariant* of f.

Precise computations for the simplest model case of germs when $\rho = 0$ (MRR 2020)

$*$ *Model cases* with residual invariant $\rho = 0$ and multiplicity $k \in \mathbb{N}$
$*$ time-one maps of simple vector fields $x' = -x^{k+1}$:

$$f_k(x) := \mathrm{Exp}(-x^{k+1}\frac{d}{dx}).\mathrm{id} = \frac{x}{(1+kx^k)^{1/k}} = x - x^{k+1} + o(x^{k+1}),\ k \in \mathbb{N}.$$

Proposition (The complex dimensions of orbits, MRR 2020)

$\zeta_{f_k}(s)$, $\mathrm{Re}(s) > \frac{k}{k+1}$, the distance zeta function of an orbit $\mathcal{O}_{f_k}(x_0)$ of a model parabolic germ.

1. $\zeta_{f_k}(s)$ can be meromorphically extended to $\mathbb{C}$,
2. the poles of $\zeta_{f_k}(s)$ located at $1 - \frac{1}{k+1} = \frac{k}{k+1}$ and at (a subset of) the set of points $1 - \frac{m}{k+1}$, $m \in \mathbb{N}$, all simple
3. the Minkowski (box) dimension of $\mathcal{O}_{f_k}(x_0)$ is $D = \frac{k}{k+1}$, the pole maximal real part

Proposition continued...

Proposition

Precise description of poles:

$$\zeta_{f_k}(s) = \frac{2^{1-s}}{s} \sum_{n=0}^{M} \tilde{Z}_k(s,n)\zeta_{\frac{1}{kx_0^k}}\left(\frac{k+1}{k}s+n\right) + R(s),$$

$R(s)$ defined and **holomorphic** for $\mathrm{Re}\, s > -\frac{Mk}{k+1}$.
The functions $\tilde{Z}_k(s,n)$ entire and
$\tilde{Z}_k(s,n) =$

$$k^{-\frac{k+1}{k}s} \sum_{m=0}^{n} (-k)^m \binom{s}{m} \sum_{i=0}^{m} (-1)^i \binom{m}{i} \sum_{j=0}^{i} (-k)^{-j} \binom{i}{j} \binom{\frac{i-m}{k}}{n+m-j}.$$

Hurwitz zeta function: $\zeta_a(s) := \sum_{j=0}^{\infty} \frac{1}{(j+a)^s},\ a > 0,\ Re(s) > 1$

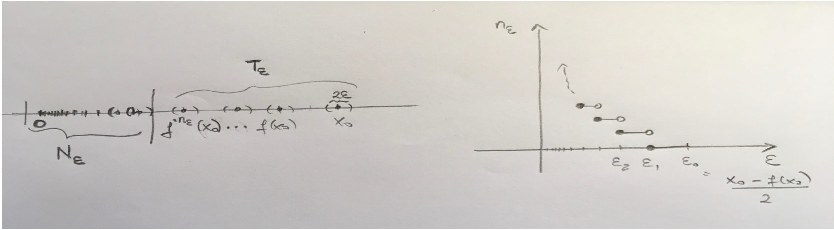
Generalized asymptotic expansion of the tube function - coefficients allowed to be oscillatory

$f(x) = x - ax^{k+1} + o(x^{k+1}) \in \mathrm{Diff}(\mathbb{R}, 0)$, $a > 0$, arbitrary parabolic germ

'Problem' noted in (R, 2014), (MRRZ, 2019):
$(*)$ the tube function $\varepsilon \mapsto V_f(\varepsilon)$ fails to have a full asymptotic expansion in power-logarithm scale,
$(*)$ oscillatory coefficient at order $O(\varepsilon^{\frac{2k+1}{k+1}})$, $\varepsilon \to 0$.

The continuous tube function-a 'dynamical smoothening' of the expansion (MRRZ 2019)

$$V_f(\varepsilon) = |\mathcal{O}_f(x_0)_\varepsilon| = |T_\varepsilon| + |N_\varepsilon| = 2\varepsilon \cdot n_\varepsilon + (f^{n_\varepsilon}(x_0) + 2\varepsilon),\ \varepsilon > 0$$



$(*)$ $\varepsilon \mapsto n_\varepsilon$ the so-called **discrete critical time** 'separating' tail and nucleus - jump function at $\varepsilon_n = \frac{f^{\circ n}(x_0) - f^{\circ(n+1)}(x_0)}{2} \to 0,\ n \to \infty$.
$(*)$ $n_\varepsilon \in \mathbb{N}$ determined by two inequalities:

$$|f^{\circ(n_\varepsilon - 1)}(x_0) - f^{\circ n_\varepsilon}(x_0)| \geq 2\varepsilon,$$
$$|f^{\circ n_\varepsilon}(x_0) - f^{\circ(n_\varepsilon + 1)}(x_0)| < 2\varepsilon.$$

($*$) the **continuous critical time** (MRRZ 2019) $\varepsilon \mapsto \tau_\varepsilon$
- an analytic, dynamical 'approximation' of n_ε
- relies on *embedding of f as the time-one map in a flow* $\{f^t:\ t\in\mathbb{R}\}$:

$$f^{\tau_\varepsilon}(x_0) - f^{\tau_\varepsilon+1}(x_0) = 2\varepsilon.$$

Note: $n_\varepsilon = \lfloor \tau_\varepsilon \rfloor + 1$, $\varepsilon > 0$

The **continuous tube function** $\varepsilon \mapsto V_f^c(\varepsilon)$

$$V_f^c(\varepsilon) = 2\varepsilon\tau_\varepsilon + (f^{\tau_\varepsilon}(x_0) + 2\varepsilon),\ \varepsilon > 0.$$

($*$) analytic in $\varepsilon \in (0,\delta)$
($*$) expansion coincides with the expansion of $\varepsilon \mapsto V_f(\varepsilon)$ up to the first oscillatory term
($*$) full asymptotic expansion in a power-log scale, no oscillatory coefficients!

Generalized asymptotic expansion of tube function-coefficients oscillatory functions

Proposition (MRR 2020)

A **generalized asymptotic expansion** of the tube function with full description of **oscillatory** coefficients:

$$V_f(\varepsilon) \sim 2^{\frac{1}{k+1}} a^{-\frac{1}{k+1}} \frac{k+1}{k} \cdot \varepsilon^{\frac{1}{k+1}} + \sum_{m=2}^{k} a_m \cdot \varepsilon^{\frac{m}{k+1}} + 2\rho \frac{k-1}{k} \cdot \varepsilon \log \varepsilon + b_{k+1}(x_0)\varepsilon +$$

$$+ \sum_{m=k+2}^{2k} \sum_{p=0}^{\lfloor \frac{m}{k} \rfloor + 1} c_{m,p} \varepsilon^{\frac{m}{k+1}} \log^p \varepsilon + \sum_{p=1}^{\lfloor \frac{2k+1}{k} \rfloor + 1} c_{2k+1,p} \varepsilon^{\frac{2k+1}{k+1}} \log^p \varepsilon +$$

$$+ \tilde{P}_{2k+1}(G(\tau_\varepsilon)) \cdot \varepsilon^{\frac{2k+1}{k+1}} + \sum_{m=2k+2}^{\infty} \sum_{p=0}^{\lfloor \frac{m}{k} \rfloor + 1} \tilde{Q}_{m,p}(G(\tau_\varepsilon)) \cdot \varepsilon^{\frac{m}{k+1}} \log^p \varepsilon, \ \varepsilon \to 0^+.$$

(∗) $\varepsilon \mapsto \tau_\varepsilon$ the so-called **continuous critical time** (MRRZ 2019)
(∗) $G : [0, +\infty) \to \mathbb{R}$ 1-**periodic**, $G(s) = 1 - s, \ s \in (0,1), \ G(0) = 0$
(∗) $\tilde{P}_{2k+1}$ resp. $\tilde{Q}_{m,p}$, **polynomials** whose coefficients in general depend on coefficients of f and initial condition x_0.

Asymptotic expansion of the continuous tube function

f parabolic

Proposition (MRR 2020)

$$V_f^c(\varepsilon) \sim 2^{\frac{1}{k+1}} a^{-\frac{1}{k+1}} \frac{k+1}{k} \cdot \varepsilon^{\frac{1}{k+1}} + \sum_{m=2}^{k} a_m \cdot \varepsilon^{\frac{m}{k+1}} + 2\rho \frac{k-1}{k} \cdot \varepsilon \log \varepsilon + b_{k+1}(x_0)\varepsilon$$
$$+ \sum_{m=k+2}^{\infty} \sum_{p=0}^{\lfloor \frac{m}{k} \rfloor + 1} c_{m,p} \varepsilon^{\frac{m}{k+1}} \log^p \varepsilon, \ \varepsilon \to 0^+.$$

$(*)$ $c_{2k+1,0}$ resp. $c_{m,p}$, $m \geq 2k+2$, $p = 0, \ldots, \lfloor \frac{m}{k} \rfloor + 1$, are *free coefficients* of polynomials $\tilde{P}_{2k+1}$ resp. $\tilde{Q}_{m,p}$

$(*)$ only the coefficient $b_{k+1}(x_0)$ depends on the initial condition x_0.

Distributional expansion of the tube function for parabolic orbits

f parabolic

Proposition (MRR 2020)

$$V_f(\varepsilon) \sim_{\mathcal{D}} 2^{\frac{1}{k+1}} a^{-\frac{1}{k+1}} \frac{k+1}{k} \cdot \varepsilon^{\frac{1}{k+1}} + \sum_{m=2}^{k} a_m \cdot \varepsilon^{\frac{m}{k+1}} + 2\rho \frac{k-1}{k} \cdot \varepsilon \log \varepsilon + b_{k+1}(x_0)\varepsilon +$$
$$+ \sum_{m=k+2}^{2k} \sum_{p=0}^{\lfloor \frac{m}{k} \rfloor +1} c_{m,p} \varepsilon^{\frac{m}{k+1}} \log^p \varepsilon + \sum_{p=1}^{\lfloor \frac{2k+1}{k} \rfloor +1} c_{2k+1,p} \varepsilon^{\frac{2k+1}{k+1}} \log^p \varepsilon +$$
$$+ d_{2k+1,0}(x_0) \cdot \varepsilon^{\frac{2k+1}{k+1}} + \sum_{m=2k+2}^{\infty} \sum_{p=0}^{\lfloor \frac{m}{k} \rfloor +1} d_{m,p}(x_0) \cdot \varepsilon^{\frac{m}{k+1}} \log^p \varepsilon, \ \ \varepsilon \to 0^+.$$

Here,
$d_{2k+1,0}(x_0) := \int_0^1 \tilde{P}_{2k+1}(s)\, ds$,
$d_{m,p}(x_0) := \int_0^1 \tilde{Q}_{m,p}(s)\, ds, \ m \geq 2k+2, \ p = 0, \dots, \lfloor \frac{m}{k} \rfloor + 1$,
the *mean values* of 1-periodic functions $\tilde{P}_{2k+1} \circ G$ and $\tilde{Q}_{m,p} \circ G$.

Fractal zeta function for the general non-model case

Arbitrary parabolic germ

$$f(x) = x - ax^{k+1} + o(x^{k+1}) \in \mathrm{Diff}(\mathbb{R}_+, 0)$$

Theorem (B MRR 2020, Complex dimensions for arbitrary parabolic orbits)

$f \in \mathrm{Diff}(\mathbb{R}_+, 0)$, of formal class (k, ρ), $k \in \mathbb{N}$, $\rho \in \mathbb{R}$.

1. *The distance zeta function $\zeta_f(s)$ can be meromorphically extended to $\mathbb{C}$.*
2. *In any open right half-plane $W_M := \{\mathrm{Re}\, s > 1 - \frac{M}{k+1}\}$, where $M \in \mathbb{N}$, $M > k+2$, given as:*

Theorem

For $s \in W_M := \{\mathrm{Re}\, s > 1 - \frac{M}{k+1}\}$*:*

$$\zeta_f(s) = (1-s)\sum_{m=1}^{k} \frac{a_m}{s - \left(1 - \frac{m}{k+1}\right)} + (1-s)\left(\frac{b_{k+1}(x_0)}{s} + \frac{a_{k+1}}{s^2}\right) + $$

$$+ (1-s)\sum_{m=k+2}^{M-1} \sum_{p=0}^{\lfloor \frac{m}{k} \rfloor + 1} \frac{(-1)^p p! \cdot c_{m,p}(x_0)}{\left(s - \left(1 - \frac{m}{k+1}\right)\right)^{p+1}} + g(s),$$

$g(s)$ *holomorphic in* W_M*.*

$*$ the coefficients in principal parts of poles real, with dependence on x_0, as noted!

$*$ related to the coefficients of the asymptotic expansion of the tube function of the orbit!

$*$ **new** wrt model: **higher-order poles** correspond to *logarithmic terms*

Formal class from complex dimensions

Corollary (MRR Formal class of a parabolic germ from complex dimensions)

Let f be a parabolic germ $f(x) = x - ax^{k+1} + o(x^{k+1})$, $a > 0$ from the formal class (k, ρ). Then ζ_f is meromorphic in $\mathbb{C}$ and the formal class is encoded in two complex dimensions:

1. *the simple pole with largest real part, $\omega_1 = 1 - \frac{1}{k+1}$, and its residue:*

$$\mathrm{Res}(\zeta_f(s), \omega_1) = \frac{a_1}{k+1} = \frac{2^{\frac{1}{k+1}} a^{-\frac{1}{k+1}}}{k},$$

2. *the double pole with largest real part, $\omega_{k+1} = 0$, and the residue:*

$$\mathrm{Res}(s \cdot \zeta_f(s), \omega_{k+1}) = a_{k+1} = 2\rho \frac{k-1}{k}.$$

Model hyperbolic orbits

- $f_a(x) = ax$, $0 < a < 1$
- $\mathcal{O}_{f_a}(x_0) = \{x_0 a^n : n \in \mathbb{N}_0\}$
- $\mathcal{L}_{f_a} = \{\ell_j = f_a^{\circ j}(x_0) - f_a^{\circ(j+1)}(x_0) = x_0(1-a)a^j : j \in \mathbb{N}_0\}$
- $$\zeta_{f_a}(s) = \frac{2^{1-s}}{s} \sum_{j=0}^{\infty} \ell_j^s = \frac{2^{1-s} x_0^s (1-a)^s}{s} \cdot \frac{1}{1-a^s}$$
- extends meromorphically to all of $\mathbb{C}$ from $\{\mathrm{Re}\, s > 0\}$
- double pole at $s = 0$ and simple poles at
$$s_k = \frac{2k\pi\mathbb{i}}{\log a}, \ k \in \mathbb{Z}$$
- $V_f(\varepsilon) = -\frac{2}{\log a}\varepsilon(-\log\varepsilon) + \varepsilon H\left(\log_a \frac{2\varepsilon}{x_0(1-a)}\right)$,
$H : [0, +\infty) \to \mathbb{R}$ is 1-periodic and bounded

Parabolic orbits vs. hyperbolic orbits and fractality

!! parabolic case: oscillations of the coefficients can be smoothened by integration
!! hyperbolic case: the oscillations are mulitiplicative periodic and cannot be smoothened distributionally

(a) parabolic orbits: $\tau_\varepsilon \sim \varepsilon^{-\frac{1}{k+1}}$, $\frac{d}{d\varepsilon}\tau_\varepsilon \sim \varepsilon^{-1-\frac{1}{k+1}}$, where $1+\frac{1}{k+1} > 1$

(b) hyperbolic orbits: $\tau_\varepsilon \sim -\log \varepsilon$, $\frac{d}{d\varepsilon}\tau_\varepsilon \sim -\varepsilon^{-1}$

The consequence:
($*$) in the parabolic case **no oscillatory coefficients in the distributional expansion** (seen in poles of zeta function as **no non-real complex dimensions**)
($*$) in the hyperbolic case **oscillatory coefficients remain** (seen in poles of zeta function as **purely imaginary complex dimensions**, similarly as for Cantor sets (LF 2013, LRZ 2017)

? who is fractal ?

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