Fractal shape optimization with applications to linear acoustics

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Table of contents

- F. Magoulès, P.T.K. Ngyuen, P. Omnes, A. Rozanova-Pierrat, *Optimal absorbtion of acoustic waves by a boundary*. SIAM J. Control Optim. Vol. 59, No. 1, (2021), pp. 561-583.

- M. Hinz, A. Rozanova-Pierrat, A. Teplyaev, *Non-Lipschitz uniform domain shape optimization in linear acoustics*. SIAM J. Control Optim. Vol. 59, No. 2 (2021), pp. 1007–1032.

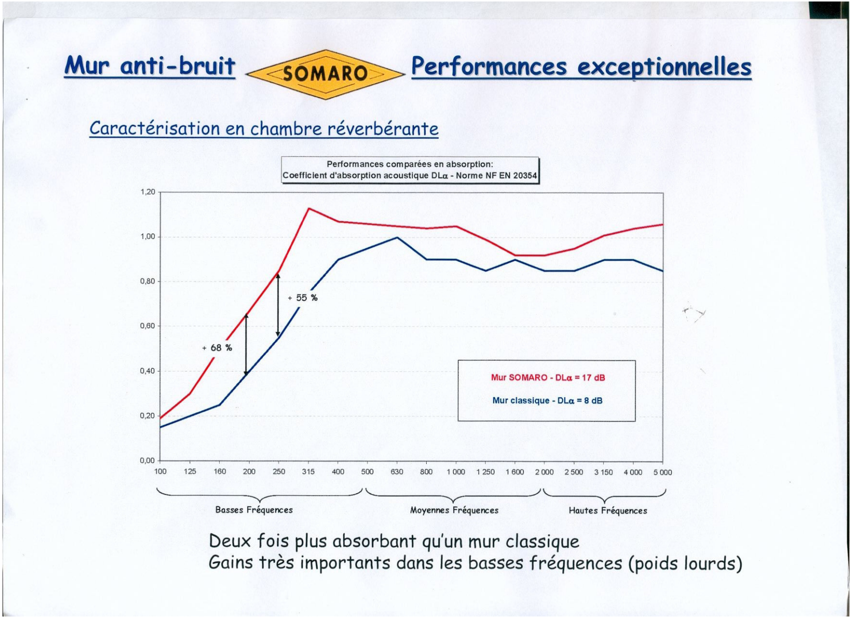
- M. R. Lancia, A. Rozanova-Pierrat, "Fractals in engineering: Theoretical aspects and Numerical approximations", ICIAM 2019 - SEMA SIMAI SPRINGER SERIES PUBLICATIONS, 2021.

Table of contents

Traffic noise absorbing wall

“Fractal wall” TM, porous material is the cement-wood (acoustic absorbent),
Patent Ecole Polytechnique-Colas, Canadian and US patent

Absorption of the “ Fractal wall”



Acoustic anechoic chambers

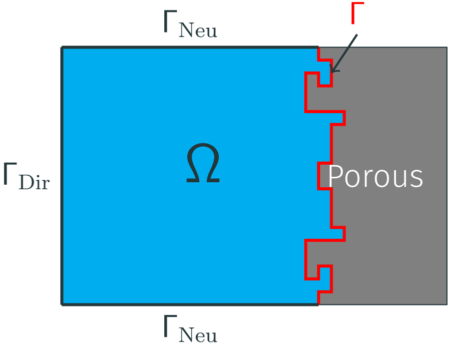
Test anechoic chamber

Microsoft anechoic chamber -20db noise level, the quietest place on earth

Test semi-anechoic chamber

Table of contents

Helmholtz problem for a fixed frequency and a noise source



$$\begin{cases} \triangle u + \omega^2 u = f(x) \quad x \in \Omega, \\ u = g(x) \quad \text{on } \Gamma_{Dir}, \qquad \dfrac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_{Neu}, \\ \dfrac{\partial u}{\partial n} + \alpha(x,\omega)\,\mathrm{Tr}\, u = 0 \quad \text{on } \Gamma, \quad \mathtt{Re}(\alpha) > 0 \text{ and } \mathtt{Im}(\alpha) < 0 \end{cases}$$

F. Magoulès, T.P.K. Nguyen, P. Omnes, ARP. SICON, 2021; M. Hinz, ARP, A. Teplyaev, SICON, 2021.

Damping by the boundary: evolutive in time model ($\mathrm{Re}(\alpha) > 0$ et $\mathrm{Im}(\alpha) < 0$)

$$\begin{cases} \partial_t^2 u - \triangle u = e^{-i\omega t} f(x), \\ u|_{\Gamma_{Dir}} = 0, \quad \left.\dfrac{\partial u}{\partial n}\right|_{\Gamma_{Neu}} = 0, \\ \frac{\partial u}{\partial n} - \frac{1}{\omega}\mathrm{Im}(\alpha(x))\,\mathrm{Tr}\,\partial_t u + \mathrm{Re}(\alpha(x))\,\mathrm{Tr}\,u|_{\Gamma} = 0, \\ u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1 \end{cases}$$

$$X(\Omega) = \left\{u \in H^1(\Omega) |\ \mathrm{Tr}\,u|_{\Gamma_{Dir}} = 0\right\} \times L^2(\Omega)$$

$$\|(u,v)\|^2_{X(\Omega)} = \int_\Omega \left(|\nabla_x u|^2 + |v|^2\right) \mathrm{d}x + \int_\Gamma \mathrm{Re}(\alpha(x))|\,\mathrm{Tr}\,u|^2 d\mu.$$

$$\partial_t \left(\|(u,\partial_t u)\|^2_{X(\Omega)}\right) = \frac{2}{\omega}\int_\Gamma \mathrm{Im}(\alpha(x))|\,\mathrm{Tr}\,\partial_t u|^2 d\mu.$$

C. Bardos, J. Rauch, Asymptotic Analysis, 1994

Table of contents

Sobolev extension domains

Definition
A domain $\Omega \subset \mathbb{R}^n$ is called a ***Sobolev extension domain*** *if there exists a bounded linear extension operator $E : H^1(\Omega) \to H^1(\mathbb{R}^n)$:*

$$\forall u \in H^1(\Omega) \quad \exists v = Eu \in H^1(\mathbb{R}^n) \text{ with } v|_\Omega = u \text{ and } C(\Omega) > 0 :$$

$$\|v\|_{H^1(\mathbb{R}^n)} \leq C\|u\|_{H^1(\Omega)}.$$

Jones [1981]: If Ω is an uniform (or (ε, ∞)-) domain, then it is Sobolev extension domain.

Locally uniform or (ε, δ)-domains ($\varepsilon > 0, 0 < \delta \leq \infty$)

Definition
An open connected subset Ω of $\mathbb{R}^n$ is an (ε, δ)-domain,

if whenever $x, y \in \Omega$ and $|x - y| < \delta$, ***(thus locally)***

there is a rectifiable arc $\gamma \subset \Omega$ with length $\ell(\gamma)$ joining x to y and satisfying

1. $\ell(\gamma) \leq \frac{|x-y|}{\varepsilon}$ ***(uniformly locally quasiconvex)*** *and*
2. $d(z, \partial\Omega) \geq \varepsilon |x - z| \frac{|y-z|}{|x-y|}$ *for $z \in \gamma$.*

Theorem ($n = 2$, Jones [1981])
A bounded and finitely connected domain Ω is (ε, ∞)-domain $\Longleftrightarrow$ its boundary consists of a finite number of points and quasicircles.

Well-posedness and irregularity of the boundary

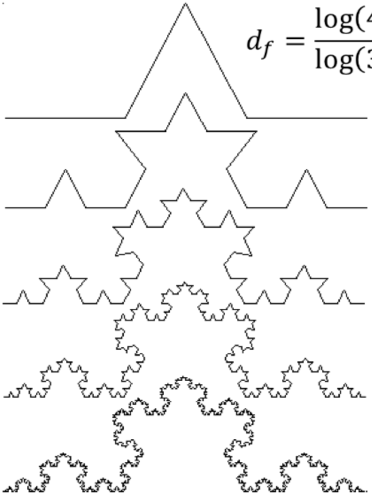
- Ω is a **Sobolev extension domain**
- with a compact boundary $\partial\Omega = \operatorname{supp}\mu$ for a positive Borel measure μ on $\mathbb{R}^n$ satisfying for $d > 0$, $d \in]n-2, n[$ the **upper d-regular** condition: there is a constant $c_d > 0$ such that

$$\mu(B_r(x)) \leq c_d r^d, \quad x \in \partial\Omega, \quad 0 < r \leq 1. \tag{1}$$

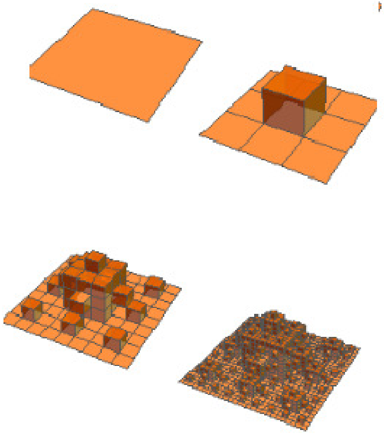
($\implies \dim_H \partial\Omega \geq d$)

Examples: union of different d-sets, multifractals, ...

Examples of self-similar fractal boundaries



$$d_f = \frac{\log(4)}{\log(3)}$$



$$2 < d = \frac{\log(13)}{\log(3)} \approx 2.33 < 3$$ (Wikipedia)

Pointwise definition of the trace operator A. Jonnson, 2009

Definition

For a Sobolev extension domain Ω of $\mathbb{R}^n$ with $\operatorname{supp}\mu = \partial\Omega$ (for an upper regular Borel measure μ),
the trace operator $\mathrm{Tr} : H^1(\Omega) \to L^2(\partial\Omega, \mu)$ is defined μ-a.e. by

$$x \in \partial\Omega \qquad \mathrm{Tr}\, u(x) = \lim_{r\to 0} \frac{1}{\lambda^n(\Omega \cap B_r(x))} \int_{\Omega\cap B_r(x)} u(y)dy.$$

$$B(\partial\Omega, \mu) := \mathrm{Tr}(H^1(\Omega)).$$

Properties of μ, $\operatorname{supp}\mu = \partial\Omega$ are important to caracterize $B(\partial\Omega, \mu)$:

$$H^{\frac{1}{2}}(\partial\Omega), \quad B^{2,2}_{1-\frac{n-d}{2}}(\partial\Omega), \quad B^{2,2}_{1}(\partial\Omega), \ \dots$$

d-sets, H. Wallin 1991
Jonnson 1997

Trace theorem on boundaries given by upper d-regular measures μ

1. Let Ω be a bounded Sobolev extension domain in $\mathbb{R}^n$.
2. Let $\partial\Omega = \operatorname{supp}\mu$ be compact, $0 < n-2 < d \leq n$ for a Borel positive measure μ s.t.

$$\exists c_d > 0 \quad \mu(B(x,r)) \leq c_d r^d, \quad x \in \partial\Omega, \quad 0 < r \leq 1. \tag{2}$$

Then

(i) $\operatorname{Tr} : H^1(\Omega) \to L^2(\partial\Omega, \mu)$ is compact operator and $\exists\, c_{\operatorname{Tr}}(n, \Omega, d, c_d) > 0$, s. t. $\|\operatorname{Tr} f\|_{L^2(\partial\Omega,\mu)} \leq c_{\operatorname{Tr}} \|f\|_{H^1(\Omega)}, \quad f \in H^1(\Omega)$.

Trace theorem on boundaries given by upper d-regular measures μ

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(ii) $B(\partial\Omega, \mu) := \mathrm{Tr}(H^1(\Omega))$ is a Hilbert space (compact and dense in $L^2(\partial\Omega, \mu)$)

$$\|\varphi\|_{B(\partial\Omega,\mu)} := \inf\{\|g\|_{H^1(\Omega)} \mid \varphi = \mathrm{Tr}\ g\}.$$

Trace theorem on boundaries given by upper d-regular measures μ

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(iii) $\exists$ a linear operator $H_{\partial\Omega} : B(\partial\Omega, \mu) \to H^1(\Omega)$ of norm one s. t. $\forall \varphi \in B(\partial\Omega, \mu)$ $\operatorname{Tr}(H_{\partial\Omega}\varphi) = \varphi$.

M. Hinz, ARP, A. Teplyaev, SIAM SICON, 2021, M. Hinz, F. Magoulès, ARP, M. Rynkovskaya, A. Teplyaev, Applied Mathematical Modelling 2021.

Some important corrolaries

Norm equivalence:

If $\mathrm{Tr} : H^1(\Omega) \to L^2(\partial\Omega, \mu)$ is compact, then the norm $\|u\|_{H^1(\Omega)}$ on $H^1(\Omega)$ is equivalent to

$$\|u\|_{\mathrm{Tr}} = \left(\int_\Omega |\nabla u|^2 \mathrm{d}x + \int_{\partial\Omega} |\mathrm{Tr} u|^2 d\mu \right)^{\frac{1}{2}} .$$

Compact embedding:

If Ω is bounded and a Sobolev extension domain, then the embedding

$$H^1(\Omega) \subset L^2(\Omega) \text{ is compact.}$$

Green formula

Thanks to multiple works of **M. R. Lancia** (*d*-sets, Jonsson measures), we obtain

Proposition
Let $\Omega \subset \mathbb{R}^n$ be a Sobolev extension domain with a compact boundary $\partial\Omega = \mathrm{supp}\mu$ an upper-regular positive Borel measure with $n-2 < d < n$.

Then for all $u, v \in H^1(\Omega)$ with $\Delta u \in L^2(\Omega)$

$$\langle \frac{\partial u}{\partial \nu}, \operatorname{Tr} v\rangle_{B'(\partial\Omega,\mu),B(\partial\Omega,\mu)} := \int_\Omega v\Delta u dx + \int_\Omega \nabla v \cdot \nabla u dx.$$

Weak well-posedness of the Helmholtz problem

Theorem

Let Ω be a bounded Sobolev extension domain and μ be a positive Borel d-upper regular measure, such that $\operatorname{supp}\mu = \partial\Omega$ is a compact in $\mathbb{R}^n$.

$$V(\Omega) = \{u \in H^1(\Omega) |\ \operatorname{Tr} u = 0 \text{ on } \Gamma_{Dir}\}$$

$$\|u\|^2_{V(\Omega,\mu)} = \int_\Omega |\nabla u|^2 dx + \int_\Gamma \mathrm{Re}(\alpha) |\operatorname{Tr} u|^2 d\mu \text{ is equivalent to } \|u\|^2_{H^1(\Omega)}$$

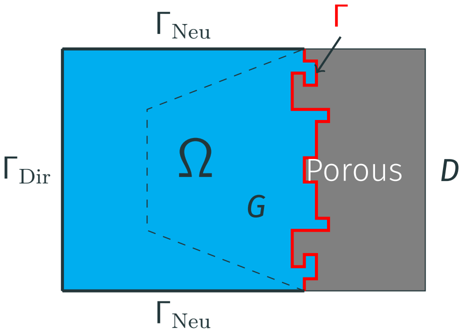
$\forall f \in L^2(\Omega)$, and $\omega > 0$ there exists a unique solution $u \in V(\Omega)$,

$$\forall v \in V(\Omega) \quad \int_\Omega \nabla u \cdot \nabla \bar{v} dx - \omega^2 \int_\Omega u\bar{v} dx + \int_\Gamma \alpha \,\mathrm{Tr}\, u \,\mathrm{Tr}\, \bar{v}\, d\mu = -\int_\Omega f\bar{v} dx$$

$$\exists C(\alpha, \omega, C_{Poincaré}(\Omega)) > 0 : \quad \|u\|_{H^1(\Omega)} \le C\|f\|_{L^2(\Omega)}$$

Table of contents

Minimization of acoustical energy for a fixed frequency and a noise source



$$\forall v \in V(\Omega) \quad \int_\Omega \nabla u \cdot \nabla \bar{v} dx - \omega^2 \int_\Omega u \bar{v} dx + \int_\Gamma \alpha \,\mathrm{Tr}\, u \,\mathrm{Tr}\, \bar{v} \, d\mu = -\int_\Omega f \bar{v} dx$$

$$J(\Omega, \mu, u(\Omega, \mu)) := A \int_\Omega |u|^2 dx + B \int_\Omega |\nabla u|^2 dx + C \int_\Gamma |\,\mathrm{Tr}\, u|^2 d\mu$$

$$\min_{\Omega \in U_{ad}(D,G,\ldots)} J(\Omega, \mu, u(\Omega, \mu))$$

Admissible class of domains U_{ad}

CHENAIS-1975 Domains in a fixed ball satisfying for $\varepsilon > 0$ the ε-cone property:

$$\forall x \in \partial\Omega, \exists \xi_x \in \mathbb{R}^n \text{ with } \|\xi_x\| = 1 \text{ such that for all } y \in \overline{\Omega} \cap B_\varepsilon(x)$$

$$C(y, \xi_x, \varepsilon) = \{z \in \mathbb{R}^n | (z-y, \xi_x) \geq \cos(\varepsilon)\|z-y\| \text{ and } 0 < \|z-y\| < \varepsilon\} \subset \Omega.$$

Admissible class of domains U_{ad}

CHENAIS-1975 Domains in a fixed ball satisfying for $\varepsilon > 0$ the ε-cone property:
Compact class for

1. the Hausdorff convergence of domains:

$$m \in \mathbb{N} \quad \Omega_m \subset D \quad \text{and} \quad d_H(D \setminus \Omega_m, D \setminus \Omega) \to 0 \quad \text{for} \quad m \to +\infty$$

2. in the sense of characteristic functions:

$$\mathbf{1}_{\Omega_m} \to \mathbf{1}_{\Omega} \quad \text{for} \quad m \to +\infty \quad \text{in} \quad L^p_{loc}(\mathbb{R}^n) \quad \forall p \in [1, \infty[$$

3. in the sense of compacts:

$$\begin{cases} \forall K \text{ compact in } \Omega \quad \Rightarrow \quad K \subset \Omega_m \\ \forall U \text{ compact in } D \setminus \overline{\Omega} \quad \Rightarrow \quad U \in D \setminus \overline{\Omega}_m \end{cases} \quad \text{for a sufficiently large } m$$

Admissible class of domains U_{ad}

CHENAIS-1975 Domains in a fixed ball satisfying for $\varepsilon > 0$ the ε-cone property:

Compact class for three type of domain convergences.

But we don't have

$$\forall f \in C(\overline{D}) \qquad \int_{\partial\Omega_m} f d\mathcal{H}^{n-1} \to \int_{\partial\Omega} f d\mathcal{H}^{n-1}.$$

We have

- the continuity of the volume
- **lower semi-continuity of perimeters**

Existence of optimal shapes for homogeneous Dirichlet problems.

Admissible class of domains U_{ad}

CHENAIS-1975 Domains in a fixed ball satisfying for $\varepsilon > 0$ the ε-cone property:

- **Compact class** for three type of domain convergences
- $E_m : H^1(\Omega_m) \to H^1(\mathbb{R}^n)$ have a norm uniformly bounded (independant on m).

Existence of optimal shapes for homogeneous Neumann problems.

Admissible class of domains U_{ad}

CHENAIS-1975 Domains in a fixed ball satisfying for $\varepsilon > 0$ the ε-cone property

BUCUR-2016 Relaxation method for Lipschitz boundaries of finite $\mathcal{H}^{n-1}$ measure
(free boundary discontinous problems)
Robin boundary problems
No-compacity $\Longrightarrow$ no existence result.

Lipschitz admissible domains: $\mu := \lambda$ (for $\Omega \subset \mathbb{R}^2$) or $\mathcal{H}^{n-1}$ (for $\Omega \subset \mathbb{R}^n$)

Definition
$\mathcal{L}ip$ is the class of all domains $\Omega \subset D$ for which

1. $\exists\, \varepsilon > 0$: $\forall$ *domains* $\Omega \in \mathcal{L}ip$ *satisfy the* ε*-cone property* [AGMON-1965, CHENAIS-1975]
2. $\exists\, \hat{c} > 0$: $\forall\, \Omega \in \mathcal{L}ip$ *and* $\forall\, x \in \Gamma$ *we have*

$$\int_{\Gamma \cap B_r(x)} d\lambda \leq \hat{c} r. \tag{3}$$

$$U_{ad}(\Omega_0, \varepsilon, \hat{c}, G) = \{\Omega \in \mathcal{L}ip \mid \Gamma_{Dir} \cup \Gamma_{Neu} \subset \partial\Omega,\ \Gamma \subset \overline{G}, \quad M_0 \leq \int_\Gamma d\lambda \leq M(\hat{c}),\ \int_\Omega \mathrm{d}x = \mathrm{Vol}(\Omega_0)\},$$

F. Magoulès, T.P.K. Nguyen, P. Omnes, ARP. SICON, 2021

Existence of optimal shape

Theorem

1. $U_{ad}(\Omega_0, \varepsilon, \hat{c}, G)$ *is compact with respect to the Hausdorff convergence, in the sense of characteristic functions in* $L^1(D)$ *and in the sense of compacts.*
2. $\omega > 0$, α, f, g *be fixed on* D*, then* $\exists\, \Omega_{opt} \in U_{ad}(\Omega_0, \varepsilon, \hat{c}, G)$ *and* $\exists$ *a finite valued* 1*-dimensional positive measure* μ^* *on* Γ_{opt} *equivalent to* λ*:*

$$\int_{\Gamma_{opt}} d\mu^* \geq \int_{\Gamma_{opt}} d\lambda,$$

$$\inf_{\Omega \in U_{ad}(\Omega_0, \varepsilon, \hat{c}, G)} J(\Omega, u(\Omega, \lambda), \lambda) = J(\Omega_{opt}, u(\Omega_{opt}, \mu^*), \mu^*).$$

3. *If* $\mu^* = \lambda$*, then* $J(\Omega_{opt}, u(\Omega_{opt}, \lambda), \lambda)$ *is the minimum on* $U_{ad}(\Omega_0, \varepsilon, \hat{c}, G)$*.*

F. Magoulès, T.P.K. Nguyen, P. Omnes, ARP. SICON, 2020

Ideas of the proof

1. Helmholtz problem is well-posed on $U_{ad}(\Omega_0, \varepsilon, \hat{c}, G)$
2. Stability constant is uniform on $U_{ad}(\Omega_0, \varepsilon, \hat{c}, G)$
3. Extensions of $H^1(\Omega)$ to $H^1(D)$ have uniform bound on $U_{ad}(\Omega_0, \varepsilon, \hat{c}, G)$ [CHENAIS-1975]
4. By compactness of $U_{ad}(\Omega_0, \varepsilon, \hat{c}, G)$, any minimizing J sequence of domains has a subsequence $(\Omega_m)_{m\in\mathbb{N}}$ converging to $\Omega^* \in U_{ad}(\Omega_0, \varepsilon, \hat{c}, G)$
5. Variational formulations (V.F.) on Ω_m converge to V. F. on Ω^* with μ^* as the measure on $\partial\Omega^*$ (weak limit of $(n-1)$-Hausdorff measures on $\partial\Omega_m$)
6. Compactness of the trace operator and the embedding $V(\Omega) \subset L^2(\Omega)$

General uniform admissible domains: $U_{ad}(D, D_o, \varepsilon, s, d, \bar{c}_s, c_d)$

Definition ((ε, ∞)-domain or uniform domain)

$$\forall x, y \in \Omega \text{ there is a rectifiable arc } \gamma \subset \Omega \text{ with length } \ell(\gamma)$$

joining x to y and satisfying

1. $\ell(\gamma) \leq \frac{|x-y|}{\varepsilon}$ ***(uniformly locally quasiconvex)*** *and*
2. $d(z, \partial\Omega) \geq \varepsilon |x-z| \frac{|y-z|}{|x-y|}$ *for* $z \in \gamma$.

Examples: Lipschitz domains, von Koch snow-flake, . . .
no-collapsing domains, NTA domains
Uniform domains are Sobolev extension domains by JONES-1981.

General uniform admissible domains: $U_{ad}(D, D_0, \varepsilon, s, d, \bar{c}_s, c_d)$

Definition
Let $D_0 \subset D \subset \mathbb{R}^n$ be non-empty bounded Lipschitz domains.

A pair (Ω, μ) is called a ***shape admissible domain*** *with parameters*

$$D, \quad D_0, \quad \varepsilon > 0, \quad n-1 \leq s < n, \quad 0 \leq d \leq s, \quad \bar{c}_s > 0, \quad c_d > 0$$

if

1. Ω *is an* (ε, ∞)*-domain:* $D_0 \subset \Omega \subset D$
2. μ *is a finite Borel measure* μ*,* $\mathrm{supp}\mu = \partial\Omega$*:*

$$\mu(B_r(x)) \leq c_d\, r^d, \quad x \in \partial\Omega, \quad 0 < r \leq 1, \quad (\Rightarrow \ \dim_H \partial\Omega \geq d)$$

$$\mu(\overline{B_r(x)}) \geq \bar{c}_s\, r^s, \quad x \in \partial\Omega, \quad 0 < r \leq 1.$$

(If $s = d$ then $\dim_H \partial\Omega = d$ and $\partial\Omega$ is d-set).

Existence of optimal shape

Theorem

1. *$U_{ad}(D, D_0, \varepsilon, s, d, \bar{c}_s, c_d)$ is compact with respect to the Hausdorff convergence, in the sense of characteristic functions in $L^1(D)$, in the sense of compacts and* ***weak convergence of boundary measures****.*
2. *$\omega > 0$, α, f, g be fixed on D, then $\exists\ (\Omega_{opt}, \mu^*) \in U_{ad}$:*

$$\min_{\Omega \in U_{ad}(D, D_0, \varepsilon, s, d, \bar{c}_s, c_d)} J(\Omega, u(\Omega, \mu), \mu) = J(\Omega_{opt}, u(\Omega_{opt}, \mu^*), \mu^*)$$

for u, the weak solution of the Helmholtz problem.

M. Hinz, ARP, A. Teplyaev, SICON, 2021.

Mosco convergence, U.Mosco, 1994

Definition
A sequence of functionals $G^m : H \to (-\infty, +\infty]$ is said to M-converge to a functional $G : H \to (-\infty, +\infty]$ in a Hilbert space H, if

1. *(lim sup condition) For every $u \in H$ there exists u_m converging strongly in H such that*
$$\overline{\lim} G^m[u_m] \leq G[u], \quad \text{as } m \to +\infty. \tag{4}$$
2. *(lim inf condition) For every v_m converging weakly to u in H*
$$\underline{\lim} G^m[v_m] \geq G[u], \quad \text{as } m \to +\infty. \tag{5}$$

Linear problems (mixed Poisson or Helmholtz problems)

- To define a quadratic form (energy or equivalent norm of H^1)

$$b_m(u_m, u_m) = \int_{\Omega_m} (|\nabla u_m|^2 + |u_m|^2)\mathrm{d}x + \int_{\partial\Omega_m} a_m |\operatorname{Tr} u_m|^2 d\mu_m$$

on $L^2(D)^2$, $\Omega_m \subset D$

- its Mosco-convergence is ensured if
  - $\Omega_m \to \Omega$ by Hausdorff and characteristic functions ($\Omega \subset D$)
  - extension $H^\sigma(\Omega_m) \to H^\sigma(D)$ is uniform on m for $0 \leq \sigma \leq 1$
  - $\forall m \in \mathbb{N}\ \|\sqrt{a_m}\operatorname{Tr}_{\partial\Omega_m} u\|_{L^2(\partial\Omega_m,\mu_m)} \leq C_\sigma \|u\|_{H^\sigma(\mathbb{R}^n)}$ for $u \in H^\sigma(\mathbb{R}^n)$ $\frac{1}{2} < \sigma \leq 1$
  - $a_m\mu_m \rightharpoonup a\mu$:

$$\forall \phi \in C(\overline{D}) \quad \int_{\partial\Omega_m} a_m \phi d\mu_m \to \int_{\partial\Omega} a\phi d\mu, \quad m \to +\infty$$

Linear problems (mixed Poisson or Helmholtz problems)

Let $(u_m)_{m\in\mathbb{N}}$ be the sequence of weak solutions on $(\Omega_m)_{m\in\mathbb{N}}$.

If

- the sequence of solutions is **uniformly bounded on** m:

$$\|(E_{\mathbb{R}^n} u_m)|_D\|_{H^1(D)} \leq C,$$

- $b_m(u_m, w) = 0$ is the variational formulation on Ω_m,
- $b_m(u_m, u_m) \xrightarrow{M} b(u, u)$ in $L^2(D)$ for $\Omega_m \to \Omega$

then

- $u|_\Omega$ (the weak limit of $E_{R^n} u_m|_D$) is the weak solution of $b(u, w) = 0$ on Ω,
- $(E_{\mathbb{R}^n} u_m)|_\Omega \to (E_{\mathbb{R}^n} u)|_\Omega$ in $H^1(\Omega)$.

A. Dekkers, ARP, A. Teplyaev, 2022; M. Hinz, ARP, A. Teplyaev, SICON, 2021

Table of contents

Conclusion

Non-Lipschitz shapes are minimizers of the energy.

M. Hinz, F. Magoulès, A. Rozanova-Pierrat, M. Rynkovskaya, A. Teplyaev, *On the existence of optimal shapes in architecture*. Applied Mathematical Modelling, Vol. 94, (2021), pp. 676–687.

Thank you very much for your attention!