Wave propagation and absorption models with a Robin boundary condition in domains with a non-Lipschitz boundary

Anna Rozanova-Pierrat

with A. Dekkers (CS), M. Hinz (Bielefeld), A. Teplyaev (Connecticut)

19 May 2022

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Table of contents

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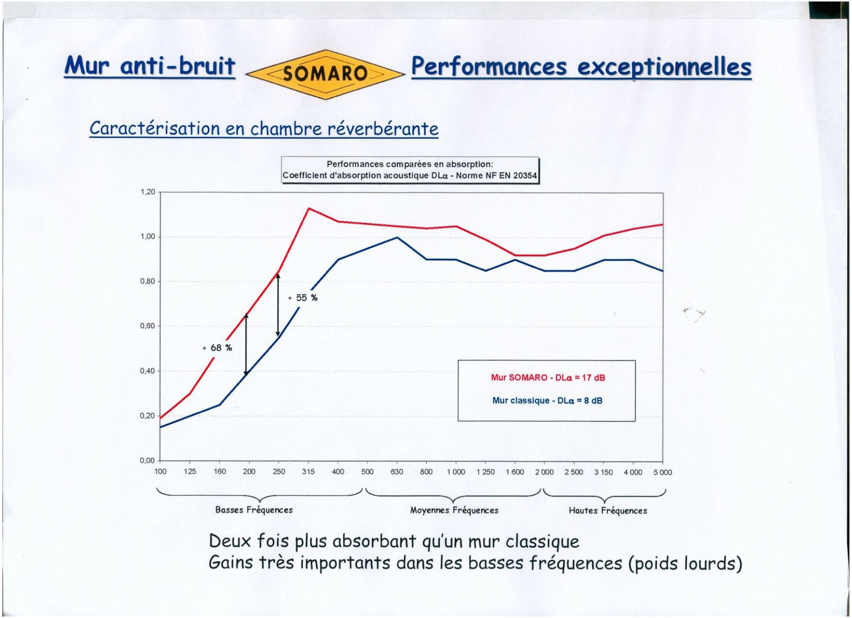
Table of contents

Traffic noise absorbing wall

“Fractal wall” TM, porous material is the cement-wood (acoustic absorbent),
Patent Ecole Polytechnique-Colas, Canadian and US patent

Absorption of the “ Fractal wall”



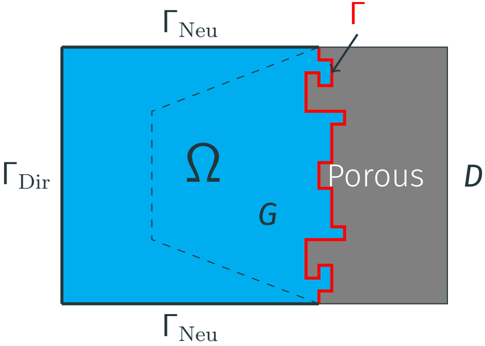
Acoustic anechoic chambers

Test anechoic chamber

Microsoft anechoic chamber -20db noise level, the quietest place on earth

Test semi-anechoic chamber

Helmholtz problem for a fixed frequency and a noise source



$$\begin{cases} \triangle u + \omega^2 u = f(x) \quad x \in \Omega, \\ u = g(x) \quad \text{on } \Gamma_{Dir}, \qquad \dfrac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_{Neu}, \\ \dfrac{\partial u}{\partial n} + \alpha(x,\omega)\,\mathrm{Tr}\, u = 0 \quad \text{on } \Gamma, \quad \mathtt{Re}(\alpha) > 0 \text{ and } \mathtt{Im}(\alpha) < 0 \end{cases}$$

F. Magoulès, T.P.K. Nguyen, P. Omnes, ARP. SICON, 2021; M. Hinz, ARP, A. Teplyaev, SICON, 2021.

Damping by the boundary: evolutive in time model ($\mathtt{Re}(\alpha) > 0$ et $\mathtt{Im}(\alpha) < 0$)

$$\begin{cases} \partial_t^2 u - \triangle u = e^{-i\omega t} f(x), \\ u|_{\Gamma_{Dir}} = 0, \quad \left.\dfrac{\partial u}{\partial n}\right|_{\Gamma_{Neu}} = 0, \\ \frac{\partial u}{\partial n} - \frac{1}{\omega} \mathtt{Im}(\alpha(x)) \operatorname{Tr} \partial_t u + \mathtt{Re}(\alpha(x)) \operatorname{Tr} u|_\Gamma = 0, \\ u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1 \end{cases}$$

$$X(\Omega) = \left\{ u \in H^1(\Omega) | \ \operatorname{Tr} u|_{\Gamma_{Dir}} = 0 \right\} \times L^2(\Omega)$$

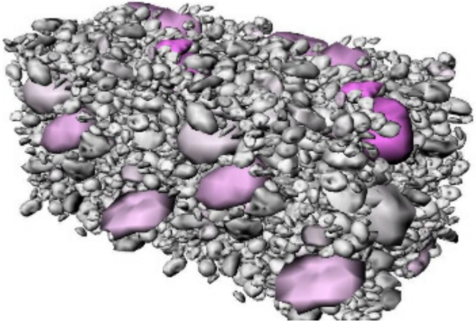
$$\|(u,v)\|^2_{X(\Omega)} = \int_\Omega \left(|\nabla_x u|^2 + |v|^2 \right) dx + \int_\Gamma \mathtt{Re}(\alpha(x)) |\operatorname{Tr} u|^2 d\mu.$$

$$\partial_t \left(\|(u, \partial_t u)\|^2_{X(\Omega)} \right) = \frac{2}{\omega} \int_\Gamma \mathtt{Im}(\alpha(x)) |\operatorname{Tr} \partial_t u|^2 d\mu.$$

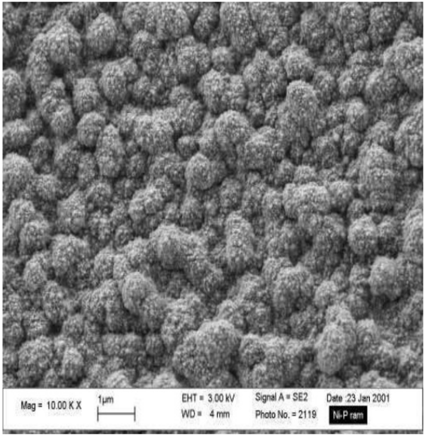
C. Bardos, J. Rauch, Asymptotic Analysis, 1994

Nature complexity and their models

Porous materials

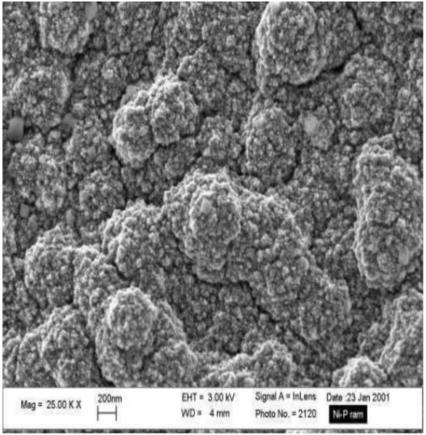


Irregularity of boundaries



$1\,\mu m$

Irregularity of boundaries



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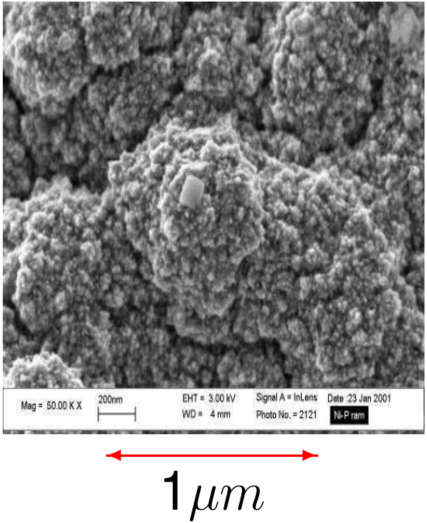

Irregularity of boundaries

Antigiogenesis of cancerous tumours

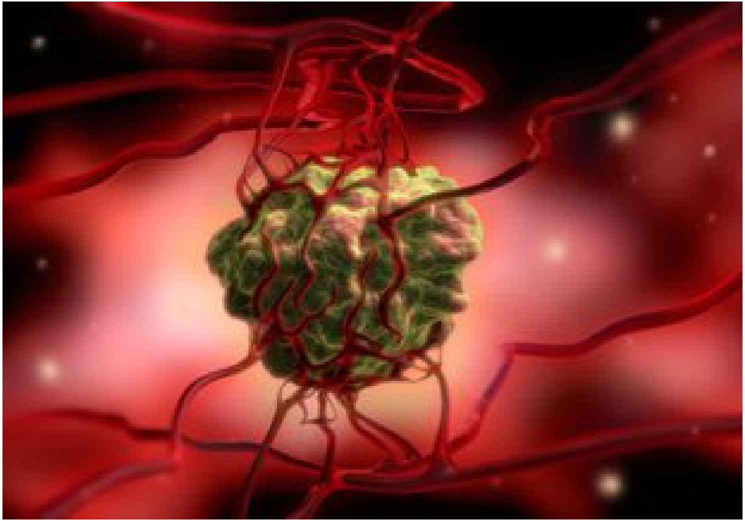

Table of contents

Dirichlet Poisson problem for $f \in \mathcal{D}(\Omega)$

For an **arbitrary** bounded $\Omega \subset \mathbb{R}^n$

$$\begin{cases} -\Delta u = f \text{ in } \Omega, & \quad (f \in L^2(\Omega)) \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1}$$

Let $H^1_0(\Omega) = \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{H^1(\Omega)}}$.

$$\exists\, !u \in H^1_0(\Omega) : \quad \forall \phi \in H^1_0(\Omega) \quad (\nabla u, \nabla \phi)_{L^2(\Omega)} = (f, \phi)_{L^2(\Omega)}$$

Regularity of the weak solution related with the regularity of the boundary for $f \in \mathcal{D}(\Omega)$

$$\boxed{\text{Regular boundary} \quad \Longleftrightarrow \quad \text{regularity of the weak solution}}$$

1. $\partial\Omega \in C^\infty$: existence of classical derivatives, classical solution

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3. **Lipschitz boundaries**: $\exists \nu$ a.e. on $\partial\Omega$, $u \in C^\infty(\Omega) \cap C(\overline{\Omega})$, but not necessarily in $H^2(\Omega)$ (yes, if Ω is convex P. Grisvard 1974)

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4. **self-similar fractal boundaries of a NTA-domain**:
   $\nexists \nu \ \forall x \in \partial\Omega$ and $\exists \frac{\partial u}{\partial \nu}$ only in the weak sense;
   $$u \in H^1_0(\Omega) \cap C^\infty(\Omega) \cap C(\overline{\Omega}), \quad \text{but} \quad u \notin H^2(\Omega)$$
   Nyström, 1996, von Koch's snowflake

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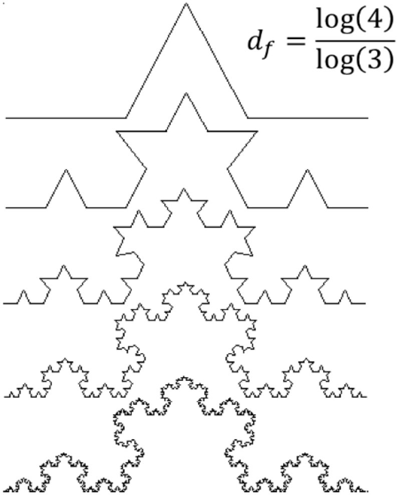
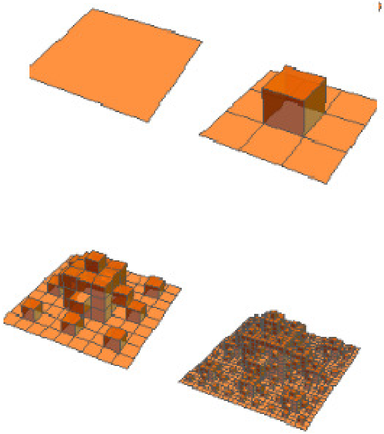
5. general case

$$u \in H^1_0(\Omega) \cap C^\infty(\Omega)$$

Examples of self-similar fractal boundaries





$2 < d = \frac{\log(13)}{\log(3)} \approx 2.33 < 3$ (Wikipedia)

Mixed boundary Poisson problem on non-Lipschitz domains: open problem

$\Omega \subset \mathbb{R}^n$ be bounded domain with a compact non-Lipschitz boundary
$\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_R$ or $\partial\Omega = \Gamma_R$

$$\begin{cases} -\Delta u = f \text{ in } \Omega, \qquad (f \in L^2(\Omega)) \\ u = 0 \text{ on } \Gamma_D, \\ \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_N, \\ \frac{\partial u}{\partial n} + a\mathrm{Tr} u = 0 \text{ on } \Gamma_R, \qquad (a > 0) \end{cases}$$

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$$V(\Omega) := \{u \in H^1(\Omega) |\ Tr_{\Gamma_D} u = 0\}.$$

endowed with the following norm

$$\|u\|^2_{V(\Omega)} = \int_\Omega |\nabla u|^2\, dx + a \int_{\Gamma_R} |Tr_{\partial\Omega} u|^2 \mathrm{d}\mu,$$

$$\forall f \in L^2(\Omega)\ \exists! u \in V(\Omega): \quad \forall v \in V(\Omega) \quad (u, v)_{V(\Omega)} = (f, v)_{L^2(\Omega)}.$$

Mixed boundary Poisson problem on non-Lipschitz domains: open problem

- **D. Daners**, Robin boundary value problems on arbitrary domains. Trans. Amer. Math. Soc. 352(9), 4207–4236 (**2000**).

  $\mu = \mathcal{H}^{n-1}$, if $\mathcal{H}^{n-1}(\Gamma_R) = +\infty$, then $\mathrm{Tr} u|_{\Gamma_R} = 0$ (Dirichlet boundary condition).

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  Von Koch type fractal boundaries in $\mathbb{R}^2$, d-measure.

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- **A. Dekkers, ARP, A. Teplyaev**, Calc. Var. (2022);

  M. Hinz, ARP, A. Teplyaev, submitted, preprint

  R^n, μ is an upper d-regular Borel measure, $n-2 < d < n$.

Table of contents

The "worst" boundary

"Sobolev admissible domains":

We consider **the Sobolev extension domains** Ω with compact boundaries $\partial\Omega$ defined by the support of a positive Borel measure μ on $\mathbb{R}^n$

$$\partial\Omega = \operatorname{supp}\mu,$$

which in addition is **upper *d*-regular** for a fixed $d \in]n-2, n[$: there is a constant $c_d > 0$ such that

$$\mu(B_r(x)) \leq c_d r^d, \quad x \in \partial\Omega, \quad 0 < r \leq 1. \tag{2}$$

($\Longrightarrow \dim_H \partial\Omega \geq d$)

Examples, remarks

- d-sets: $\dim_H \partial\Omega = d > 0$
  $\exists c_1, c_2 > 0$,

$$c_1 r^d \leq \mu(\partial\Omega \cap \overline{B_r(x)}) \leq c_2 r^d, \quad \text{for } \forall\, x \in \partial\Omega,\ 0 < r \leq 1,$$

- Lipschitz and more regular boundaries
- bounded dimension boundaries

$$n - 2 < \dim_H \partial\Omega < n$$

Definition of $W^{k,p}$-extension domains

Definition

*A domain $\Omega \subset \mathbb{R}^n$ is called a $W^{k,p}$**-extension domain** (for $k \in \mathbb{N}^*$, $1 \leq p \leq \infty$) if there exists a bounded linear extension operator $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n)$:*

$$\forall u \in W^{k,p}(\Omega) \quad \exists v = Eu \in W^{k,p}(\mathbb{R}^n) \text{ with } v|_\Omega = u \text{ and } C(k,p,\Omega) > 0 :$$

$$\|v\|_{W^{k,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{k,p}(\Omega)}.$$

Equivalently, there exists a linear continuous trace/restriction operator

$$\mathrm{Tr} : W^{k,p}(\mathbb{R}^n) \to W^{k,p}(\Omega).$$

Geometrical properties of Ω ensuring that Ω is a $W^{k,p}$-extension domain ????

Known $W^{k,p}$-extension domains

Theorem
If a domain $\Omega \in D \Longrightarrow \Omega$ is a $W^{k,p}$-extension domain.

1. Calderon [1961], Stein [1970] :

$$D = \{\text{Lipschitz domains}\} =: D_{Lip}$$

2. Jones [1981] : $(D_{loc\ unif} \supsetneq D_{Lip})$

$$D = \{\text{locally uniform or } (\varepsilon, \delta)\text{-domains}\} =: D_{loc\ unif}$$

Theorem ($n = 2$, Jones [1981])
Let $D_{\mathbb{R}^2} = D_{loc\ unif} \cap \{\text{finitely connected domains in } \mathbb{R}^2\}$. Then

$$\Omega \in D_{\mathbb{R}^2} \quad \Longleftrightarrow \quad \Omega \text{ is a } W^{k,p}\text{-extension domain.}$$

Herron, Koskela [1991]: a bounded $\Omega \in D_{loc\ unif}$ is an uniform domain.

Locally uniform or (ε, δ)-domains ($\varepsilon > 0$, $0 < \delta \leq \infty$)

Definition
An open connected subset Ω of $\mathbb{R}^n$ is an (ε, δ)-domain,

if whenever $x, y \in \Omega$ and $|x - y| < \delta$, ***(thus locally)***

there is a rectifiable arc $\gamma \subset \Omega$ with length $\ell(\gamma)$ joining x to y and satisfying

1. $\ell(\gamma) \leq \frac{|x-y|}{\varepsilon}$ ***(uniformly locally quasiconvex)*** *and*
2. $d(z, \partial\Omega) \geq \varepsilon|x - z|\frac{|y-z|}{|x-y|}$ *for $z \in \gamma$.*

Theorem ($n = 2$, Jones [1981])
A bounded and finitely connected domain $\Omega \in D_{loc\ unif} \iff$ its boundary consists of a finite number of points and quasicircles.

d-Sets and (ε, δ)-domains A. Jonsson, H. Wallin,1984

- Ω is an n-set or satisfies "the measure density condition"

$$\exists c > 0\ \forall x \in \Omega,\ \forall r \in]0,1]\ \lambda^n(B_r(x) \cap \Omega) \geq C\lambda^n(B_r(x)) = cr^n.$$

- An n-set Ω cannot be "thin" close to its boundary $\partial\Omega$.
- n-sets $\supsetneq D_{loc\ unif} \supsetneq D_{Lip}$.

Optimal class of $W^{k,p}$-extension domains

Theorem (Hajłasz, Koskela, and Tuominen [2008])
A domain $\Omega \subset \mathbb{R}^n$ is a $W^{k,p}$-extension domain

1. *for* $1 \leq p < \infty$, $k \geq 1$, $k \in \mathbb{N} \Longrightarrow \Omega$ *is an* n*-set.*
2. *for* $p = \infty$ *and* $k = 1 \Longleftrightarrow \Omega$ *is uniformly locally quasiconvex.*

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2. *for* $p = \infty$ *and* $k = 1 \Longleftrightarrow \Omega$ *is uniformly locally quasiconvex.*
3. *for* $1 < p < \infty$, $k \ge 1$, $k \in \mathbb{N} \Longleftrightarrow \Omega$ *is an* n*-set and* $W^{k,p}(\Omega) = C^{k,p}(\Omega)$ *with norms' equivalence.*

By $C^{k,p}(\Omega)$ is denoted the **space of the fractional sharp maximal functions**:
$C^{k,p}(\Omega) = \{f \in L^p(\Omega) | f^\sharp_{k,\Omega} \in L^p(\Omega)\}$, where

$$f^\sharp_{k,\Omega}(x) = \sup_{r>0} r^{-k} \inf_{P \in \mathcal{P}^{k-1}} \frac{1}{\lambda^n(B_r(x))} \int_{B_r(x)\cap\Omega} |f - P| dy,$$

with the norm $\|f\|_{C^{k,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|f^\sharp_{k,\Omega}\|_{L^p(\Omega)}$.

Trace operator A. Jonnson, 2009

Definition
For a Sobolev extension domain Ω of $\mathbb{R}^n$ with $\operatorname{supp}\mu = \partial\Omega$ (for an upper regular Borel measure μ),
the trace operator $\operatorname{Tr} : H^1(\Omega) \to L^2(\partial\Omega, \mu)$ is defined μ-a.e. by

$$x \in \partial\Omega \qquad \operatorname{Tr} u(x) = \lim_{r\to 0} \frac{1}{\lambda^n(\Omega \cap B_r(x))} \int_{\Omega\cap B_r(x)} u(y)dy.$$

$$B(\partial\Omega, \mu) := \operatorname{Tr}(H^1(\Omega)).$$

Properties of μ, $\operatorname{supp}\mu = \partial\Omega$ are important to caracterize $B(\partial\Omega, \mu)$:

$$H^{\frac{1}{2}}(\partial\Omega), \quad B^{2,2}_{1-\frac{n-d}{2}}(\partial\Omega), \quad B^{2,2}_{1}(\partial\Omega), \ \ldots$$

---

d-sets, H. Wallin 1991
Jonnson 1997

Trace theorem on boundaries given by upper d-regular measures μ

1. Let Ω be a bounded $W^{1,2}(\Omega)$-extension domain in $\mathbb{R}^n$
2. Let $\partial\Omega = \operatorname{supp}\mu$ be compact, $0 < n-2 < d \leq n$ for a Borel positive measure μ s.t.

$$c_d > 0 \quad \mu(B(x,r)) \leq c_d r^d, \quad x \in \partial\Omega, \quad 0 < r \leq 1. \tag{3}$$

Then

(i) $\mathrm{Tr} : H^1(\Omega) \to L^2(\partial\Omega, \mu)$ is compact operator and $\exists\, c_{\mathrm{Tr}}(n, \Omega, d, c_d) > 0$, s. t. $\|\mathrm{Tr} f\|_{L^2(\partial\Omega,\mu)} \leq c_{\mathrm{Tr}} \|f\|_{H^1(\Omega)}\,, \quad f \in H^1(\Omega).$

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(ii) $B(\partial\Omega, \mu) := \mathrm{Tr}(H^1(\Omega))$ is a Hilbert space (compact and dense in $L^2(\partial\Omega, \mu)$)

$$\|\varphi\|_{B(\partial\Omega,\mu)} := \inf\{\|g\|_{H^1(\Omega)} \mid \varphi = \mathrm{Tr}\ g\}.$$

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(iii) $\exists$ a linear operator $H_{\partial\Omega} : B(\partial\Omega,\mu) \to H^1(\Omega)$ of norm one s. t. $\forall \varphi \in B(\partial\Omega,\mu)$ $\operatorname{Tr}(H_{\partial\Omega}\varphi) = \varphi$.

M. Hinz, ARP, A. Teplyaev, SIAM SICON, 2021, M. Hinz, F. Magoulès, ARP, M. Rynkovskaya, A. Teplyaev, Applied Mathematical Modelling 2021.

Some important corrolaries

Norm equivalence:
If $\mathrm{Tr} : H^1(\Omega) \to L^2(\partial\Omega, \mu)$ is compact, then the norm $\|u\|_{H^1(\Omega)}$ on $H^1(\Omega)$ is equivalent to

$$\|u\|_{\mathrm{Tr}} = \left(\int_\Omega |\nabla u|^2 \mathrm{d}x + \int_{\partial\Omega} |\mathrm{Tr} u|^2 d\mu \right)^{\frac{1}{2}} .$$

Compact embedding:
If Ω is bounded and a Sobolev extension domain, then the embedding

$$H^1(\Omega) \subset L^2(\Omega) \text{ is compact.}$$

Green formula

Thanks to multiple works of M. R. Lancia (***d***-sets, Jonsson measures), we obtain

Proposition
Let $\Omega \subset \mathbb{R}^n$ be a Sobolev extension domain with a compact boundary $\partial\Omega = \mathrm{supp}\mu$ an upper-regular positive Borel measure with $n-2<d<n$.

Then for all $u, v \in H^1(\Omega)$ with $\Delta u \in L^2(\Omega)$

$$\langle \frac{\partial u}{\partial \nu}, \mathrm{Tr}\, v\rangle_{B'(\partial\Omega,\mu),B(\partial\Omega,\mu)} := \int_\Omega v\Delta u dx + \int_\Omega \nabla v \cdot \nabla u dx.$$

Remark
$\Delta u \in L^2(\Omega)$: $\exists f \in L^2(\Omega)$ *s.t.* $-\Delta u = f$ *with for example* $\frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0$.

$\Longrightarrow$ u *is the weak solution of the Neumann Poisson problem.*

Mixed boundary Poisson problem

For $\Omega \subset \mathbb{R}^n$ a Sobolev extension domain with a compact boundary $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_R = \mathrm{supp}\mu$ with an upper regular Borel mesure μ and compact Γ_D, Γ_R, s. t. $\mu(\Gamma_D \cap \Gamma_N) = \mu(\Gamma_D \cap \Gamma_R) = \mu(\Gamma_N \cap \Gamma_R) = 0$.

$$\begin{cases} -\Delta u = f \text{ in } \Omega, & (f \in L^2(\Omega)) \\ u = 0 \text{ on } \Gamma_D, & \\ \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_N, & \\ \frac{\partial u}{\partial n} + a\mathrm{Tr} u = 0 \text{ on } \Gamma_R, & (a > 0) \end{cases}$$

$$V(\Omega) := \{u \in H^1(\Omega) |\ Tr_{\Gamma_D} u = 0\}.$$

endowed with the following norm

$$\|u\|^2_{V(\Omega)} = \int_\Omega |\nabla u|^2\, dx + a \int_{\Gamma_R} |Tr_{\partial\Omega} u|^2 \mathrm{d}\mu,$$

$$\forall f \in L^2(\Omega)\ \exists! u \in V(\Omega) : \quad \forall v \in V(\Omega) \quad (u, v)_{V(\Omega)} = (f, v)_{L^2(\Omega)}.$$

A. Dekkers, ARP, A. Teplyaev, Calculus of Variations and Partial Differential Equations, 2022

Weak well-posedness of the Helmholtz problem

Let μ be a positive Borel measure: $\operatorname{supp}\mu = \partial\Omega$ is a compact in $\mathbb{R}^n$.

$$V(\Omega) = \{u \in H^1(\Omega) |\ \operatorname{Tr} u = 0 \text{ on } \Gamma_{Dir}\}$$

$$\|u\|^2_{V(\Omega,\mu)} = \int_\Omega |\nabla u|^2 dx + \int_\Gamma \mathrm{Re}(\alpha)|\operatorname{Tr} u|^2 d\mu \text{ equivalent to } \|u\|^2_{H^1(\Omega)}$$

$\forall f \in L^2(\Omega)$, and $\omega > 0$ there exists a unique solution $u \in V(\Omega)$,

$$\forall v \in V(\Omega) \quad \int_\Omega \nabla u \cdot \nabla \bar{v} dx - \omega^2 \int_\Omega u\bar{v} dx + \int_\Gamma \alpha \,\mathrm{Tr}\, u \,\mathrm{Tr}\, \bar{v}\, d\mu = -\int_\Omega f\bar{v} dx$$

$$\exists C(\alpha, \omega, C_{Poincaré}(\Omega)) > 0 : \quad \|u\|_{H^1(\Omega)} \le C\|f\|_{L^2(\Omega)}$$

Methods for evolutive in time problems

- Galerkin method based on the spectral problem of $-\Delta$
- To work in the Hilbert space of the weak solutions of the Poisson problem:

$$\mathcal{D}(-\Delta) = \{u \in H^1(\Omega) | \quad -\Delta u \in L^2(\Omega) : \\ \exists f \in L^2(\Omega) \quad \forall v \in V(\Omega) \quad (u,v)_{V(\Omega)} = (f,v)_{L^2(\Omega)}\}$$

- Fix point type theorems of functional analysis
- Approximation by the solutions on regular boundaries

  (with converging (extension) sequence of initial conditions; $\rightharpoonup H^1(\mathbb{R}^n)$)

  - $\Omega_m \to \Omega$ in the sense of Hausdorff and caracteristic functions in D;
  - Mosco convergence; $VF_m(v_m, \phi) \to VF(u, \phi)\ \forall \phi \in H(D)$
  - uniform on m linear bounded extension $E : H^1(\Omega_m) \to H^1(D)$
  - $(Ev_m)_{m\in\mathbb{N}}$ is uniformly bounded on m
  - $\forall t \geq 0 \quad Ev_{m_k}|_\Omega \to u$ in $H^1(\Omega)$

Table of contents

Westervelt problem and known results (PhD of A. Dekkers)

$$\begin{cases} \partial_t^2 u - c^2\Delta u - \nu\Delta\partial_t u = \alpha u\partial_t^2 u + \alpha(\partial_t u)^2 + f \quad on \quad]0,T] \times \Omega, \\ u = 0 \quad \text{on } \Gamma_D \times [0,T], \\ \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_N \times [0,T], \\ \frac{\partial u}{\partial n} + au = 0 \quad \text{on } \Gamma_R \times [0,T], \\ u(0) = u_0, \quad \partial_t u(0) = u_1. \end{cases}$$

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Bounded domain with C^2 boundary:

- B. Kaltenbacher, I. Lasiecka, 2009, 2012 ($\partial\Omega = \Gamma_D$ non homogeneous) 2011 (Robin or Neumann non homogeneous) $n \leq 3$;
- S. Meyer, M. Wilke, 2013 (Dirichlet non homogeneous case, all n, $W^{k,p}$).

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In the Non-Lipschitz case, no access to

- the H^2-regularity (thus high energy a priori estimates)
- Nyström: $w \in H_0^1(\Omega),\ -\Delta w = f \in L^2(\Omega) \quad \|\nabla w\|_{L^6(\Omega)} \not\lesssim C\|\Delta w\|_{L^2(\Omega)}$

Westervelt problem and known results (PhD of A. Dekkers)

$$\begin{cases} \partial_t^2 u - c^2\Delta u - \nu\Delta\partial_t u = \alpha u\partial_t^2 u + \alpha(\partial_t u)^2 + f \quad \textit{on} \quad [0,T]\times\Omega, \\ u = 0 \quad \text{on } \Gamma_D \times [0,T], \\ \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_N \times [0,T], \\ \frac{\partial u}{\partial n} + au = 0 \quad \text{on } \Gamma_R \times [0,T], \\ u(0) = u_0, \quad \partial_t u(0) = u_1. \end{cases}$$

Domain Ω	Linear equation	Nonlinear equation
$\partial\Omega = \Gamma_D$ in $\mathbb{R}^2$	arbitrary	NTA or limit of NTA domains
$\partial\Omega = \Gamma_D$ in $\mathbb{R}^3$	arbitrary	arbitrary
$\Gamma_R \neq \varnothing$ in $\mathbb{R}^2$ or $\mathbb{R}^3$	Sobolev admissible	Sobolev admissible

Estimate of $\|u\|_{L^\infty(\Omega)}$

Theorem
Let Ω be a bounded domain and $f \in L^p(\Omega)$ $p \geq 2$, then for u weak solution of the Poisson problem

$$\|u\|_{L^\infty(\Omega)} \leq C\|f\|_{L^p(\Omega)} = C\|\Delta u\|_{L^p(\Omega)}.$$

1. *If $\partial\Omega = \Gamma_{Dir}$*
   - *$\Omega \subset \mathbb{R}^2$ NTA domains (Nyström (1994)),*
   - *$\Omega \subset \mathbb{R}^3$ arbitrary domain (Xie (1991)).*
2. *If $\partial\Omega = \Gamma_{Rob}$ and $\Omega \subset \mathbb{R}^n$*
   - *Daners (2000): $p > n$ for $n-1$-dimensional boundaries, $C = \tilde{C}\max\left(1, \frac{1}{a}\right)$*
   - *A. Dekkers, ARP: $p \geq 2$ for Sobolev admissible domains;*
3. *If $\partial\Omega = \Gamma_{Rob} \cup \Gamma_{Dir} \cup \Gamma_{Neu}$, $\Omega \subset \mathbb{R}^n$*
   - *A. Dekkers, ARP, A. Teplyaev, 2022: $p \geq 2$, if Ω is (ε, ∞)-domain, then $C = C(\varepsilon, n, C_P)$, but not on a.*

Mixed problem for the Westervelt equation, $\nu > 0$, $p = 2$

Theorem
Let Ω be bounded Sobolev admissible domain of $\mathbb{R}^2$ or $\mathbb{R}^3$.
For all $\phi \in L^2(\mathbb{R}^+; V(\Omega))$ with $u(0) = u_0 \in \mathcal{D}(-\Delta)$ and $\partial_t u(0) = u_1 \in V(\Omega)$, $f \in L^2(\mathbb{R}^+; L^2(\Omega))$,

$$\|f\|_{L^2(\mathbb{R}^+;L^2(\Omega))} + \|u_0\|_{\mathcal{D}(-\Delta)} + \|u_1\|_{V(\Omega)} \leq \frac{\nu}{C_2} r, \tag{4}$$

$$\int_0^{+\infty} (\partial_t^2 u, \phi)_{L^2(\Omega)} + c^2 (u, \phi)_{V(\Omega)} + \nu (\partial_t u, \phi)_{V(\Omega)} ds - \int_0^{+\infty} \alpha (u \partial_t^2 u + (\partial_t u)^2 + f, \phi)_{L^2(\Omega)} ds = 0,$$

$$\exists!\, u \in X^2 := H^1(\mathbb{R}^+; \mathcal{D}(-\Delta)) \cap H^2(\mathbb{R}^+; L^2(\Omega)) :$$

$$\exists r_* > 0 : \quad \forall r \in [0, r_*[\quad (4) \Rightarrow \quad \|u\|_{X^2} \leq 2r.$$

Application of M.F. Sukhinin's Theorem $Lu + \Phi(u) = F$

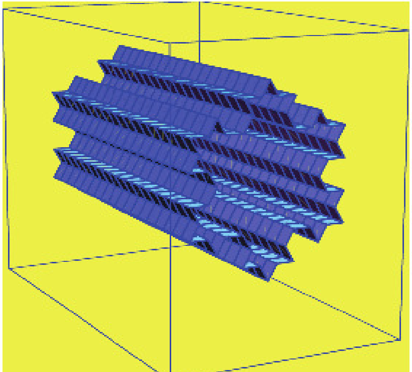
Definition for functionals and bilinear forms, U.Mosco, 1994

Definition
A sequence of functionals $G^m : H \to (-\infty, +\infty]$ is said to M-converge to a functional $G : H \to (-\infty, +\infty]$ in a Hilbert space H, if

1. *(lim sup condition) For every $u \in H$ there exists u_m converging strongly in H such that*
$$\overline{\lim} G^m[u_m] \leq G[u], \quad \text{as } m \to +\infty. \tag{5}$$
2. *(lim inf condition) For every v_m converging weakly to u in H*
$$\underline{\lim} G^m[v_m] \geq G[u], \quad \text{as } m \to +\infty. \tag{6}$$

Approximation of solutions on fractal domains by solutions on prefractal domains (irregular by regular)

- Von Koch 2D mixtures (mixed Poisson problem, R. Capitanelli, A. Vivaldi, 2010, 2011)
- cylindrical von Koch domain 3D (Venttsel problem, M. R. Lancia, P. Vernole, 2010)



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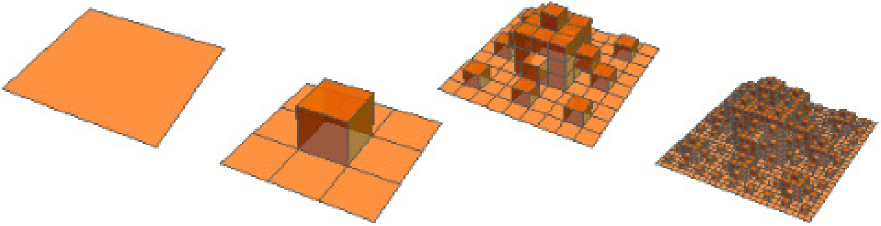

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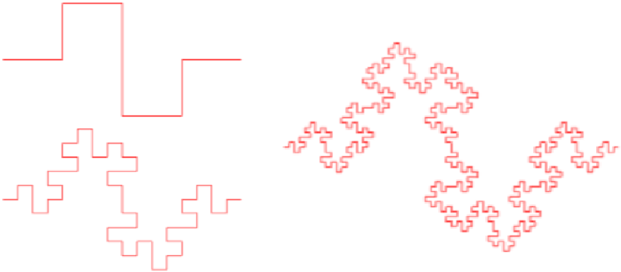

Linear problems (mixed Poisson or Helmholtz problems)

- To define a quadratic form (energy or equivalent norm of H^1)

$$b_m(u_m, u_m) = \int_{\Omega_m} (|\nabla u_m|^2 + |u_m|^2)\mathrm{d}x + \int_{\partial\Omega_m} a_m |\operatorname{Tr} u_m|^2 d\mu_m$$

on $L^2(D)^2$, $\Omega_m \subset D$

- its Mosco-convergence is ensured if
  - $\Omega_m \to \Omega$ by Hausdorff and characteristic functions ($\Omega \subset D$)
  - extension $H^\sigma(\Omega_m) \to H^\sigma(D)$ is uniform on m for $0 \le \sigma \le 1$
  - $\forall m \in \mathbb{N}$ $\|\sqrt{a_m}\operatorname{Tr}_{\partial\Omega_m} u\|_{L^2(\partial\Omega_m, \mu_m)} \le C_\sigma \|u\|_{H^\sigma(\mathbb{R}^n)}$ for $u \in H^\sigma(\mathbb{R}^n)$ $\frac{1}{2} < \sigma \le 1$
  - $a_m\mu_m \rightharpoonup a\mu$:

$$\forall \phi \in C(\overline{D}) \quad \int_{\partial\Omega_m} a_m \phi d\mu_m \to \int_{\partial\Omega} a\phi d\mu, \quad m \to +\infty$$

Linear problems (mixed Poisson or Helmholtz problems)

Let $(u_m)_{m\in\mathbb{N}}$ be the sequence of weak solutions on $(\Omega_m)_{m\in\mathbb{N}}$.

If

- the sequence of solutions is **uniformly bounded on** m:

$$\|(E_{\mathbb{R}^n}u_m)|_D\|_{H^1(D)} \leq C,$$

- $b_m(u_m, w) = 0$ is the variational formulation on Ω_m,
- $b_m(u_m, u_m) \overset{M}{\to} b(u, u)$ in $L^2(D)$ for $\Omega_m \to \Omega$

then

- $u|_\Omega$ (the weak limit of $E_{R^n}u_m|_D$) is the weak solution of $b(u, w) = 0$ on Ω,
- $(E_{\mathbb{R}^n}u_m)|_\Omega \to (E_{\mathbb{R}^n}u)|_\Omega$ in $H^1(\Omega)$.

A. Dekkers, ARP, A. Teplyaev, 2022; M. Hinz, ARP, A. Teplyaev, SICON, 2021

For the Westervelt mixed problem, $\nu > 0$, $p = 2$

Let $\Omega_m \subset D\ \forall m \in \mathbb{N}$ ($a_m = 1/\lambda^{n-1}(\Gamma_m) \to 0$, $\mu_m = \lambda^{n-1}$, μ)

$$
\begin{aligned}
F_m[u,\phi] := & \int_0^T \int_{\Omega_m} [\partial_t^2 u\phi + \nabla u \nabla \phi + \nu \nabla \partial_t u \nabla \phi]\, d\lambda^n dt \\
& + \int_0^T \int_{\Gamma_m} a_m [Tr_{\partial\Omega_m} u + \nu Tr_{\partial\Omega_m} \partial_t u]\, Tr_{\partial\Omega_m} \phi\, d\lambda^{n-1} dt \\
& + \int_0^T \int_{\Omega_m} [-\alpha(u\partial_t^2 u) - \alpha(\partial_t u)^2 + f]\phi\, d\lambda^n dt
\end{aligned}
$$

For all $u \in L^2([0,T]; L^2(D))$, fixed $\phi \in L^2([0,T], H^1(D))$

$$
\overline{F}_m[u,\phi] = \begin{cases} F_m[u,\phi] & \text{if } u \in H^1(]0,T[, H^1(D)) \cap H^2(]0,T[; L^2(D)), \\ +\infty & \text{otherwise} \end{cases}
$$

A. Dekkers, ARP, A. Teplyaev, 2022

For the Westervelt mixed problem, $\mathbb{R}^2$, $\mathbb{R}^3$, $\nu > 0$, $p = 2$

Theorem

1. $(u \mapsto \overline{F}_m[u,\phi]) \overset{M}{\to} (u \mapsto \overline{F}[u,\phi])$ *in* $L^2([0,T];L^2(D))$

2. $\forall \phi \in L^2([0,T];H^1(D))$ *if* $v_m \rightharpoonup u$ *in* $H(D) = H^1(]0,T[,H^1(D)) \cap H^2(]0,T[;L^2(D))$,

$$\textit{then} \quad F_m[v_m,\phi] \underset{m\to+\infty}{\longrightarrow} F[u,\phi]$$

3. $\partial\Gamma_{Dir,\Omega_m} = \partial\Gamma_{Dir,\Omega} = \partial\Gamma_{Dir,D}$

$$(E_{\mathbb{R}^n}u_{0,m})|_\Omega \underset{m\to+\infty}{\rightharpoonup} u_0, \quad (E_{\mathbb{R}^n}u_{1,m})|_\Omega \underset{m\to+\infty}{\rightharpoonup} u_1 \text{ in } H^1(\Omega),$$

then $(E_{\mathbb{R}^n}u_m)|_D \rightharpoonup u^*$ *in* $H(D)$ *with* $u^*|_\Omega = u$ $((E_{\mathbb{R}^n}u_m)|_\Omega \to u$ *in* $H^1(\Omega))$

Table of contents

Conclusion

Solving PDEs on domains with Non-Lipschitz boundaries.

Approximation of d-sets.

Rough boundaries are the energy minimizers.

Thank you very much for your attention!