NON-LINEAR EQUATIONS ON FRACTALS

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The Joint Meeting of The AMS-EMS-SMF
July 18th-22nd, 2022, Grenoble, France
Special Session on Fractal Geometry in Pure and Applied Mathematics
To Sasha

My coconspirator extraordinaire!

Bob Sturmfels

Sasha Teplyaev (UConn)
Welcome!

Planning has begun for Fractals 7 (June 9-13, 2020). The purpose of this conference, held every three years, is to bring together mathematicians who are already working in the area of analysis and probability on fractals with students and researchers from related areas. Information will be posted here as it becomes available.

Financial support will be provided to a limited number of participants to cover the cost of housing in Cornell single dormitory rooms and partially support other travel expenses. Students and junior researchers from underrepresented groups in STEM are particularly encouraged to apply for travel funding. Well-established researchers are encouraged to use their own travel funding; the NSF expects that most funds will be expended on otherwise unfunded mathematicians.

Registration details will be publicized once available.

All general inquiries can be sent to: fractals_math@cornell.edu

Conference Organizers:

- Robert Strichartz (chair), Cornell University
- Patricia Alonso Ruiz, Texas A&M University
- Michael Hinz, Bielefeld University
- Luke Rogers, University of Connecticut
- Alexander Teplyaev, University of Connecticut
abstract of the talk

The talk will discuss several approaches to study non-linear non-Lipschitz PDE-like equations on fractals, in particular the joint work with Michael Hinz and Anna Rozanova-Pierrat, and also [Hinz-Meinert], [Falconer-Hu], [Strichartz et al.]. I also will discuss a general infinite dimensional framework in a joint work with Maria Gordina and Michael Röckner **Ornstein-Uhlenbeck processes with singular drifts: integral estimates and Girsanov densities.** We consider a perturbation of a Hilbert space-valued Ornstein-Uhlenbeck process by a class of singular nonlinear non-autonomous maximal monotone time-dependent drifts. The only further assumption on the drift is that it is bounded on balls in the Hilbert space uniformly in time. First we introduce a new notion of generalized solutions for such equations, which we call pseudo-weak solutions, and prove that they always exist and obtain pathwise estimates in terms of the data of the equation. We show that pseudo-weak solutions have continuous sample paths and the laws are absolutely continuous with respect to the law of the original Ornstein-Uhlenbeck process. We obtain integrability estimates of the associated Girsanov densities. Some of our results concern non-random equations as well, and results are new even in finite-dimensional autonomous settings.
Non-linear equations on fractals: outline of the talk

- Introduction
  - Kigami ((1989) – current)
  - Strichartz et al. (2004 – )
  - Falconer–Hu (1999–2012)
  - Lancia, Vélez-Santiago, Vernole (2019)
  - Hinz–Meinert (2020–2022)
  - joint with Michael Hinz and Anna Rozanova-Pierrat (2021 – current)
  - joint with Maria Gordina and Michael Röckner

  Ornstein-Uhlenbeck processes with singular drifts:
  integral estimates and Girsanov densities
  - selected technical details

- Motivation (as time permits)

This is a part of the broader program to develop **probabilistic, spectral and vector analysis on singular spaces** by carefully building approximations by graphs or manifolds.
Kigami ((1989) – current)

2:00–2:45pm Jun Kigami (Kyoto University, Japan) Conductive Homogeneity of Compact Metric Spaces and Construction of $p$-Energy

In the ordinary theory of Sobolev spaces on domains of $\mathbb{R}^n$, the $p$-energy is defined as the integral of $|\nabla f|^p$. In this paper, we try to construct a $p$-energy on compact metric spaces as a scaling limit of discrete $p$-energies on a series of graphs approximating the original space. In conclusion, we propose a notion called conductive homogeneity under which one can construct a reasonable $p$-energy if $p$ is greater than the Ahlfors regular conformal dimension of the space. In particular, if $p = 2$, then we construct a local regular Dirichlet form and show that the heat kernel associated with the Dirichlet form satisfies upper and lower sub-Gaussian type heat kernel estimates. As examples of conductively homogeneous spaces, we present new classes of square-based self-similar sets and rationally ramified Sierpinski crosses, where no diffusions were constructed before.
Strichartz et al. (2004 – )

P.E. Herman, R. Peirone, R.S. Strichartz,  
**p-energy and p-harmonic functions on Sierpinski gasket type fractals.**  

...

R. Shimizu,  
**Construction of p-energy and associated energy measures on the Sierpinski carpet,**  
arXiv:2110.13902

Shiping Cao, Qingsong Gu, Hua Qiu  
**p-energies on p.c.f. self-similar sets**  
arXiv:2112.10932

Fabrice Baudoin, Li Chen  
**Sobolev spaces and Poincaré inequalities on the Vicsek fractal**  
arXiv:2207.02949
Herman, P. Edward [Herman, Paul Edward] (1-CHI); Peirone, Roberto (1-ROME2); Strichartz, Robert S. (1-CRNL)

$p$-energy and $p$-harmonic functions on Sierpinski gasket type fractals. (English summary)


For $1 < p < \infty$ the classical $p$-energy of a suitable function $u$ is defined to be the integral of $|\nabla u|^p$, where $\nabla$ is the gradient. The techniques in [J. Kigami, *Analysis on fractals*, Cambridge Univ. Press, Cambridge, 2001; MR1840042] for the case $p = 2$ are extended. That is, the fractal is approximated by a sequence of “grids” endowed with their discrete $p$-energies. A nonlinear renormalization problem is solved to find an appropriate scaling factor and a corresponding self-similar discrete $p$-energy. This makes it possible to construct a limiting model, analogous to the case $p = 2$, via an increasing sequence of discrete $p$-energies. The existence proof relies on Schauder’s fixed point theorem. The uniqueness of the $p$-energy remains open. A function is called $p$-harmonic when it minimizes the $p$-energy subject to boundary conditions. Its uniqueness is again only conjectured. Numerical approximations for the Sierpiński gasket are presented. The existence proof is generalized to “weakly completely symmetric fractals” in the sense of [R. Peirone, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) **3** (2000), no. 2, 431–460; MR1769995].

Volker Metz
Falconer–Hu (1999–2012)

K. Falconer, J. Hu, Jiaxin; Y. Sun,
**Inhomogeneous parabolic equations on unbounded metric measure spaces.**

K. Falconer, J. Hu,
**Nonlinear diffusion equations on unbounded fractal domains.**

K. Falconer, J. Hu,
**Non-linear elliptical equations on the Sierpinski gasket.**

K. Falconer,
**Semilinear PDEs on self-similar fractals.**
This paper is concerned with weak solutions to the Cauchy problem for the semi-linear parabolic equation

\[ u_t = \Delta u + u^p + f(x), \quad t > 0, \ x \in M, \quad u(0, x) = \phi(x), \]

where \( p > 0 \) and the functions \( f, \phi: M \to \mathbb{R} \) are measurable and nonnegative, with \( (M, d) \) a locally compact separable metric space and \( \mu \) a Radon measure on \( M \) with full support.

Depending on the fractal dimension \( \alpha \) and the walk dimension \( \beta \) of \( M \), local and global existence and regularity properties of solutions to (1) are investigated. Conditions for these properties are imposed on the heat kernel as well as on the functions \( f, \phi \). In particular, conditions for non-existence and local/global existence of solutions to (1) are provided. The regularity of solutions of (1) is studied in terms of Hölder continuity exponents.
Semilinear PDEs on Self-Similar Fractals

K. J. Falconer

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Abstract: A Laplacian may be defined on self-similar fractal domains in terms of a suitable self-similar Dirichlet form, enabling discussion of elliptic PDEs on such domains. In this context it is shown that semilinear equations such as $\Delta u + u^p = 0$, with zero Dirichlet boundary conditions, have non-trivial non-negative solutions if $0 < \nu \leq 2$ and $p > 1$, or if $\nu > 2$ and $1 < p < (\nu + 2)/(\nu - 2)$, where $\nu$ is the “intrinsic dimension” or “spectral dimension” of the system. Thus the intrinsic dimension takes the rôle of the Euclidean dimension in the classical case in determining critical exponents of semilinear problems.
Lancia, Vélez-Santiago, Vernole (2019)


Summary: "We investigate the solvability of the Ambrosetti-Prodi problem for the $p$-Laplace operator $\Delta_p$ with Venttsel’ boundary conditions on a two-dimensional open bounded set with Koch-type boundary, and on an open bounded three-dimensional cylinder with Koch-type fractal boundary. Using a priori estimates, regularity theory and a sub-supersolution method, we obtain a necessary condition for the non-existence of solutions (in the weak sense), and the existence of at least one globally bounded weak solution. Moreover, under additional conditions, we apply the Leray-Schauder degree theory to obtain results about multiplicity of weak solutions.”
Hinz–Meinert (2020–2022)

Michael Hinz, Melissa Meinert,
**Approximation of partial differential equations on compact resistance spaces.**

Michael Hinz, Melissa Meinert,
**On the viscous Burgers equation on metric graphs and fractals.**

Burgers equation and Cole–Hopf transform:

\[ u' = \Delta u - \langle u, \nabla \rangle u \]

\[ u' = -dd^* u - d(u^2)/2 \]

Main difficulty: there are no continuous vector fields on fractals!
Define a “universal” space of finite energy (or \( L^2 \)) differential forms \( \mathcal{H} \)
joint with Michael Hinz and Anna Rozanova-Pierrat (2021 – current)

- large compact classes of fractal domains, including fractal shape optimization
- large collections of fractal PDE, including non-linear

Adrien Dekkers, Anna Rozanova-Pierrat, T.
Mixed boundary valued problems for linear and nonlinear wave equations in domains with fractal boundaries.

Michael Hinz, Anna Rozanova-Pierrat, Anna; T.
Non-Lipschitz uniform domain shape optimization in linear acoustics.

Michael Hinz, Frédéric Magoulès, Anna Rozanova-Pierrat, Marina Rynkovskaya, T.
On the existence of optimal shapes in architecture.
Part 2: Singular Stochastic Partial Differential Equations (SPDE) in Hilbert Spaces

On a singular spaces, such as a fractal with a Dirichlet form, there are several natural Hilbert spaces:

- $L^2_\mu$ with respect to a singular measure
- $H^1 = W^{1,2} = \text{Dom} \mathcal{E}$ the space of finite energy functions that are in $L^2_\mu$
- $\text{Dom}(\Delta_\nu)$ where $\nu$ is possibly a different singular measure
  - the situation is especially delicate when $\mu \perp \nu$

Conclusion: we are interested in SPDE on Hilbert spaces with most relaxed assumptions on the coefficients.

M. Gordina, M. Röckner, T.

Singular perturbations of Ornstein-Uhlenbeck processes: integral estimates and Girsanov densities.
Probability Theory and Related Fields 178 (2020)
Non-smooth perturbations of the Ornstein-Uhlenbeck operators


\begin{equation}
(1) \quad dX_t = (AX_t + F(t, X_t)) \, dt + \sigma dW_t, \quad X_0 = x \in H
\end{equation}

(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}) \text{ a filtered probability space}

\( H \) \text{ a separable Hilbert space}

\( W_t \) \text{ a cylindrical Wiener process in } H

\((A, D_A)\) \text{ the generator of a } C_0\text{-semigroup } e^{tA} \text{ such that there is an } \omega > 0 \text{ such that for any } x \in D_A

\[ \langle Ax, x \rangle \leq -\omega |x|^2 \]

\( \sigma \) \text{ and } \sigma^{-1} \text{ are in } B(H) \text{ with } \sigma \text{ being self-adjoint and positive. Moreover, there is an } \alpha > 0 \text{ such that}

\[ \int_0^\infty (1 + t^{-\alpha}) \| e^{tA} \sigma \|_{HS}^2 dt < \infty. \]
$F(t, \cdot) : [0, \infty) \times D_F(t, \cdot) \to 2^H$ a family of maps such that for any $t \in [0, \infty)$ we have $D_F(t, \cdot) = D_F \subset H$.

$F(t, \cdot)$ an $m$-dissipative map: for any $x_1, x_2 \in D_F$

$$\langle y_1 - y_2, x_1 - x_2 \rangle \leq 0,$$

for all $y_1 \in F(t, x_1), y_2 \in F(t, x_2), t \in [0, \infty)$, and for any $\gamma > 0$ and $t \in [0, \infty)$

$$\text{Range} \left( \gamma I - F(t, \cdot) \right) = H.$$

$F_0(t, x)$ it is known that for any $(t, x) \in [0, \infty) \times D_F$, the set $F(t, x)$ is non-empty, closed and convex, and so for any $x \in D_F$

$$F_0(t, x) := \{ y \in F(t, x) : |y| = \inf \{|z|, z \in F(t, x)\} \}$$

is a well-defined map.

$a(u)$ an increasing function $a : [0, \infty) \to [0, \infty)$ such that

$$|F_0(t, x)| \leq a(|x|), (t, x) \in [0, \infty) \times H,$$

where we allow for $\lim_{u \to \infty} a(u) = \infty$. 

\[\text{Sasha Teplyaev (UConn) NON-LINEAR EQUATIONS ON FRACTALS July 20, 2022 15 / 55}\]
Yosida approximation to $F$:

for any $\alpha > 0$

$$F_\alpha (t, x) := \frac{1}{\alpha} (J_\alpha (x) - x), \ x \in H,$$

where

$$J_\alpha (x) := (I - \alpha F)^{-1} (x), \ I (x) = x.$$ 

Then each $F_\alpha$ is single-valued, dissipative, Lipschitz continuous and satisfies

$$\lim_{\alpha \to 0} F_\alpha (t, x) = F_0 (t, x),$$

$$|F_\alpha (t, x)| \leq |F_0 (t, x)|$$

for $x \in D(F)$, and $F_\alpha$ satisfy the same growth condition as $F_0$. 
\[ dX_t^\alpha = (AX_t + F^\alpha (t, X_t^\alpha)) \, dt + \sigma dW_t \]

\[ X_0^\alpha = x \in H \]

● **Moments of** \( X_t^\alpha (x) \)

Stochastic \( W_{x,A,\sigma} (t) := e^{tA}x + \int_0^t e^{(t-s)A}\sigma dW (s) \)

convolution

\( W_{0,A,\sigma} (t) \) is a Gaussian random variable with values in \( H \) with the mean 0 and the covariance operator \( Q_t x = \int_0^t e^{sA}\sigma^2 e^{sA^*} x ds \).

Moment space \( \mathcal{M} \) the space of \( C^2 \)-functions \( \varphi : (0, \infty) \rightarrow [0, \infty) \) such that

- \( \varphi \) is an increasing convex function;
- the limit \( \frac{u\varphi'(u)}{\varphi(u)} \xrightarrow{u \rightarrow \infty} L_\varphi \) exists, and \( L_\varphi \in [1, \infty] \).

Remark. This definition is motivated by de la Vallée-Poussin Theorem (a criterion for uniform integrability).

Examples of functions in \( \mathcal{M} \).

\( \varphi (u) = u^p, \quad p \geq 1 \) with \( L_\varphi = p \);
\( \varphi (u) = e^u \) with \( L_\varphi = \infty \);
\( \varphi (u) = u \ln (u + 1) \) with \( L_\varphi = 1 \).
Theorem ($\varphi$-moments). Consider (2) with $F^\alpha$ being dissipative, Lipschitz continuous and satisfying

$$|F^\alpha(t, x)| \leq a(|x|), (t, x) \in [0, \infty) \times H.$$

Then for any $\varphi \in \mathcal{M}$ the following (uniform) estimate holds

$$\mathbb{E}^x \varphi \left( |X_t^\alpha(x)|^2 \right) \leq \frac{e^{-\omega t}}{2} \varphi (4|x|^2) + \mathbb{E}^x K_\varphi + \mathbb{E}^x K_{\varphi, \omega, a(\cdot)}.$$
where $K_{\phi}$ and $K_{\phi,\omega,a(\cdot)}$ are the following random functions

$$K_{\phi,\omega,a(\cdot)}(t) := \frac{\omega t}{2} \varphi \left( \frac{a \left( \sup_{s \in [0,t]} W_{0,A,\sigma}(s) \right)}{\omega^2} \right)^2,$$

$$K_{\phi}(t) := \frac{1}{2} \varphi \left( 4 \left| \sup_{s \in [0,t]} W_{0,A,\sigma}(s) \right|^2 \right).$$

Note that these functions are finite a.s.

Examples of functions $\phi \in \mathcal{M}$ and $a(u)$:

- $\phi(u)$ and $a(u)$ are polynomials;
- $\phi(u)$ is an exponential function and $a(u)$ is a linear function (Fernique’s Theorem).
Theorem (Uniform integrability of Girsanov densities). For any growth function $a(\cdot)$ and any fixed $x$ and $t$, there is an increasing positive unbounded function $\Psi(\cdot)$ such that

$$E\rho_\alpha(x, t)\Psi(\rho_\alpha(x, t)) \leq 1$$

Remark. The function $\psi$ does not depend on $\alpha$! Examples of functions $\Psi$: if $a(u)$ is of polynomial growth, then $\Psi(u)$ can be chosen to be a power of $\log$; if $a(u)$ is of exponential growth, then $\Psi(u)$ can be chosen to be an iterated $\log$.

Method of proof: localization (stopping times) for the stochastic convolution $W_{x, A, \sigma}(t)$ and $\int_0^t e^{-\omega(t-s)}a(\|W_{0, A, \sigma}(s)\|)\,ds$
Lemma

Consider

\[ X_t = Z_t + W_{0,A,\sigma}(t) \]

\[ dZ_t = (AZ_t + F(t, Z_t + W_{0,A,\sigma}(t))) \, dt, \quad Z_0 = x, \]

with coefficients satisfying our Assumptions and its mild solution \( Z_t^x \) that can be obtained as a weak limit solutions of regularized equation. Then \( |Z_t| \) is an absolutely continuous function almost surely satisfying

\[ \frac{d}{dt} (|Z_t^x|) \leq -\omega |Z_t^x| + a (|W_{0,A,\sigma}(t)|) \]

for almost all \( t \geq 0 \), and

\[ |Z_t^x| \leq |x| e^{-\omega t} + \int_0^t e^{-\omega(t-s)} a (|W_{0,A,\sigma}(s)|) \, ds \]

for all \( t \geq 0 \).
Application

\[ dX_t = (AX_t + F(X_t)) \, dt + \sigma dW_t, \quad X_0 = x \in H \]

\[ k \in D(A), \; \varepsilon > 0, \; t_0 \in (0, \; T], \; 0 \leq t \leq t_0, \]

\[ F_{\alpha, \varepsilon}^k(t, x) := -\frac{\varepsilon t}{t_0} A k + \frac{\varepsilon}{t_0} k + F_{\alpha} \left( x - \frac{\varepsilon t}{t_0} k \right), \]

\[ F^{k, \varepsilon}(t, x) := -\frac{\varepsilon t}{t_0} A k + \frac{\varepsilon}{t_0} k + F \left( x - \frac{\varepsilon t}{t_0} k \right), \]

\[ G^{k, \varepsilon}(t, x) := \sigma^{-1} \left( F_{\alpha, \varepsilon}^k(t, x) - F_{\alpha}(t, x) \right). \]

**Theorem (Quasi-invariance).** At any fixed time \( t_0 \in [0, \; T] \) the distribution of \( X_{\alpha}(t_0) \) is quasi-invariant under translations by elements in \( D(A) \). More precisely, for any unit vector \( k \in D(A), \; 0 < \varepsilon < \min\{t_0, \; 1\} \) and \( t_0 \in (0, \; T] \), the distribution of the random variable \( X_{\alpha}(t_0, x) + \varepsilon k \) has a density, \( \rho_{t_0, \varepsilon, k}(y) \), with respect to the distribution of the random variable \( X_{\alpha}(t_0, x) \).
Further directions:

- smoothness results for an invariant measure for (1)
- closability of the gradient
- looking for examples with an invariant measure which is not absolutely continuous with respect to the Gaussian measure
References (partial list)


Quasi-invariance and its applications

\((X, \mathcal{B})\) a measure space

\(G\) a group of automorphisms of \((X, \mathcal{B})\)

\(\mu\) a Radon measure on \((X, \mathcal{B})\) is said to be quasi-invariant under the action by \(G\) if the transformed measure \(\mu_g (A) := \mu \left( g^{-1} A \right)\), \(A \in \mathcal{B}\), is equivalent to the measure \(\mu\).

\[ J_g (x) \frac{d\mu_g}{d\mu} (x), x \in X \] is the Radon-Nikodym derivative

Example \(X = W\) is an abstract Wiener space with the Gaussian measure \(\mu\), and \(G\) is the group of translations by elements in the Cameron-Martin subspace \(H\).
• Integration by parts formula when $G$ is the group of translations.

\[ j_h(x) \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} J_{\varepsilon h}(x), \quad x \in X, \text{ then} \]

\[ \int_X \frac{f(x + \varepsilon h) - f(x)}{\varepsilon} d\mu(x) = \int_X f(x) \frac{J_{\varepsilon h}(x) - 1}{\varepsilon} d\mu(x), \]

\[ \int_X D_h f(x) d\mu(x) = \int_X f(x) j_h(x) d\mu(x). \]

• Quasi-invariance as smoothness of measures in infinite dimensions.

• Regularity of invariant measures in infinite dimensions and for degenerate semigroups; closability of the gradient (joint with M. Röckner)
Technical details: Pseudo-weak convergence

For \((S, \mathcal{F}, \mu)\) is \(\sigma\)-finite measure space, \(L^0(S, \mu; H)\) we denotes the \(\mathcal{F}\)-measurable functions defined up to sets of \(\mu\)-measure zero and equipped with the topology of convergence in measure. We use \(\underset{n \rightarrow \infty}{\longrightarrow}\) for the weak convergence in Banach spaces.

Suppose \(f, f_n \in L^0(S, \mu; H), \ n \in \mathbb{N}\). We say that \(\{f_n\}_{n=1}^{\infty}\) converges pseudo-weakly to \(f\), denoted by

\[
\underset{n \rightarrow \infty}{\longrightarrow} f_n \underset{\psi}{\longrightarrow} f,
\]

if

\[
\psi(f_n) \underset{n \rightarrow \infty}{\longrightarrow} \psi(f) \text{ in } L^2(S, \mu_A; H) \text{ for any } A \in \mathcal{F} \text{ with } \mu(A) < \infty \tag{1}
\]

and for some \(\psi: H \rightarrow H\) defined by

\[
\psi(h) := \begin{cases} 
\frac{h}{|h|_H} \psi_0(|h|_H), & \text{if } h \neq 0, \\
0, & \text{if } h = 0,
\end{cases}
\]

where \(\psi_0: \mathbb{R}_+ \rightarrow \mathbb{R}_+\) is a strictly increasing continuous function with \(\psi_0(0) = 0\). In this case we say that \(f\) is a \(\psi\)-pseudo-weak limit of the sequence \(\{f_n\}_{n=1}^{\infty}\).
The definition of $\psi$-pseudo-weak limits depends on the choice of function $\psi$ which we usually fix. Typical examples for such a function $\psi$ are $\psi(h) = h$ or

$$
\psi(h) = \frac{h}{1 + |h|_H}, \ h \in H.
$$

(3)

**Proposition.** Suppose $f, f_n \in L^0(S, \mu; H)$. Then for any bounded $\psi : H \rightarrow H$ we have that $f_n \xrightarrow{n \rightarrow \infty} f$ if and only if

$$
\int_A \langle \psi(f_n) - \psi(f), h \rangle \ d\mu \xrightarrow{n \rightarrow \infty} 0
$$

(4)

for any $h \in H$ and any $A \in \mathcal{F}$ with $\mu(A) < \infty$.

For a fixed $\psi$ the pseudo-weak limit is unique. The topology of $L^0(S, \mu; H)$ defined by convergence in measure

$$
\lim_{n \rightarrow \infty} \mu \left( \{|f_n - f|_H > \varepsilon\} \cap A \right) = 0 \quad \text{for all } \varepsilon > 0, \ A \in \mathcal{F}, \ \mu(A) < \infty
$$

implies pseudo-weak convergence, but these two types of convergence are not equivalent in general.
The following assertion is an easy consequence of the Banach-Saks Theorem applied to the Hilbert space $L^2(S, \mu; H)$ or, more elementarily, of Fatou’s Lemma.

**Proposition**

Suppose $f, f_n \in L^2(S, \mu; H), n \in \mathbb{N}$, and

$$f_n \xrightarrow[n \to \infty]{} f,$$

then

$$|f|_H \leq \limsup_{n \to \infty} |f_n|_H \quad \mu\text{-a.e.}$$
Let \( f, f_n \in L^0(S, \mu; H) \), \( n \in \mathbb{N} \), such that

\[
\begin{array}{ccc}
    f_n & \xrightarrow{\psi} & f \\
    n \to \infty & & \\
\end{array}
\]

Then

\[
|f|_H \leq \limsup_{n \to \infty} |f_n|_H \text{ \( \mu \)-a.e.}
\]

Let \( \psi \) in the definition of pseudo-weak convergence be bounded. If \( f_n \in L^0(S, \mu; H) \), \( n \in \mathbb{N} \), are such that

\[
\sup_{n \in \mathbb{N}} |f_n|_H < \infty \text{ \( \mu \)-a.e.,}
\]

then there exists \( f \in L^0(S, \mu; H) \) such that for some subsequence \( \{n_k\}_{k \in \mathbb{N}} \)

\[
\begin{array}{ccc}
    f_{n_k} & \xrightarrow{\psi} & f \\
    k \to \infty & & \\
\end{array}
\]
Yosida approximations to $A$

We need the Yosida approximations $A_\alpha$ to $A$ for small $\alpha$. Surprisingly, it is not easy to find a reference. Following Hille–Yosida, let $\rho(A)$ be the resolvent set, then the resolvent of $A$ is defined as

$$R_{\lambda} (A) := (\lambda I - A)^{-1}, \quad \lambda \in \rho(A) \in B(H),$$

$$R_{\lambda} (A) : H \longrightarrow D_A.$$

Recall that for $\lambda > 0$ we have $\|R_{\lambda} (A)\| \leq 1/\lambda$. In addition,

$$\lambda R_{\lambda} (A) x \xrightarrow[\lambda \to \infty]{} x, \quad x \in H. \quad (6)$$

Note that $AR_{\lambda} (A) x = R_{\lambda} (A) Ax, \quad x \in D_A$. Finally, the Yosida approximations to $A$ are defined by

$$A_\alpha x := \frac{1}{\alpha} AR^{\frac{1}{\alpha}} (A) x, \quad x \in H. \quad (7)$$
The Yosida approximations $A_\alpha$ to $A$ satisfy the following properties, see Brezis book 2011, where

$$J_\alpha := (I - \alpha A)^{-1},$$

$J_\alpha \in B(H)$, $\|J_\alpha\| \leq 1$.

$$A_\alpha x \xrightarrow[\alpha \to 0]{} Ax, \quad x \in D_A,$$

$|A_\alpha x|_H \leq |Ax|_H$, $x \in D_A$,

$$A_\alpha x = J_\alpha Ax, \quad x \in D_A,$$

$A_\alpha \in B(H),$

$\|A_\alpha\| \leq \frac{1}{\alpha},$

$$A_\alpha = AJ_\alpha = \frac{1}{\alpha}(J_\alpha - I).$$

**Proposition.** Under Assumption ??

$$\|J_\alpha\| \leq 1/(1 + \alpha \beta)$$

and

$$\langle A_\alpha x, x \rangle \leq -\beta_\alpha |x|_H^2$$

for all $x \in H$, where

$$\beta_\alpha := \frac{\beta}{1 + \alpha \beta}.$$
To define pseudo-weak solutions we use the Yosida approximation to $F$: fix $t \in [0, \infty)$ and set $F := F(t, \cdot)$. Then for any $\alpha > 0$ we define

$$F_\alpha := \frac{1}{\alpha} (J_\alpha (x) - x), x \in H,$$

where

$$J_\alpha (x) := (I - \alpha F)^{-1} (x), \quad I (x) = x,$$

which is a nonlinear generalization of (8). Then each $F_\alpha$ is single-valued, dissipative, Lipschitz continuous with Lipschitz constant less than $\frac{2}{\alpha}$ and satisfies

$$\lim_{\alpha \to 0} F_\alpha (x) = F_0 (x), x \in D_F,$$

$$|F_\alpha (x)|_H \leq |F_0 (x)|_H, x \in D_F.$$
end of Part 2
Part 3: Motivation

From Self-Similar Groups to Self-Similar Sets and Spectra

Rostislav Grigorchuk, Volodymyr Nekrashevych & Zoran Šunić
Random Walks on Sierpiński Graphs: Hyperbolicity and Stochastic Homogenization

Vadim A. Kaimanovich
Groups and analysis on fractals

V. Nekrashevych, A. Teplyaev • Published 2005 • Mathematics

We describe relation between analysis on fractals and the theory of self-similar groups. In particular, we focus on the construction of the Laplacian on limit sets of such groups in several concrete examples, and in the general p.c.f. case. We pose a number of open questions.
What are Hausdorff and spectral dimensions of a self-similar set?
For the circle, $d_S = 1$
For Riemannian $d$-manifolds, $d_S = d_H = d$

In general, $d_S$ can be defined using the asymptotics of eigenvalues or, equivalently, asymptotics of the heat kernel.

If $d_s$ is well defined, then

\[ \text{recurrence of the diffusion} \iff d_S < 2 \]

in which case we sometimes can prove Kigami’s formula

\[ d_S = 2 \frac{d_{H,R}}{d_{H,R} + 1} \]

where $d_{H,R}$ is the effective resistance Hausdorff dimension.
On the Sierpinski gasket (S. Goldstein 1984)

\[ d_{\text{topo}} = 1 < d_S = \frac{\log 9}{\log 5} < d_H = \frac{\log 3}{\log 2} \]

On the basilica Julia set we formally computed (Rogers-T, 2010)

\[ d_S = \frac{4}{3} \]

On the Sierpinski carpet \( \exists! d_S \) (Barlow, Bass, Kumagai, T. 1989-2010)

\[ d_{\text{topo}} = 1 < d_{H, \text{topo}} = 1 + \frac{\log 2}{\log 3} < d_S < d_H = \frac{\log 8}{\log 3} \]
François Englert
From Wikipedia, the free encyclopedia

François Baron Englert (French: [ɑ̃ɡlɛʁ]; born 6 November 1932) is a Belgian theoretical physicist and 2013 Nobel prize laureate (shared with Peter Higgs). He is Professeur emeritus at the Université libre de Bruxelles (ULB) where he is member of the Service de Physique Théorique. He is also a Sackler Professor by Special Appointment in the School of Physics and Astronomy at Tel Aviv University and a member of the Institute for Quantum Studies at Chapman University in California. He was awarded the 2010 J. J. Sakurai Prize for Theoretical Particle Physics (with Gerry Guralnik, C. R. Hagen, Tom Kibble, Peter Higgs, and Robert Brout), the Wolf Prize in Physics in 2004 (with Brout and Higgs) and the High Energy and Particle Prize of the European Physical Society (with Brout and Higgs) in 1997 for the mechanism which unifies short and long range interactions by generating massive gauge vector bosons. He has made contributions in statistical physics, quantum field theory, cosmology, string theory and supergravity.[4] He is the recipient of the 2013 Prince of Asturias Award in technical and scientific research, together with Peter Higgs and the CERN.
METRIC SPACE-TIME AS FIXED POINT
OF THE RENORMALIZATION GROUP EQUATIONS
ON FRACTAL STRUCTURES

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We take a model of foamy space-time structure described by self-similar fractals. We study the propagation of a scalar field on such a background and we show that for almost any initial conditions the renormalization group equations lead to an effective highly symmetric metric at large scale.
Fig. 1. The first two iterations of a 2-dimensional 3-fractal.
Fig. 5. The plane of 2-parameter homogeneous metrics on the Sierpinski gasket. The hyperbole $\alpha = -\beta / (\beta + 1)$ separates the domain of euclidean metrics from minkowskian metrics and corresponds – except at the origin – to 1-dimensional metrics. $M_1, M_2, M_3$ denote unstable minkowskian fixed geometries while $E$ corresponds to the stable euclidean fixed point. The unstable fixed points $0_1, 0_2$ and $0_3$ associated to 0-dimensional geometries are located at the origin and at infinity on the $(\alpha, \beta)$ coordinates axis. The six straight lines are subsets invariant with respect to the recursion relation but repulsive in the region where they are dashed. The first points of two sequences of iterations are drawn. Note that for one of them the 10th point $(\alpha = -56.4, \beta = -52.5)$ is outside the frame of the figure.
Fig. 10. A metrical representation of the two first iterations of a 2-dimensional 2-fractal corresponding to the euclidean fixed point. Vertices are labelled according to fig. 4.
Figure 6.4. Geometric interpretation of Proposition 6.1.

We first recall from [Ki4] some facts about limits of resistance networks. Although we state all the results of this section for the Sierpiński gasket, they can be applied to general pcf fractals with only minor changes.

Let \( E(\cdot, \cdot) \) be defined by (1.2) for any function \( f \) on \( V^* \), where \( E_n \) is a compatible sequence of Dirichlet forms on \( \Gamma_n \).

Proposition 7.1. Every point of \( V^* = \bigcup_{n \geq 0} V_n \) has positive capacity.

Proof. Let \( x \in V^* \). Then \( x \in V_n \) for some \( n \). The capacity of \( \{x\} \) with respect to \( E \) is the same as that with respect to \( E_n \) because of the compatibility of the sequence of networks. The latter capacity is positive because \( V_n \) is a finite set. □

The effective resistance is defined for any \( x, y \in V^* \) by

\[
R(x, y) = \left( \min_{u} E(u, u) : u(x) = 1, u(y) = 0 \right) - 1.
\]

(7.1)

Here the minimum is taken over all functions on \( V^* \). Note that \( x, y \in V_n \) for large enough \( n \) and that (7.1) does not change if \( E \) is replaced by \( E_n \), because of the compatibility condition (see [Ki4], Proposition 2.1.11). By Theorem 2.1.14 in [Ki4], \( R(x, y) \) is a metric on \( V^* \). In what follows we will write \( R \)-continuity, \( R \)-closure etc. for continuity, closure etc. with respect to the effective resistance metric \( R \). It is known that if \( E(u, u) < \infty \) then \( u \) is \( R \)-continuous ([Ki4], Theorem 2.2.6(1)). The main ingredient in the proof of this fact is the inequality

\[
|u(x) - u(y)|^2 \leq R(x, y) E(u, u).
\]

(7.2)

Let \( \Omega \) be the \( R \)-completion of \( V^* \). We can conclude from (7.2) that if \( u \) is a function on \( V^* \) such that \( E(u, u) < \infty \) then \( u \) has a unique continuation.

Figure 6.4. Geometric interpretation of Proposition 6.1.
The Spectral Dimension of the Universe is Scale Dependent

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We measure the spectral dimension of universes emerging from nonperturbative quantum gravity, defined through state sums of causal triangulated geometries. While four dimensional on large scales, the quantum universe appears two dimensional at short distances. We conclude that quantum gravity may be “self-renormalizing” at the Planck scale, by virtue of a mechanism of dynamical dimensional reduction.

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Quantum gravity as an ultraviolet regulator?—A shared hope of researchers in otherwise disparate approaches to quantum gravity is that the microstructure of space and time may provide a physical regulator for the ultraviolet infinities encountered in perturbative quantum field theory.
other hand, the “short-distance spectral dimension,” obtained by extrapolating Eq. (12) to $\sigma \rightarrow 0$ is given by

$$D_s(\sigma = 0) = 1.80 \pm 0.25,$$

and thus is compatible with the integer value two.

Random Geometry and Quantum Gravity
A thematic semestre at Institut Henri Poincaré
14 April, 2020 - 10 July, 2020
Organizers : John BARRETT, Nicolas CURIEN, Razvan GURAU, Renate LOLL, Gregory MIERMONT, Adrian TANASA
Fractal space-times under the microscope: 
a renormalization group view on Monte Carlo data

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ABSTRACT: The emergence of fractal features in the microscopic structure of space-time is a common theme in many approaches to quantum gravity. In this work we carry out a detailed renormalization group study of the spectral dimension $d_s$ and walk dimension $d_w$ associated with the effective space-times of asymptotically safe Quantum Einstein Gravity (QEG). We discover three scaling regimes where these generalized dimensions are approximately constant for an extended range of length scales: a classical regime where $d_s = d, d_w = 2$, a semi-classical regime where $d_s = 2d/(2+d), d_w = 2 + d$, and the UV-fixed point regime where $d_s = d/2, d_w = 4$. On the length scales covered by three-dimensional Monte Carlo simulations, the resulting spectral dimension is shown to be in very good agreement with the data. This comparison also provides a natural explanation for the apparent puzzle between the short distance behavior of the spectral dimension reported from Causal Dynamical Triangulations (CDT), Euclidean Dynamical Triangulations (EDT), and Asymptotic Safety.

KEYWORDS: Models of Quantum Gravity, Renormalization Group, Lattice Models of Gravity, Nonperturbative Effects
Fractal space-times under the microscope:
A Renormalization Group view on Monte Carlo data

Martin Reuter and Frank Saueressig

a classical regime where $d_s = d, d_w = 2$, a semi-classical regime where $d_s = 2d/(2 + d), d_w = 2 + d$, and the UV-fixed point regime where $d_s = d/2, d_w = 4$. On the length scales covered
Causal dynamical triangulations

25,971 views  Jan 26, 2013  Causal dynamical triangulation (CDT) is a lattice model of quantum gravity. In two space-time dimensions (instead of the four we live in) it
Dynamical triangulation of the 2-torus

1,435 views  Sep 7, 2013  This video illustrates a Monte Carlo simulation for two-dimensional quantum gravity on a torus. Starting with a regular triangulation of the torus repeatedly a so-called flip move is performed on a randomly chosen edge. The triangulations obtained after a large
Dynamical triangulation of the 2-torus

1,435 views  Sep 7, 2013  This video illustrates a Monte Carlo simulation for two-dimensional quantum gravity on a torus. Starting with a regular triangulation of the torus repeatedly a so-called flip move is performed on a randomly chosen edge. The triangulations obtained after a large
Figure 1. Barycentric subdivision of a 2-simplex, the graphs $G^T_0$, $G^T_1$ and $G^T_2$.

Figure 2. Adjacency (dual) graph $G_2$, in bold, and the barycentric subdivision graph pictured together with the thin image of $G^T_2$. 
Figure 3. On the left: the graph $G_4^T$ for barycentric subdivision of a 2-simplex. On the right: the adjacency (dual) graph $G_4$. 
Solution to the wave equation,
$t = 0.366$

Initial condition:
\[
\begin{cases}
  u(x, 0) = \delta(x); \\
  \frac{\partial u}{\partial t}(x, 0) = 0.
\end{cases}
\]
\[
\begin{cases}
  u(x, 0) = 0; \\
  \frac{\partial u}{\partial t}(x, 0) = \delta(x).
\end{cases}
\]
Brownian motion:
Thiele (1880), Bachelier (1900)
Einstein (1905), Smoluchowski (1906)
Wiener (1920'), Doob, Feller, Levy, Kolmogorov (1930'),
Doeblin, Dynkin, Hunt, Ito ...

\[ \text{distance} \sim \sqrt{\text{time}} \]

“Einstein space–time relation for Brownian motion”

Wiener process in \( \mathbb{R}^n \) satisfies \( \frac{1}{n} \mathbb{E} |W_t|^2 = t \) and has a
Gaussian transition density:

\[
p_t(x, y) = \frac{1}{(4\pi t)^{n/2}} \exp \left( -\frac{|x - y|^2}{4t} \right)
\]
De Giorgi-Nash-Moser estimates for elliptic and parabolic PDEs;

Li-Yau (1986) type estimates on a geodesically complete Riemannian manifold with $\text{Ricci} \geq 0$:

$$p_t(x, y) \sim \frac{1}{V(x, \sqrt{t})} \exp \left( -c \frac{d(x, y)^2}{t} \right)$$

$$\text{distance} \sim \sqrt{\text{time}}$$
Gaussian:

\[
p_t(x, y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x - y|^2}{4t}\right)
\]

Li-Yau Gaussian-type:

\[
p_t(x, y) \sim \frac{1}{V(x, \sqrt{t})} \exp\left(-c \frac{d(x, y)^2}{t}\right)
\]

Sub-Gaussian:

\[
p_t(x, y) \sim \frac{1}{t^{d_H/d_w}} \exp\left(-c \left(\frac{d(x, y)^{d_w}}{t}\right)^{1\over d_w-1}\right)
\]

\[\textit{distance} \sim \textit{time}^{1\over d_w}\]
Brownian motion on $\mathbb{R}^d$: $\mathbb{E}|X_t - X_0| = ct^{1/2}$.

Anomalous diffusion: $\mathbb{E}|X_t - X_0| = o(t^{1/2})$, or (in regular enough situations),

$$\mathbb{E}|X_t - X_0| \approx t^{1/d_w}$$

with $d_w > 2$.

Here $d_w$ is the so-called walk dimension (should be called “walk index” perhaps).

This phenomena was first observed by mathematical physicists working in the transport properties of disordered media, such as (critical) percolation clusters.
\[ p_t(x, y) \sim \frac{1}{t^{d_H/d_w}} \exp \left( -c \frac{d(x, y)^{1/d_w}}{t^{1/d_w-1}} \right) \]

\textit{distance} \sim \textit{(time)}^{\frac{1}{d_w}}

\[ d_H = \text{Hausdorff dimension} \]
\[ \frac{1}{\gamma} = d_w = \text{“walk dimension” (} \gamma = \text{diffusion index)} \]
\[ \frac{2d_H}{d_w} = d_S = \text{“spectral dimension” (diffusion dimension)} \]

First example: Sierpiński gasket; Kusuoka, Fukushima, Kigami, Barlow, Bass, Perkins (mid 1980’—)
A part of an infinite Sierpiński gasket.
Figure: An illustration to the computation of the spectrum on the infinite Sierpiński gasket. The curved lines show the graph of the function $\mathcal{R}(\cdot)$.


On the infinite Sierpiński gasket the spectrum of the Laplacian consists of a **dense** set of eigenvalues $\mathcal{R}^{-1}(\Sigma_0)$ of infinite multiplicity and a **singularly continuous** component of spectral multiplicity one supported on $\mathcal{R}^{-1}(\mathcal{J}_R)$. 
Half-line example

Transition probabilities in the $pq$ random walk. Here $p \in (0, 1)$ and $q = 1 - p$.

$$\begin{align*}
(\Delta_p f)(x) &= \begin{cases} 
  f(0) - f(1), & \text{if } x = 0 \\
  f(x) - qf(x - 1) - pf(x + 1), & \text{if } 3^{-m(x)}x \equiv 1 \pmod{3} \\
  f(x) - pf(x - 1) - qf(x + 1), & \text{if } 3^{-m(x)}x \equiv 2 \pmod{3}
\end{cases}
\end{align*}$$

Theorem (J.P.Chen, T., 2016)

If $p \neq \frac{1}{2}$, the Laplacian $\Delta_p$ on $\ell^2(\mathbb{Z}_+)$ has purely singularly continuous spectrum. The spectrum is the Julia set, a topological Cantor set of Lebesgue measure zero, of the polynomial $R(z) = \frac{z(z^2 - 3z + (2 + pq))}{pq}$

This is a simple, possibly the simplest, quasi-periodic example related to the recent results of A.Avila, D.Damanik, A.Gorodetski, S.Jitomirskaya, Y.Last, B.Simon et al.
Spectral zeta function

**Theorem.** (Derfel-Grabner-Vogl, Steinhurst-T., Chen-T.-Tsougkas, Kajino (2007–2017)) For a large class of **finitely ramified symmetric fractals** the spectral zeta function

\[
\zeta(s) = \sum \lambda_j^{s/2}
\]

has a meromorphic continuation from the half-plane \( \text{Re}(s) > d_s \) to \( \mathbb{C} \). Moreover, all the poles and residues are computable from the geometric data of the fractal. Here \( \lambda_j \) are the eigenvalues if the unique symmetric Laplacian.

- **Example:** \( \zeta(s) \) is the Riemann zeta function up to a trivial factor in the case when our fractal is \([0, 1]\).
- In more complicated situations, such as the Sierpiński gasket, there are infinitely many non-real poles, which can be called complex spectral dimensions, and are related to oscillations in the spectrum.
Poles (white circles) of the spectral zeta function of the Sierpiński gasket.

\[ d_s = \frac{\log 9}{\log 5} \]

\[ d_R = \frac{\log 4}{\log 5} \]
The question of existence of groups with intermediate growth, i.e. subexponential but not polynomial, was asked by Milnor in 1968 and answered in the positive by Grigorchuk in 1984. There are still open questions in this area, and a complete picture of which orders of growth are possible, and which are not, is missing.

The Basilica group is a group generated by a finite automation acting on the binary tree in a self-similar fashion, introduced by R. Grigorchuk and A. Zuk in 2002, does not belong to the closure of the set of groups of subexponential growth under the operations of group extension and direct limit.

In 2005 L. Bartholdi and B. Virag further showed it to be amenable, making the Basilica group the 1st example of an amenable but not subexponentially amenable group (also “Münchhausen trick” and amenability of self-similar groups by V.A. Kaimanovich).
The basilica Julia set, the Julia set of $z^2 - 1$ and the limit set of the basilica group of exponential growth (Grigorchuk, Žuk, Bartholdi, Virág, Nekrashevych, Kaimanovich, Nagnibeda et al.).
In 2005, V. Nekrashevych described the Basilica as the iterated monodromy group, and there exists a natural way to associate it to the Basilica fractal (Nekrashevych+T., 2008).

In Schreier graphs of the Basilica group (2010), Nagnibeda et al. classified up to isomorphism all possible limits of finite Schreier graphs of the Basilica group.

In Laplacians on the Basilica Julia set (2010), L. Rogers+T. constructed Dirichlet forms and the corresponding Laplacians on the Basilica fractal in two different ways: by imposing a self-similar harmonic structure and a graph-directed self-similar structure on the fractal.

In 2012-2015, Dong, Flock, Molitor, Ott, Spicer, Totari and Strichartz provided numerical techniques to approximate eigenvalues and eigenfunctions on families of Laplacians on the Julia sets of \( z^2 + c \).
Spectral Analysis of the Basilica Graphs
Basilica Julia Set and the Schreier graph $\Gamma_4$
pictures taken from paper by Nagnibeda et. al.

$\Gamma_4$
Spectral Analysis of the Basilica Graphs

Replacement Rule and the Graphs $G_n$

```
G_0
  b     a
  a

G_1
  b   b   a

G_2
  a   a
  b   b

G_3
  b   b   a   a
          a
          a
```
Cumulative Distribution of Eigenvalues, Level 13
One can define a Dirichlet to Neumann map for the two boundary points of the graphs $G_n$. One can construct a dynamical system to determine these maps (which are two by two matrices). The dynamical system allows us to prove the following.

**Theorem**

In the Hausdorff metric, $\limsup_{n \to \infty} \sigma(L^{(n)})$ has a gap that contains the interval $(2.5, 2.8)$.

**Theorem (arXiv:1908.10505)**

In the Hausdorff metric, $\limsup_{n \to \infty} \sigma(L^{(n)})$ has infinitely many gaps.
Spectral Analysis of the Basilica Graphs

Infinite Blow-ups of $G_n$

**Definition**

Let $\{k_n\}_{n \in \mathbb{N}}$ be a strictly increasing subsequence of the natural numbers. For each $n$, embed $G_{k_n}$ in some isomorphically subgraph of $G_{k_{n+1}}$. The corresponding infinite blow-up is $G_\infty := \bigcup_{n \geq 0} G_{k_n}$.

**Assumption**

The infinite blow-up $G_\infty$ satisfies:

- For $n \geq 1$, the long path of $G_{k_{n-1}}$ is embedded in a loop $\gamma_n$ of $G_{k_n}$.
- Apart from $l_{k_{n-1}}$ and $r_{k_{n-1}}$, no vertex of the long path can be the 3, 6, 9 or 12 o’clock vertex of $\gamma_n$.
- The only vertices of $G_{k_n}$ that connect to vertices outside the graph are the boundary vertices of $G_{k_n}$.
Theorem

(1) $\sigma(L^{(kn)}|_{\ell^2_{a,kn,\gamma n}}) = \sigma(L_0^{(jn)})$.

(2) The spectrum of $L^{(\infty)}$ is pure point. The set of eigenvalues of $L^{(\infty)}$ is

$$\bigcup_{n \geq 0} \sigma(L_0^{(jn)}) = \bigcup_{n \geq 0} c_{jn}^{-1}\{0\},$$

where the polynomials $c_n$ are the characteristic polynomials of $L_0^{(n)}$, as defined in the previous proposition.

(3) Moreover, the set of eigenfunctions of $L^{(\infty)}$ with finite support is complete in $\ell^2$. 
TECHNICAL DETAILS
Fix $p, q > 0$, $p + q = 1$, and define probabilistic Laplacians $\Delta_n$ on the segments $[0, 3^n]$ of $\mathbb{Z}_+$ inductively as a generator of the random walks:

$$0 \quad 1$$

$$1 \quad q \quad p$$

$$1 \quad q \quad p \quad p \quad q$$

$$1 \quad q \quad p \quad p \quad q \quad p \quad q$$

$$1 \quad q \quad p \quad p \quad q \quad p \quad q \quad p \quad q$$

and let $\Delta = \lim_{n \to \infty} \Delta_n$ be the corresponding probabilistic Laplacian on $\mathbb{Z}_+$. 

2
If $z \neq -1 \pm p$ and $R(z) = z(z^2 + 3z + 2 + pq)/pq$, then

$R(z) \in \sigma(\Delta_n) \iff z \in \sigma(\Delta_{n+1})$

Theorem (Joe P. Chen and T., JMP 2016). $\sigma(\Delta) = J_R$, the Julia set of $R(z)$.

If $p = q$, then $\sigma(\Delta) = [-2, 0]$, spectrum is a.c.

If $p \neq q$, then $\sigma(\Delta)$ is a Cantor set of Lebesgue measure zero, spectrum is singularly continuous.


Bellissard and Simon, Cantor spectrum for the almost Mathieu equation J. Funct. Anal. 48 (1982), 408–419.
There are uncountably many “random” self-similar Laplacians $\Delta$ on $\mathbb{Z}$:

For a sequence $\mathcal{K} = \{k_j\}_{j=1}^{\infty}$, $k_j \in \{0, 1, 2\}$, let

$$X_n = -\sum_{j=1}^{n} k_j 3^j$$

and $\Delta_n$ is a probabilistic Laplacian on $[X_n, X_n + 3^n]$:

In the previous example $k_j = 0$ for all $j$.

**Theorem.**

For any sequence $\mathcal{K}$ we have $\sigma(\Delta) = J_R$. The same is true for the Dirichlet Laplacian on $\mathbb{Z}_+$ (when $k_j \equiv 0$).
R. Grigorchuk and Z. Sunik, *Asymptotic aspects of Schreier graphs and Hanoi Towers groups.*
Sierpiński 3-graph
(Hanoi Towers-3 group)

Sierpiński 4-graph
(standard)
These three polynomials are conjugate:

Sierpiński 3-graph (Hanoi Towers-3 group): \( f(x) = x^2 - x - 3 \)
\( f(3) = 3, \ f'(3) = 5 \)

Sierpiński 4-graph, “adjacency matrix” Laplacian: \( P(\lambda) = 5\lambda - \lambda^2 \)
\( P(0) = 0, \ P'(0) = 5 \)

Sierpiński 4-graph, probabilistic Laplacian: \( R(z) = 4z^2 + 5z \)
\( R(0) = 0, \ R'(0) = 5 \)
Theorem. Eigenvalues and eigenfunctions on the Sierpiński 3-graphs and Sierpiński 4-graphs are in one-to-one correspondence, with the exception of the eigenvalue $z = -\frac{3}{2}$ for the 4-graphs.

\[
4z^2 + 5z
\]

\[
\frac{4}{3}z^2 + \frac{8}{3}z
\]

\[
2z^2 + 4z
\]
Sierpiński 3-graph (Hanoi Towers-3 group)
\[ R(z) = 2z^2 + 4z \]

Sierpiński 4-graph (standard)
\[ R(z) = \frac{4}{3}z^2 + \frac{8}{3}z \]
Let $\mathcal{H}$ and $\mathcal{H}_0$ be Hilbert spaces, and $U : \mathcal{H}_0 \to \mathcal{H}$ be an isometry.

**Definition.** We call an operator $H$ **spectrally similar** to an operator $H_0$ with functions $\varphi_0$ and $\varphi_1$ if

$$U^*(H - z)^{-1}U = (\varphi_0(z)H_0 - \varphi_1(z))^{-1}$$

In particular, if $\varphi_0(z) \neq 0$ and $R(z) = \varphi_1(z)/\varphi_0(z)$, then

$$U^*(H - z)^{-1}U = \frac{1}{\varphi_0(z)}(H - R(z))^{-1}.$$

If $H = \begin{pmatrix} S & \bar{X} \\ X & Q \end{pmatrix}$ then

$$S - zI_0 - \bar{X}(Q - zI_1)^{-1}X = \varphi_0(z)H_0 - \varphi_1(z)I_0$$

**Theorem (Malozemov and T.).** If $\Delta$ is the graph Laplacian on a self-similar symmetric infinite graph, then

$$J_R \subseteq \sigma(\Delta_\infty) \subseteq J_R \cup D_\infty$$

where $D_\infty$ is a discrete set and $J_R$ is the Julia set of the rational function $R$. 


Let $\Delta$ be the probabilistic Laplacian (generator of a simple random walk) on the Sierpiński lattice. If $z \neq -\frac{3}{2}, -\frac{5}{4}, -\frac{1}{2}$, and $R(z) = z(4z + 5)$, then

$$R(z) \in \sigma(\Delta) \iff z \in \sigma(\Delta)$$

$$\sigma(\Delta) = J_R \cup D$$

where $D \overset{\text{def}}{=} \{-\frac{3}{2}\} \cup \left( \bigcup_{m=0}^{\infty} R^{-m}\{-\frac{3}{4}\} \right)$

and $J_R$ is the Julia set of $R(z)$. 

\[
\begin{align*}
0 & \quad 3/2 \\
-3/4 & \quad 5/4
\end{align*}
\]
There are uncountably many nonisomorphic Sierpiński lattices.

**Theorem (T).** The spectrum of $\Delta$ is pure point. Eigenfunctions with finite support are complete.
Let \( \Delta^{(0)} \) be the Laplacian with zero (Dirichlet) boundary condition at \( \partial L \). Then the compactly supported eigenfunctions of \( \Delta^{(0)} \) are \textbf{not} complete\footnote{eigenvalues in \( \mathcal{E} \) is not the whole spectrum).}

Let \( \partial L^{(0)} \) be the set of two points adjacent to \( \partial L \) and \( \omega^{(0)}_{\Delta} \) be the spectral measure of \( \Delta^{(0)} \) associated with \( \prod_{\partial L(0)} \). Then \( \text{supp}(\omega^{(0)}_{\Delta}) = \mathcal{J}_R \) has Lebesgue measure zero and

\[
\frac{d(\omega^{(0)}_{\Delta} \circ R_{1,2})}{d\omega^{(0)}_{\Delta}}(z) = \frac{(8z + 5)(2z + 3)}{(2z + 1)(4z + 5)}
\]
Three contractions \( F_1, F_2, F_3 : \mathbb{R}^1 \to \mathbb{R}^1, \) \( F_j(x) = \frac{1}{3}(x+p_j), \) with fixed points \( p_j = 0, \frac{1}{2}, 1. \) The interval \( I=\left[0, 1\right] \) is a unique compact set such that

\[
I = \bigcup_{j=1,2,3} F_j(I)
\]

The boundary of \( I \) is \( \partial I = V_0 = \{0, 1\} \) and the discrete approximations to \( I \) are \( V_n = \bigcup_{j=1,2,3} F_j(V_{n-1}) = \left\{ \frac{k}{3^n} \right\}_{k=0}^{3^n} \)

\[
V_0 = \partial I:
\]

\[
V_1:
\]

\[
V_2:
\]
Definition. The *discrete Dirichlet (energy) form* on $V_n$ is

$$E_n(f) = \sum_{x,y \in V_n} (f(y) - f(x))^2$$

and the *Dirichlet (energy) form* on $I$ is $E(f) = \lim_{n \to \infty} 3^n E_n(f) = \int_0^1 |f'(x)|^2 dx$

Definition. A function $h$ is *harmonic* if it minimizes the energy given the boundary values.

Proposition. $3E_{n+1}(f) \geq E_n(f)$ and $3E_{n+1}(h) = E_n(h) = 3^{-n} E(h)$ for a harmonic $h$.

Proposition. The Dirichlet (energy) form on $I$ is *self-similar* in the sense that

$$E(f) = 3 \sum_{j = 1, 2, 3} E(f \circ F_j)$$
Definition. The *discrete Laplacians* on $V_n$ are

$$\Delta_n f(x) = \frac{1}{2} \sum_{y \sim x} f(y) - f(x), \quad x \in V_n \setminus V_0$$

and the Laplacian on $I$ is $\Delta f(x) = \lim_{n \to \infty} 9^n \Delta_n f(x) = f''(x)$

Gauss–Green (integration by parts) formula:

$$\mathcal{E}(f) = - \int_0^1 f \Delta f dx + f f' \bigg|_0^1$$

Spectral asymptotics: Let $\rho(\lambda)$ be the *eigenvalue counting function* of the Dirichlet or Neumann Laplacian $\Delta$:

$$\rho(\lambda) = \# \{ j : \lambda_j < \lambda \}.$$  

Then

$$\lim_{\lambda \to \infty} \frac{\rho(\lambda)}{\lambda^{d_s/2}} = \frac{1}{\pi}$$

where $d_s = 1$ is the spectral dimension.
**Definition.** The *spectral zeta function* is $\zeta_\Delta(s) = \sum_{\lambda_j \neq 0} (-\lambda_j)^{-s/2}$

Its poles are the *complex spectral dimensions*.

Let $R(z)$ be a polynomial of degree $N$ such that its Julia set $J_R \subset (-\infty, 0]$, $R(0) = 0$ and $c = R'(0) > 1$.

**Definition.** The *zeta function of* $R(z)$ for $\text{Re}(s) > d_R = \frac{2 \log N}{\log c}$ is

$$\zeta^z_0(s) = \lim_{n \to \infty} \sum_{z \in R^{-n} \{z_0\}} (-c^n z)^{-s/2} = \sum \lambda_j^{-s/2}$$

**Theorem.** $\zeta^z_0(s) = \frac{f_1(s)}{1 - NC^{-s/2}} + f_2^z_0(s)$, where $f_1(s)$ and $f_2^z_0(s)$ are analytic for $\text{Re}(s) > 0$. If $J_R$ is totally disconnected, then this meromorphic continuation extends to $\text{Re}(s) > -\varepsilon$, where $\varepsilon > 0$.

In the case of polynomials this theorem has been improved by Grabner et al.
$d_R \in$ the poles of $\zeta_{R}^{z_0} \subseteq \left\{ \frac{2\log N + 4i\pi}{\log c} : n \in \mathbb{Z} \right\}$
Theorem. $\zeta_{\Delta}(s) = \zeta^0_R(s)$ where $R(z) = z(4z^2 + 12z + 9)$.

The Riemann zeta function $\zeta(s)$ satisfies $\zeta(s) = \pi^s \zeta^0_R(s)$ The only complex spectral dimension is the pole at $s = 1$.

A sketch of the proof: If $z \neq -\frac{1}{2}, -\frac{3}{2}$, then

$$R(z) \in \sigma(\Delta_n) \iff z \in \sigma(\Delta_{n+1})$$

and so $\zeta_{\Delta}(s) = \zeta^0_R(s)$ since the eigenvalues $\lambda_j$ of $\Delta$ are limits of the eigenvalues of $9^n \Delta_n$.

Also $\lambda_j = -\pi^2 j^2$ and so

$$\zeta_{\Delta}(s) = \sum_{j=1}^{\infty} \left(\pi^2 j^2\right)^{-s/2} = \pi^{-s} \zeta(s)$$

where $\zeta(s)$ is the Riemann zeta function. \textit{Q.E.D.}
Definition. $\Delta_\mu$ is $\mu$–Laplacian if

$$E(f) = \int_0^1 |f'(x)|^2 \, dx = - \int_0^1 f \Delta_\mu f \, d\mu + f f'|^1_0.$$ 

Definition. A probability measure $\mu$ is self-similar with weights $m_1, m_2, m_3$ if

$$\mu = \sum_{j=1,2,3} m_j \mu \circ F_j.$$ 

Proposition. $\Delta_\mu f(x) = \frac{f''}{\mu} = \lim_{n \to \infty} \left(1 + \frac{2}{pq}\right)^n \Delta_n f(x).$ 

$$\Delta_n f \left(\frac{k}{3^n}\right) = \begin{cases} p f \left(\frac{k-1}{3^n}\right) + q f \left(\frac{k+1}{3^n}\right) - f \left(\frac{k}{3^n}\right) \\ q f \left(\frac{k-1}{3^n}\right) + p f \left(\frac{k+1}{3^n}\right) - f \left(\frac{k}{3^n}\right) \end{cases}$$ 

where $m_1 = m_3, \ p = \frac{m_2}{m_1 + m_2}, \ q = \frac{m_1}{m_1 + m_2},$ and

![Diagram showing distribution of $m_1, m_2, m_3$ with arrows indicating transitions between $1, q, p$ and their inverses.](image-url)
Spectral asymptotics: If \( \rho(\lambda) \) is the eigenvalue counting function of the Dirichlet or Neumann Laplacian \( \Delta_\mu \), then

\[
0 < \liminf_{\lambda \to \infty} \frac{\rho(\lambda)}{\lambda^{d_s/2}} \leq \limsup_{\lambda \to \infty} \frac{\rho(\lambda)}{\lambda^{d_s/2}} < \infty
\]

where the spectral dimension is

\[
d_s = \frac{\log 9}{\log (1 + \frac{2}{pq})} \leq 1.
\]

All the inequalities are strict if and only if \( p \neq q \).

**Proposition.** \( R(z) \in \sigma(\Delta_n) \iff z \in \sigma(\Delta_{n+1}) \)

where \( z \neq -1 \pm p \) and \( R(z) = z(z^2 + 3z + 2 + pq)/pq \).

Note that \( R'(0) = 1 + \frac{2}{pq} \), and \( d_s = d_R \).

**Theorem.** \( \zeta_{\Delta_\mu}(s) = \zeta_R^0(s) \)
Three contractions \( F_1, F_2, F_3 : \mathbb{R}^2 \to \mathbb{R}^2 \),
\( F_j(x) = \frac{1}{2}(x + p_j) \), with fixed points \( p_1, p_2, p_3 \).

The Sierpiński gasket is a unique compact set \( S \) such that
\[
S = \bigcup_{j=1, 2, 3} F_j(S)
\]
Definition. The *boundary* of $S$ is
\[ \partial S = V_0 = \{ p_1, p_2, p_3 \} \]
and *discrete approximations* to $S$ are
\[ V_n = \bigcup_{j=1,2,3} F_j(V_{n-1}) \]

\[ V_0 : \]
\[ V_1 : \]
\[ V_2 : \]
Definition. The discrete Dirichlet (energy) form on $V_n$ is
\[ \mathcal{E}_n(f) = \sum_{x,y \in V_n} (f(y) - f(x))^2 \]
and the Dirichlet (energy) form on $S$ is
\[ \mathcal{E}(f) = \lim_{n \to \infty} \left( \frac{5}{3} \right)^n \mathcal{E}_n(f) \]

Definition. A function $h$ is harmonic if it minimizes the energy given the boundary values.

Proposition. \[ \frac{5}{3} \mathcal{E}_{n+1}(f) \geq \mathcal{E}_n(f) \]
\[ \frac{5}{3} \mathcal{E}_{n+1}(h) = \mathcal{E}_n(h) = \left( \frac{5}{3} \right)^{-n} \mathcal{E}(h) \] for a harmonic $h$.

Theorem (Kigami). $\mathcal{E}$ is a local regular Dirichlet form on $S$ which is self-similar in the sense that
\[ \mathcal{E}(f) = \frac{5}{3} \sum_{j = 1, 2, 3} \mathcal{E}(f \circ F_j) \]
Definition. The discrete Laplacians on \( V_n \) are
\[
\Delta_n f(x) = \frac{1}{4} \sum_{y \in V_n \atop y \sim x} f(y) - f(x), \quad x \in V_n \setminus V_0
\]
and the Laplacian on \( S \) is
\[
\Delta_\mu f(x) = \lim_{n \to \infty} 5^n \Delta_n f(x)
\]
if this limit exists and \( \Delta_\mu f \) is continuous.

Gauss–Green (integration by parts) formula:
\[
\mathcal{E}(f) = -\int_S f \Delta_\mu f d\mu + \sum_{p \in \partial S} f(p) \partial_n f(p)
\]
where \( \mu \) is the normalized Hausdorff measure, which is self-similar with weights \( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \):
\[
\mu = \frac{1}{3} \sum_{j = 1, 2, 3} \mu \circ F_j.
\]
Spectral asymptotics: If $\rho(\lambda)$ is the eigenvalue counting function of the Dirichlet or Neumann Laplacian $\Delta_\mu$, then

$$0 < \liminf_{\lambda \to \infty} \frac{\rho(\lambda)}{\lambda^{d_s/2}} < \limsup_{\lambda \to \infty} \frac{\rho(\lambda)}{\lambda^{d_s/2}} < \infty$$

where the spectral dimension is

$$1 < d_s = \frac{\log 9}{\log 5} < 2.$$ 

Proposition. $R(z) \in \sigma(\Delta_n) \iff z \in \sigma(\Delta_{n+1})$ where $z \neq -\frac{1}{2}, -\frac{3}{4}, -\frac{5}{4}$ and $R(z) = z(5 + 4z)$.

Theorem (Fukushima, Shima). Every eigenvalue of $\Delta_\mu$ has a form

$$\lambda = 5^m \lim_{n \to \infty} 5^n R^{-n}(z_0)$$

where $R^{-n}(z_0)$ is a preimage of $z_0 = -\frac{3}{4}, -\frac{5}{4}$ under the $n$-th iteration power of the polynomial $R(z)$. The multiplicity of such an eigenvalue is $C_1 3^m + C_2$. 
Theorem. Zeta function of the Laplacian on the Sierpiński gasket is

\[ \zeta_{\Delta_\mu}(s) = \frac{1}{2} \zeta_{R}^{-\frac{3}{4}}(s) \left( \frac{1}{5^{s/2} - 3} + \frac{3}{5^{s/2} - 1} \right) + \frac{1}{2} \zeta_{R}^{-\frac{5}{4}}(s) \left( \frac{3 \cdot 5^{-s/2}}{5^{s/2} - 3} - \frac{5^{-s/2}}{5^{s/2} - 1} \right) \]
**Definition.** If $\mathcal{L}$ is a fractal string, that is, a disjoint collection of intervals of lengths $l_j$, then its geometric zeta function is $\zeta_\mathcal{L}(s) = \sum l_j^s$.

**Theorem (Lapidus).** If $A = -\frac{d^2}{dx^2}$ is a Neumann or Dirichlet Laplacian on $\mathcal{L}$, then $\zeta_A(s) = \pi^{-s} \zeta(s) \zeta_\mathcal{L}(s)$.

**Example: Cantor self-similar fractal string.**

If $\mathcal{L}$ is the complement of the middle third Cantor set in $[0, 1]$, then the complex spectral dimensions are 1 and $\left\{\frac{\log 2+2in\pi}{\log 3} : n \in \mathbb{Z}\right\}$,

$$\zeta_\mathcal{L}(s) = \frac{1}{1-2\cdot 3^{-s}}, \quad \zeta_A(s) = \zeta(s) \frac{\pi^{-s}}{1-2\cdot 3^{-s}}$$
Definition. A post critically finite (p.c.f.) self-similar set $F$ is a compact connected metric space with a finite boundary $\partial F \subset F$ and contractive injections $\psi_i : F \to F$ such that

$$F = \Psi(F) = \bigcup_{i=1}^{k} \psi_i(F)$$

and

$$\psi_v(F) \cap \psi_w(F) \subseteq \psi_v(\partial F) \cap \psi_w(\partial F),$$

for any two different words $v$ and $w$ of the same length. Here for a finite word $w \in \{1, \ldots, k\}^m$ we define $\psi_w = \psi_{w_1} \circ \ldots \circ \psi_{w_m}$.

We assume that $\partial F$ is a minimal such subset of $F$. We call $\psi_w(F)$ an $m$-cell. The p.c.f. assumption is that every boundary point is contained in a single 1-cell.

Theorem (Kigami, Lapidus). The spectral dimension of the Laplacian $\Delta_\mu$ is the unique solution of the equation

$$\sum_{i=1}^{k} (r_i \mu_i)^{d_\delta/2} = 1$$
Conjecture. On every p.c.f. fractal $F$ there exists a local regular Dirichlet form $\mathcal{E}$ which gives positive capacity to the boundary points and is self-similar in the sense that

$$\mathcal{E}(f) = \sum_{i=1}^{k} \rho_i \mathcal{E}(f \circ \psi_i)$$

for a set of positive refinement weights $\rho = \{\rho_i\}_{i=1}^{k}$.

Definition. The group $G$ of acts on a finitely ramified fractal $F$ if each $g \in G$ is a homeomorphism of $F$ such that $g(V_n) = V_n$ for all $n \geq 0$.

Proposition. Suppose a group $G$ of acts on a self-similar finitely ramified fractal $F$ and $G$ restricted to $V_0$ is the whole permutation group of $V_0$. Then there exists a unique, up to a constant, $G$-invariant self-similar resistance form $\mathcal{E}$ with equal energy renormalization weights $\rho_i$ and

$$\mathcal{E}_0(f, f) = \sum_{x, y \in V_0} (f(x) - f(y))^2.$$ 

Moreover, for any $G$-invariant self-similar measure $\mu$ the Laplacian $\Delta_{\mu}$ has the spectral self-similarity property (a.k.a. spectral decimation).
end of the talk :-) 

Thank you!