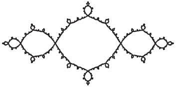
Spectral analysis on self-similar graphs, fractals, and groups

Alexander Teplyaev
University of Connecticut



** (part ii) **

July 11-12, 2022
CentraleSupélec
Université Paris-Saclay

https://alexander-teplyaev.uconn.edu

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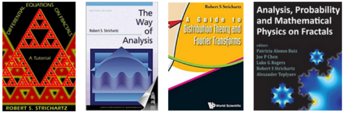
Anna Akhmatova *To the Londoners* *1940* — 2022 original .jpg

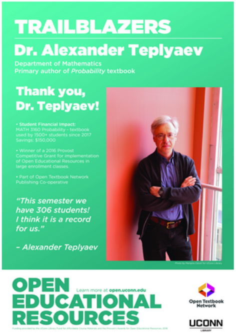
Robert Strichartz

1943-2021

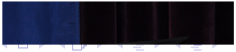


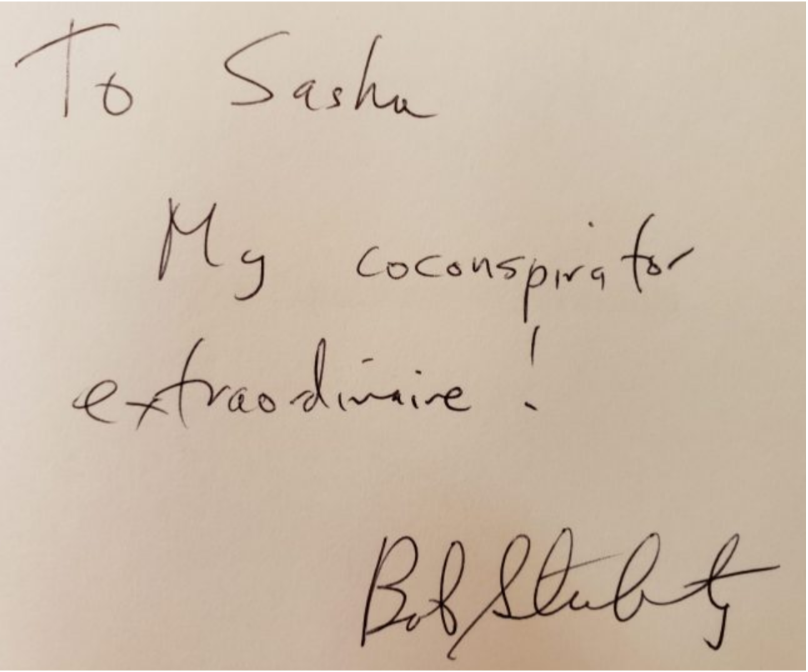
7th Cornell Conference on Analysis, Probability, and Mathematical Physics on Fractals

OER Probability



Ulysses Andrews 2017




To Sasha
My coconspirator
extraordinaire!

7th Cornell Conference on Analysis, Probability, and Mathematical Physics on Fractals: June 4–8, 2022

Home » 7th Cornell Conference on Analysis, Probability, and Mathematical Physics on Fractals

Welcome!

Planning has begun for Fractals 7 (June 9-13, 2020). The purpose of this conference, held every three years, is to bring together mathematicians who are already working in the area of analysis and probability on fractals with students and researchers from related areas. Information will be posted here as it becomes available.

Financial support will be provided to a limited number of participants to cover the cost of housing in Cornell single dormitory rooms and partially support other travel expenses. Students and junior researchers from underrepresented groups in STEM are particularly encouraged to apply for travel funding. Well-established researchers are encouraged to use their own travel funding; the NSF expects that most funds will be expended on otherwise unfunded mathematicians.

Registration details will be publicized once available.

All general inquiries can be sent to: fractals_math@cornell.edu

Conference Organizers:

- Robert Strichartz (chair), Cornell University
- Patricia Alonso Ruiz, Texas A&M University
- Michael Hinz, Bielefeld University
- Luke Rogers, University of Connecticut
- Alexander Teplyaev, University of Connecticut

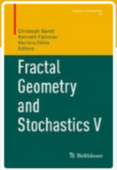
outline of the talk:

- Introduction and motivation.
- **Algebraic applications: spectrum of the Laplacian on the Basilica Julia set** (with Rogers, Brzoska, George, Jarvis arXiv:1908.10505).
- selected technical details (if time permits)

This is a part of the broader program to develop **probabilistic, spectral and vector analysis on singular spaces** by **carefully building approximations by graphs or manifolds**.

abstract of the talk

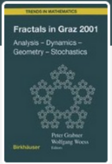
The lectures will describe how fractal research appears naturally in many areas of pure and applied mathematics and other sciences and engineering. Fractal shape optimization allows for better wave absorption properties, which is a joint work with Anna Rozanova-Pierrat (CentraleSupélec) and Michael Hinz (Bielefeld). Moreover, spectral theory, geometry of graphs, and dynamical systems are used to analyze spectral properties of the random walk generator on finitely ramified self-similar graphs, fractals, and some self-similar groups. In particular, pure point or singular continuous spectrum appears naturally in such settings. The standard examples include the Sierpinski triangle, the Vicsek tree, and the Schreier graphs of the Hanoi and basilica self-similar groups studied by Grigorchuk, Zuk, Nekrashevych, Bartholdi, Lyubich, Dang et al. We will discuss spectral dimensions, self-similar random walks and their diffusion limits, and the role of symmetries and finite ramification in computing the spectrum explicitly.



Fractal Geometry and Stochastics V pp 175–207 | Cite as

From Self-Similar Groups to Self-Similar Sets and Spectra

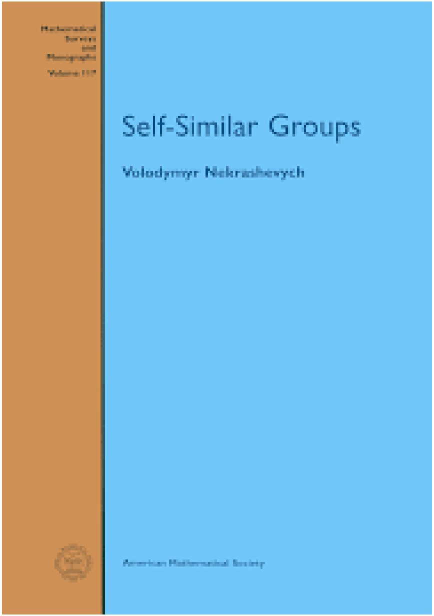
Rostislav Grigorchuk ✉, Volodymyr Nekrashevych & Zoran Šunić



Fractals in Graz 2001 pp 145–183 | Cite as

Random Walks on Sierpiński Graphs: Hyperbolicity and Stochastic Homogenization

Vadim A. Kaimanovich


Self-Similar Groups
Volodymyr Nekrashevych

arXiv > math > arXiv:2010.00675

Search...

Help | A

Mathematics > Group Theory

[Submitted on 1 Oct 2020 (v1), last revised 13 Jan 2021 (this version, v2)]

Self-similar groups and holomorphic dynamics: Renormalization, integrability, and spectrum

Nguyen-Bac Dang, Rostislav Grigorchuk, Mikhail Lyubich

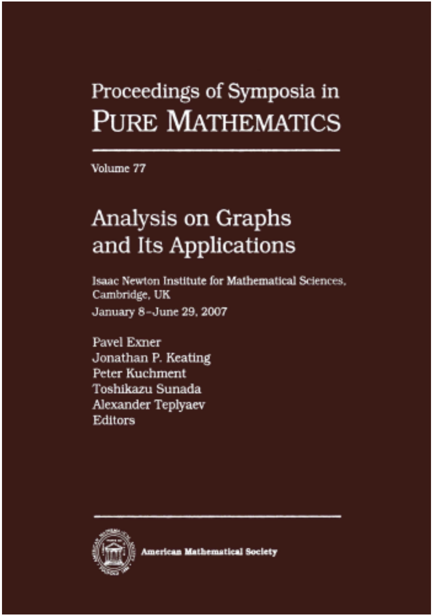
In this paper, we explore the spectral measures of the Laplacian on Schreier graphs for several self-similar groups (the Grigorchuk, Lamplighter, and Hanoi groups) from the dynamical and algebro-geometric viewpoints. For these graphs, classical Schur renormalization transformations act on appropriate spectral parameters as rational maps in two variables. We show that the spectra in question

J. Fractal Geom. 4 (2017), 369–424
DOI 10.4171/JFG/55

Journal of Fractal Geometry

Ends of Schreier graphs and cut-points of limit spaces of self-similar groups

Ievgen Bondarenko,[1] Daniele D'Angeli,[2] and Tatiana Nagnibeda[3]


Proceedings of Symposia in
PURE MATHEMATICS
Volume 77
Analysis on Graphs
and Its Applications
Isaac Newton Institute for Mathematical Sciences,
Cambridge, UK
January 8–June 29, 2007
Pavel Exner
Jonathan P. Keating
Peter Kuchment
Toshikazu Sunada
Alexander Teplyaev
Editors
American Mathematical Society

DOI: 10.1090/pspum/077/2459868 • Corpus ID: 279025

Groups and analysis on fractals

V. Nekrashevych, A. Teplyaev • Published 2005 • Mathematics

We describe relation between analysis on fractals and the theory of self-similar groups. In particular, we focus on the construction of the Laplacian on limit sets of such groups in several concrete examples, and in the general p.c.f. case. We pose a number of open questions.

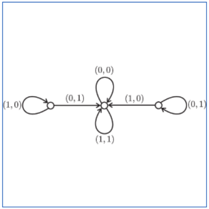


Figure 1

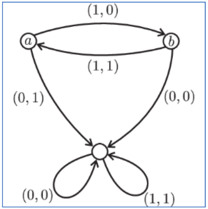


Figure 2

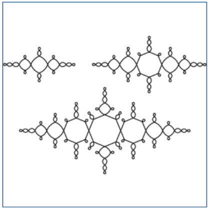


Figure 3

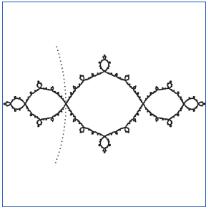


Figure 4

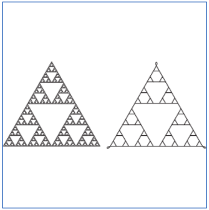


Figure 6

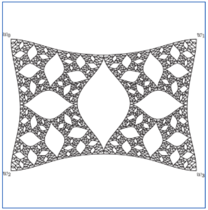


Figure 7

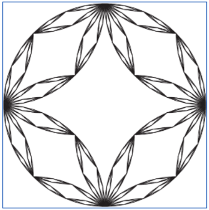


Figure 8

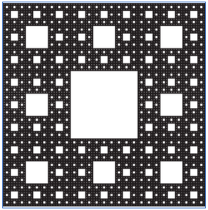


Figure 9

What are Hausdorff and spectral dimensions of a self-similar set?

For the circle, $d_S = 1$
For Riemannian d-manifolds, $d_S = d_H = d$

In general, d_S can be defined using the asymptotics of eigenvalues or, equivalently, asymptotics of the heat kernel.

If d_s is well defined, then

$$\textit{recurrence of the diffusion} \iff d_S < 2$$

in which case we sometimes can prove Kigami's formula

$$d_S = 2\frac{d_{H,R}}{d_{H,R}+1}$$

where $d_{H,R}$ is the **effective resistance Hausdorff dimension**.

On the Sierpinski gasket (S.Goldstein 1984)

$$d_{topo} = 1 < d_S = \frac{\log 9}{\log 5} < d_H = \frac{\log 3}{\log 2}$$

On the basilica Julia set we formally computed (Rogers-T, 2010)

$$d_S = \frac{4}{3}$$

On the Sierpinski carpet $\exists! d_S$ (Barlow, Bass, Kumagai, T. 1989-2010)

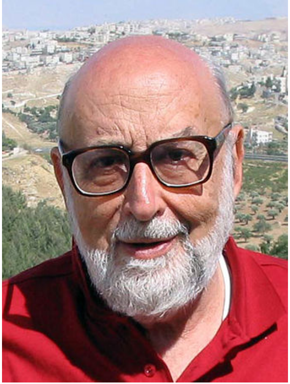
$$d_{topo} = 1 < d_{H,topo} = 1 + \frac{\log 2}{\log 3} < d_S < d_H = \frac{\log 8}{\log 3}$$

François Englert

From Wikipedia, the free encyclopedia

François Baron Englert (French: [ɑ̃glɛʁ]; born 6 November 1932) is a Belgian theoretical physicist and 2013 Nobel prize laureate (shared with Peter Higgs). He is Professor emeritus at the Université libre de Bruxelles (ULB) where he is member of the Service de Physique Théorique. He is also a Sackler Professor by Special Appointment in the School of Physics and Astronomy at Tel Aviv University and a member of the Institute for Quantum Studies at Chapman University in California. He was awarded the 2010 J. J. Sakurai Prize for Theoretical Particle Physics (with Gerry Guralnik, C. R. Hagen, Tom Kibble, Peter Higgs, and Robert Brout), the Wolf Prize in Physics in 2004 (with Brout and Higgs) and the High Energy and Particle Prize of the European Physical Society (with Brout and Higgs) in 1997 for the mechanism which unifies short and long range interactions by generating massive gauge vector bosons. He has made contributions in statistical physics, quantum field theory, cosmology, string theory and supergravity.[4] He is the recipient of the 2013 Prince of Asturias Award in technical and scientific research,

François Englert



François Englert in Israel, 2007

Nuclear Physics B280 [FS 18] (1987) 147–180
North-Holland, Amsterdam

METRIC SPACE-TIME AS FIXED POINT OF THE RENORMALIZATION GROUP EQUATIONS ON FRACTAL STRUCTURES

F. ENGLERT, J.-M. FRÈRE[1] and M. ROOMAN[2]

Physique Théorique, C.P. 225, Université Libre de Bruxelles, 1050 Brussels, Belgium

Ph. SPINDEL

Faculté des Sciences, Université de l'Etat à Mons, 7000 Mons, Belgium

Received 19 February 1986

We take a model of foamy space-time structure described by self-similar fractals. We study the propagation of a scalar field on such a background and we show that for almost any initial conditions the renormalization group equations lead to an effective highly symmetric metric at large scale.

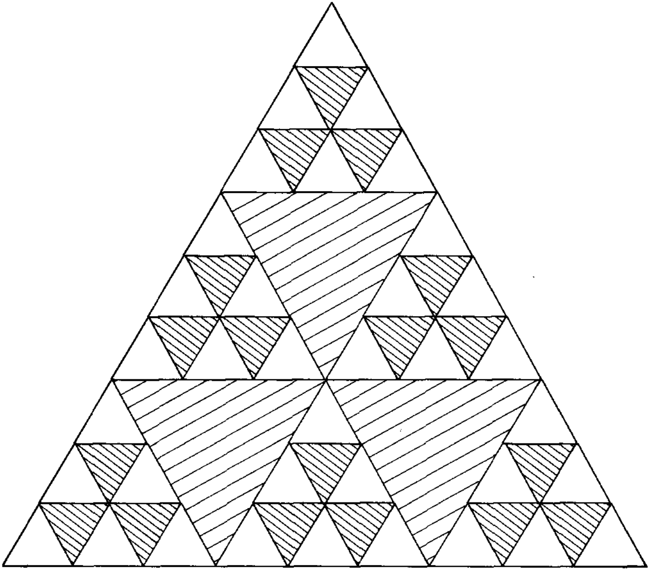


Fig. 1. The first two iterations of a 2-dimensional 3-fractal.

PRL **95,** 171301 (2005) PHYSICAL REVIEW LETTERS week ending 21 OCTOBER 2005

The Spectral Dimension of the Universe is Scale Dependent

J. Ambjørn,[1,3,*] J. Jurkiewicz,[2,†] and R. Loll[3,‡]

[1]*The Niels Bohr Institute, Copenhagen University, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark*
[2]*Mark Kac Complex Systems Research Centre, Marian Smoluchowski Institute of Physics, Jagellonian University, Reymonta 4, PL 30-059 Krakow, Poland*
[3]*Institute for Theoretical Physics, Utrecht University, Leuvenlaan 4, NL-3584 CE Utrecht, The Netherlands*
(Received 13 May 2005; published 20 October 2005)

We measure the spectral dimension of universes emerging from nonperturbative quantum gravity, defined through state sums of causal triangulated geometries. While four dimensional on large scales, the quantum universe appears two dimensional at short distances. We conclude that quantum gravity may be "self-renormalizing" at the Planck scale, by virtue of a mechanism of dynamical dimensional reduction.

DOI: 10.1103/PhysRevLett.95.171301 PACS numbers: 04.60.Gw, 04.60.Nc, 98.80.Qc

Quantum gravity as an ultraviolet regulator?—A shared hope of researchers in otherwise disparate approaches to quantum gravity is that the microstructure of space and time may provide a physical regulator for the ultraviolet infinities encountered in perturbative quantum field theory. ... *tral dimension*, a diffeomorphism-invariant quantity obtained from studying diffusion on the quantum ensemble of geometries. On large scales and within measuring accuracy, it is equal to four, in agreement with earlier measurements of the large-scale dimensionality based on the other hand, the "short-distance spectral dimension," obtained by extrapolating Eq. (12) to $\sigma \to 0$ is given by

$$D_S(\sigma = 0) = 1.80 \pm 0.25, \qquad (15)$$

and thus is compatible with the integer value two.

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Published: *December 2, 2011*

Fractal space-times under the microscope: a renormalization group view on Monte Carlo data

Martin Reuter and Frank Saueressig

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Abstract: The emergence of fractal features in the microscopic structure of space-time is a common theme in many approaches to quantum gravity. In this work we carry out a detailed renormalization group study of the spectral dimension d_s and walk dimension d_w associated with the effective space-times of asymptotically safe Quantum Einstein Gravity (QEG). We discover three scaling regimes where these generalized dimensions are approximately constant for an extended range of length scales: a classical regime where $d_s = d, d_w = 2$, a semi-classical regime where $d_s = 2d/(2+d), d_w = 2+d$, and the UV-fixed point regime where $d_s = d/2, d_w = 4$. On the length scales covered by three-dimensional Monte Carlo simulations, the resulting spectral dimension is shown to be in very good agreement with the data. This comparison also provides a natural explanation for the apparent puzzle between the short distance behavior of the spectral dimension reported from Causal Dynamical Triangulations (CDT), Euclidean Dynamical Triangulations (EDT), and Asymptotic Safety.

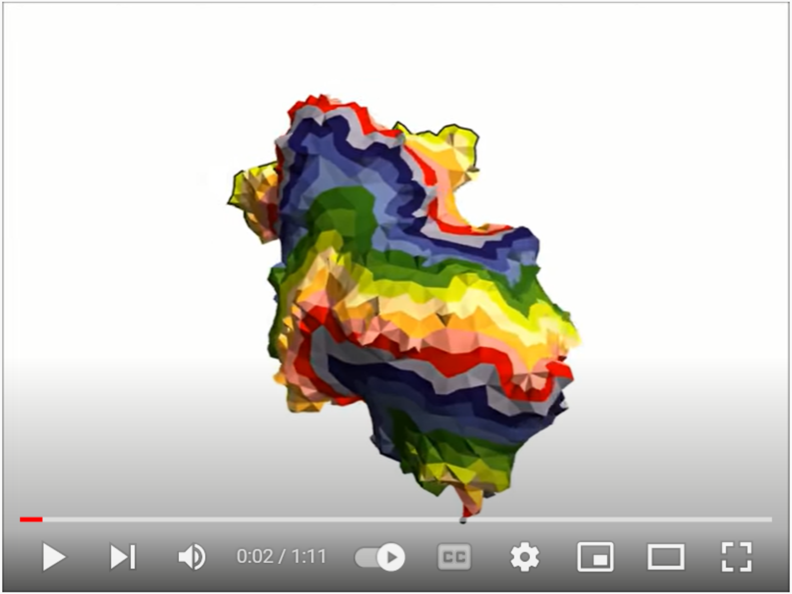
Keywords: Models of Quantum Gravity, Renormalization Group, Lattice Models of Gravity, Nonperturbative Effects

JHEP12(2011)012

Fractal space-times under the microscope: A Renormalization Group view on Monte Carlo data

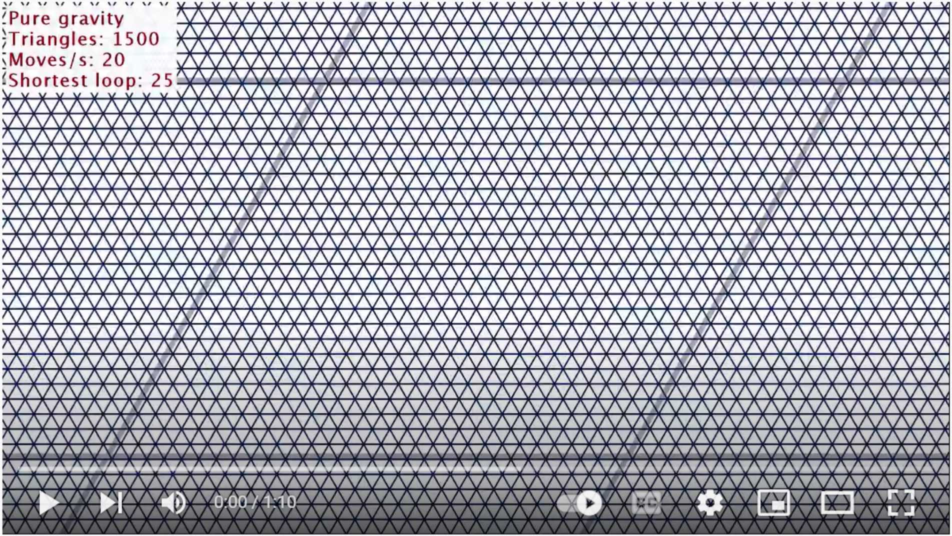
Martin Reuter and Frank Saueressig

a classical regime where $d_s = d, d_w = 2$, a semi-classical regime where $d_s = 2d/(2+d), d_w = 2+d$, and the UV-fixed point regime where $d_s = d/2, d_w = 4$. On the length scales covered



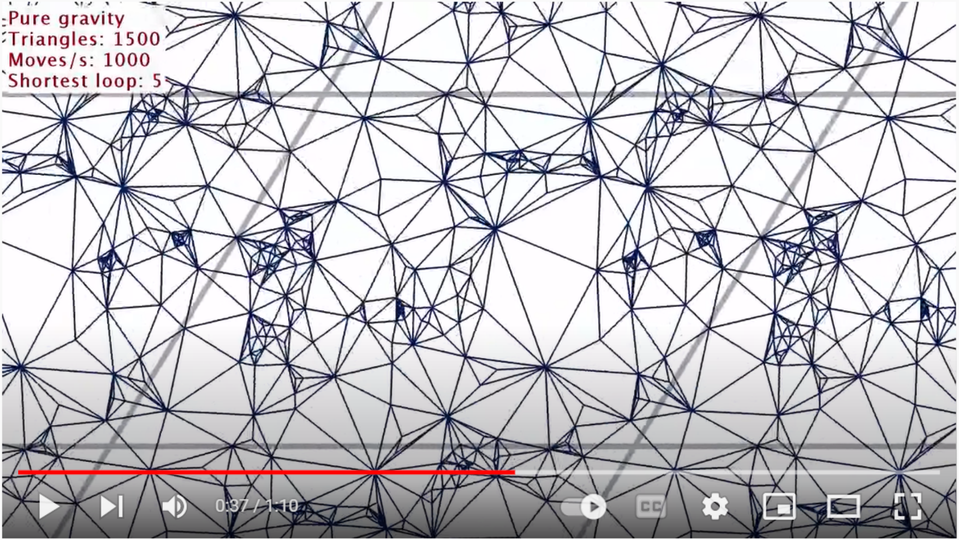
Causal dynamical triangulations

25,971 views Jan 26, 2013 Causal dynamical triangulation (CDT) is a lattice model of quantum gravity. In two space-time dimensions (instead of the four we live in) it



Dynamical triangulation of the 2-torus

1,435 views Sep 7, 2013 This video illustrates a Monte Carlo simulation for two-dimensional quantum gravity on a torus. Starting with a regular triangulation of the torus repeatedly a so-called flip move is performed on a randomly chosen edge. The triangulations obtained after a large



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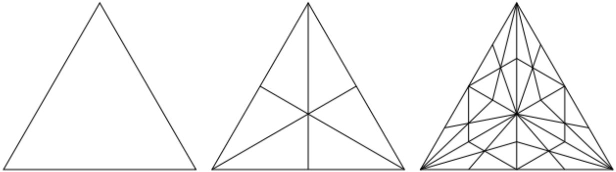


FIGURE 1. Barycentric subdivision of a 2-simplex, the graphs G_0^T, G_1^T and G_2^T.

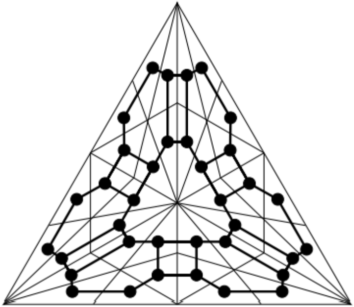


FIGURE 2. Adjacency (dual) graph G_2, in bold, and the barycentric subdivision graph pictured together with the thin image of G_2^T.

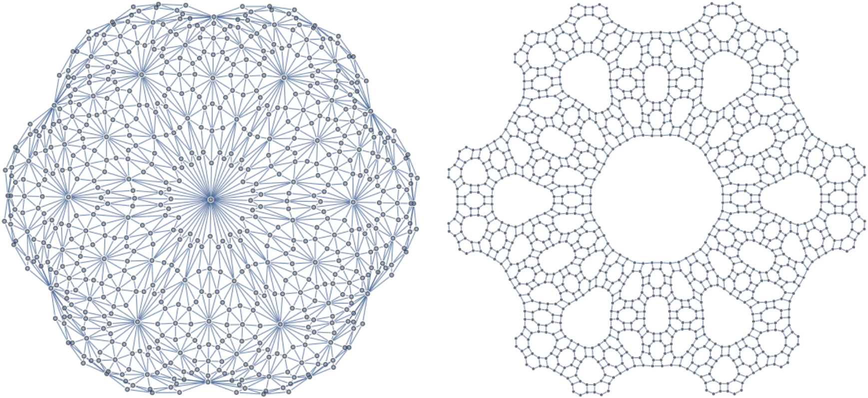


FIGURE 3. On the left: the graph G_4^T for barycentric subdivision of a 2-simplex. On the right: the adjacency (dual) graph G_4.

https://teplyaev.math.uconn.edu/fractalwave/

p: 0.2 n_0: 4 n: 8 Interval: uni r m

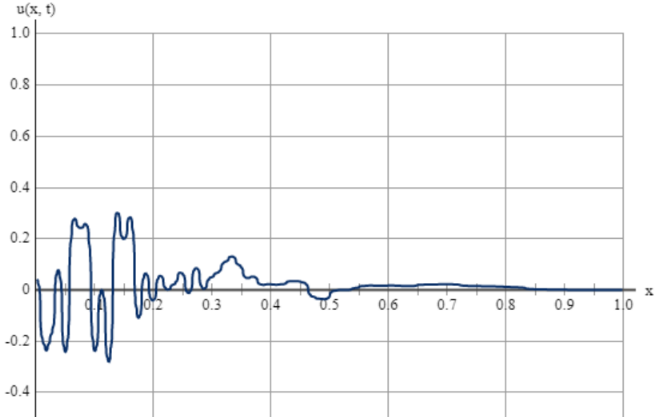
Solution to the wave equation,
t = 0.366

Speed:

Initial condition:

$$\begin{cases} u(x,0) = \tilde{\delta}(x); \\ \frac{\partial u}{\partial t}(x,0) = 0. \end{cases}$$

$$\begin{cases} u(x,0) = 0; \\ \frac{\partial u}{\partial t}(x,0) = \tilde{\delta}(x). \end{cases}$$



Brownian motion:
Thiele (1880), Bachelier (1900)
Einstein (1905), Smoluchowski (1906)
Wiener (1920'), Doob, Feller, Levy, Kolmogorov (1930'),
Doeblin, Dynkin, Hunt, Ito ...

$$distance \sim \sqrt{time}$$

"Einstein space–time relation for Brownian motion"

Wiener process in $\mathbb{R}^n$ satisfies $\frac{1}{n}\mathbb{E}|W_t|^2 = t$ and has a Gaussian transition density:

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

- **De Giorgi-Nash-Moser estimates for elliptic and parabolic PDEs;**
- **Li-Yau (1986) type estimates** on a geodesically complete Riemannian manifold with $\boldsymbol{Ricci} \geqslant \mathbf{0}$:

$$p_t(x,y) \sim \frac{1}{V(x,\sqrt{t})} \exp\left(-c\frac{d(x,y)^2}{t}\right)$$

$$distance \sim \sqrt{time}$$

Gaussian:

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

Li-Yau Gaussian-type:

$$p_t(x,y) \sim \frac{1}{V(x,\sqrt{t})} \exp\left(-c\frac{d(x,y)^2}{t}\right)$$

Sub-Gaussian:

$$p_t(x,y) \sim \frac{1}{t^{d_H/d_w}} \exp\left(-c\left(\frac{d(x,y)^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right)$$

$$distance \sim (time)^{\frac{1}{d_w}}$$

Brownian motion on $\mathbb{R}^d$: $\mathbb{E}|X_t - X_0| = ct^{1/2}$.

Anomalous diffusion: $\mathbb{E}|X_t - X_0| = o(t^{1/2})$, or (in regular enough situations),

$$\mathbb{E}|X_t - X_0| \approx t^{1/d_w}$$

with $d_w > 2$.

Here d_w is the so-called **walk dimension** (should be called **"walk index"** perhaps).

This phenomena was first observed by mathematical physicists working in the transport properties of disordered media, such as (critical) percolation clusters.

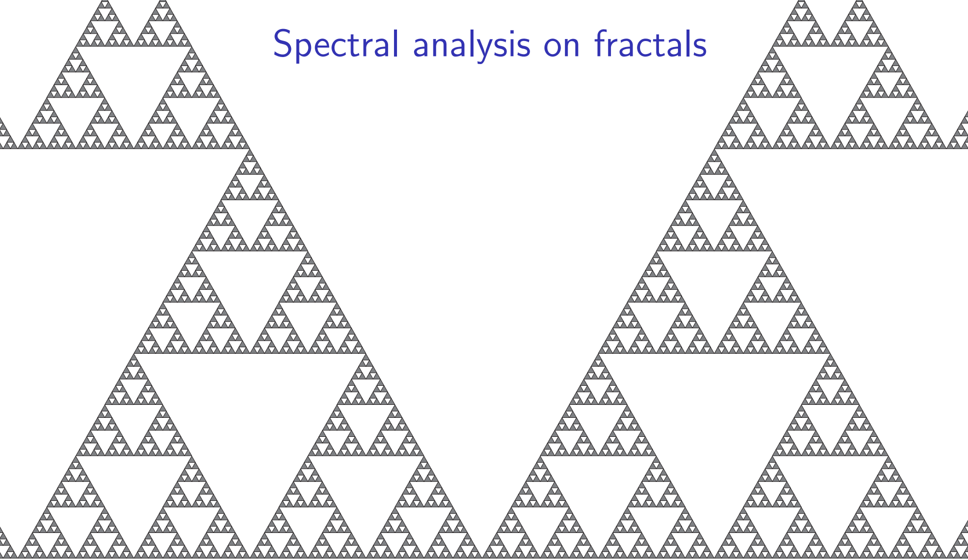
$$p_t(x,y) \sim \frac{1}{t^{d_H/d_w}} \exp\left(-c\frac{d(x,y)^{\frac{d_w}{d_w-1}}}{t^{\frac{1}{d_w-1}}}\right)$$

$$distance \sim (time)^{\frac{1}{d_w}}$$

$$d_H = \text{Hausdorff dimension}$$
$$\frac{1}{\gamma} = d_w = \text{“walk dimension” } (\gamma\text{=diffusion index})$$
$$\frac{2d_H}{d_w} = d_S = \text{“spectral dimension” (diffusion dimension)}$$

First example: Sierpiński gasket; Kusuoka, Fukushima, Kigami, Barlow, Bass, Perkins (mid 1980'—)

Spectral analysis on fractals



A part of an infinite Sierpiński gasket.

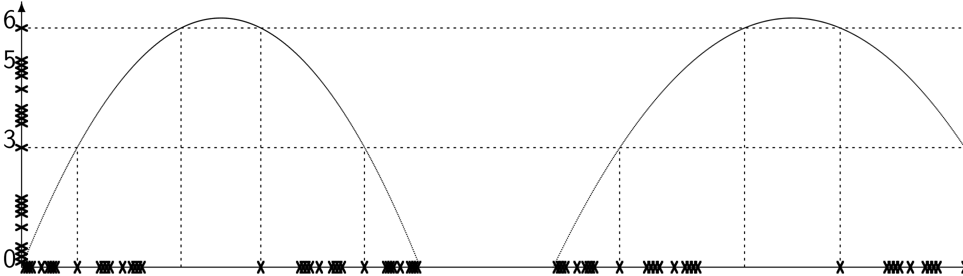
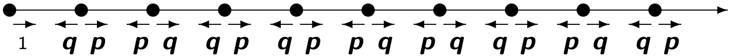


Figure: An illustration to the computation of the spectrum on the infinite Sierpiński gasket. The curved lines show the graph of the function $\mathfrak{R}(\cdot)$.

Theorem (Rammal, Toulouse 1983, Béllissard 1988, Fukushima, Shima 1991, T. 1998, Quint 2009)

On the infinite Sierpiński gasket the spectrum of the Laplacian consists of a **dense set of eigenvalues $\mathfrak{R}^{-1}(\Sigma_0)$ of infinite multiplicity** *and a* **singularly continuous component of spectral multiplicity one supported on $\mathfrak{R}^{-1}(\mathcal{J}_R)$**.

Half-line example



Transition probabilities in the pq random walk. Here $p \in (0, 1)$ and $q = 1 - p$.

$$(\Delta_p f)(x) = \begin{cases} f(0) - f(1), & \text{if } x = 0 \\ f(x) - qf(x-1) - pf(x+1), & \text{if } 3^{-m(x)}x \equiv 1 \pmod 3 \\ f(x) - pf(x-1) - qf(x+1), & \text{if } 3^{-m(x)}x \equiv 2 \pmod 3 \end{cases}$$

Theorem (J.P.Chen, T., 2016)

If $p \neq \frac{1}{2}$, the Laplacian Δ_p on $\ell^2(\mathbb{Z}_+)$ has **purely singularly continuous spectrum**. *The spectrum is the Julia set, a* **topological Cantor set of Lebesgue measure zero**, *of the polynomial* $R(z) = \dfrac{z(z^2 - 3z + (2 + pq))}{pq}$

This is a simple, possibly the simplest, quasi-periodic example related to the recent results of A.Avila, D.Damanik, A.Gorodetski, S.Jitomirskaya, Y.Last, B.Simon et al.

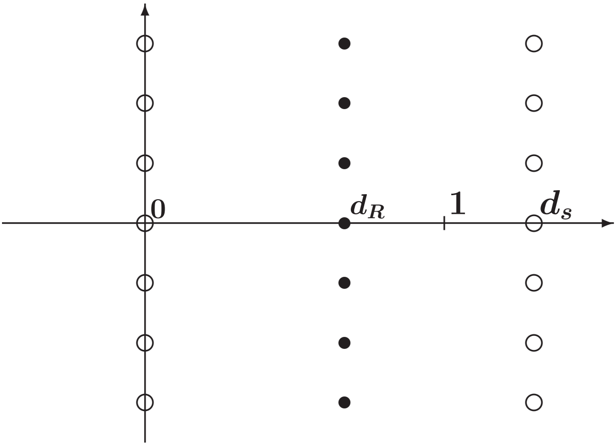
Spectral zeta function

Theorem. (Derfel-Grabner-Vogl, Steinhurst-T., Chen-T.-Tsougkas, Kajino (2007–2017)) For a large class of **finitely ramified symmetric fractals** the spectral zeta function

$$\zeta(s) = \sum \lambda_j^{s/2}$$

has a meromorphic continuation from the half-pain $Re(s) > d_S$ to $\mathbb{C}$. Moreover, all the poles and residues are computable from the geometric data of the fractal. Here λ_j are the eigenvalues if the unique symmetric Laplacian.

- Example: $\zeta(s)$ is the Riemann zeta function up to a trivial factor in the case when our fractal is $[0, 1]$.
- In more complicated situations, such as the Sierpiński gasket, there are infinitely many non-real poles, which can be called complex spectral dimensions, and are related to oscillations in the spectrum.



$$d_s = \frac{\log 9}{\log 5}$$

$$d_R = \frac{\log 4}{\log 5}$$

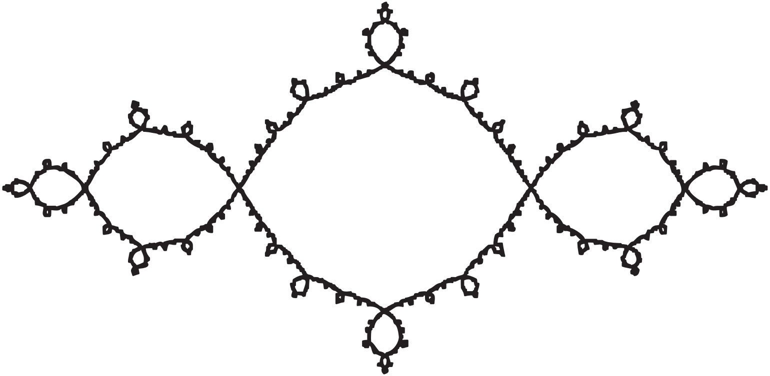
Poles (white circles) of the spectral zeta function of the Sierpiński gasket.

Spectral Analysis of the Basilica Graphs (with Luke Rogers, Toni Brzoska, Courtney George, Samantha Jarvis)

The question of existence of groups with **intermediate growth, i.e. subexponential but not polynomial**, was asked by **Milnor in 1968** and answered in the positive by **Grigorchuk in 1984**. There are still open questions in this area, and a complete picture of which orders of growth are possible, and which are not, is missing.

The Basilica group is a group generated by a finite automation acting on the binary tree in a self-similar fashion, introduced by **R. Grigorchuk and A. Zuk in 2002**, does not belong to the closure of the set of groups of subexponential growth under the operations of group extension and direct limit.

In 2005 L. Bartholdi and B. Virag further showed it to be amenable, making the Basilica group **the 1st example of an amenable but not subexponentially amenable** group (also **"Münchhausen trick" and amenability of self-similar groups by V.A. Kaimanovich**).



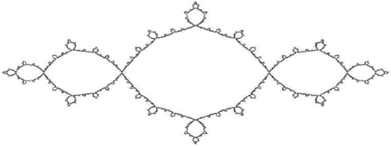
The basilica Julia set, the Julia set of $z^2 - 1$ and the limit set of the basilica group of exponential growth (Grigorchuk, Żuk, Bartholdi, Virág, Nekrashevych, Kaimanovich, Nagnibeda et al.).

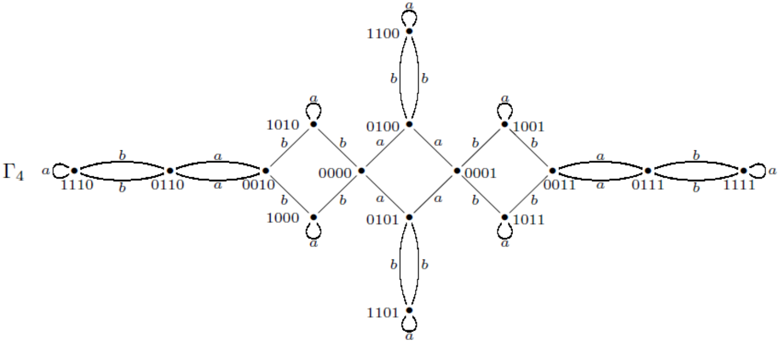
In 2005, V. Nekrashevych described the Basilica as the iterated monodromy group, and there exists a natural way to associate it to the Basilica fractal (**Nekrashevych+T., 2008**).

In **Schreier graphs of the Basilica group (2010), Nagnibeda et al.** classified up to isomorphism all possible limits of finite Schreier graphs of the Basilica group.

In **Laplacians on the Basilica Julia set (2010), L. Rogers+T.** constructed Dirichlet forms and the corresponding Laplacians on the Basilica fractal in two different ways: by imposing a self-similar harmonic structure and a graph-directed self-simliar structure on the fractal.

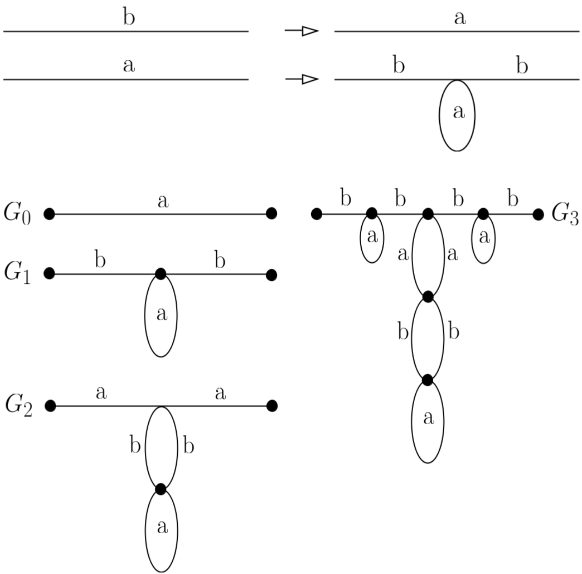
In **2012-2015, Dong, Flock, Molitor, Ott, Spicer, Totari and Strichartz** provided numerical techniques to approximate eigenvalues and eigenfunctions on families of Laplacians on the Julia sets of $z^2 + c$.



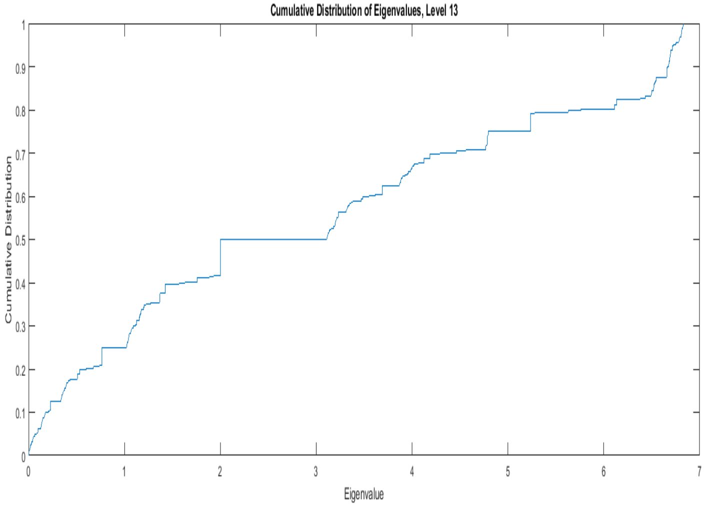


pictures taken from paper by Nagnibeda et. al.

Replacement Rule and the Graphs G_n



Distribution of Eigenvalues, Level 13



One can define a Dirichlet to Neumann map for the two boundary points of the graphs G_n. One can construct a dynamical system to determine these maps (which are two by two matrices). The dynamical system allows us to prove the following.

Theorem

In the Hausdorff metric, $\limsup_{n\to\infty} \sigma(L^{(n)})$ has a gap that contains the interval $(2.5, 2.8)$.

Theorem (arXiv:1908.10505)

In the Hausdorff metric, $\limsup_{n\to\infty} \sigma(L^{(n)})$ has infinitely many gaps.

Infinite Blow-ups of G_n

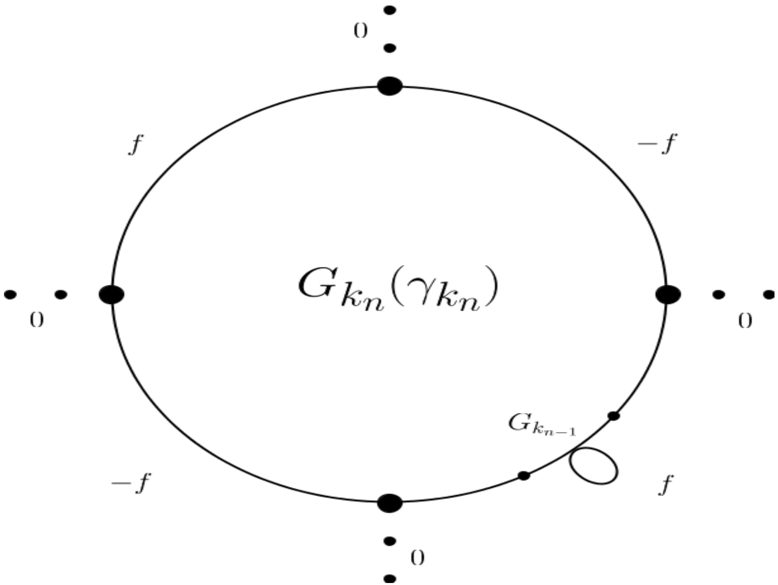
Definition

Let $\{k_n\}_{n\in\mathbb{N}}$ be a strictly increasing subsequence of the natural numbers. For each n, embed G_{k_n} in some isomorphic subgraph of $G_{k_{n+1}}$. The corresponding infinite blow-up is $G_\infty := \cup_{n\geq 0} G_{k_n}$.

Assumption

The infinite blow-up G_∞ satisfies:

- For $n \geq 1$, the long path of $G_{k_{n-1}}$ is embedded in a loop γ_n of G_{k_n}.
- Apart from $l_{k_{n-1}}$ and $r_{k_{n-1}}$, no vertex of the long path can be the $3, 6, 9$ or 12 o'clock vertex of γ_n.
- The only vertices of G_{k_n} that connect to vertices outside the graph are the boundary vertices of G_{k_n}.


0
f
$-f$
$G_{k_n}(\gamma_{k_n})$
0
0
$G_{k_{n-1}}$
$-f$
f
0

Theorem

(1) $\sigma(L^{(k_n)}|_{\ell^2_{a,k_n,\gamma_n}}) = \sigma(L_0^{(j_n)})$.

(2) The spectrum of $L^{(\infty)}$ is pure point. The set of eigenvalues of $L^{(\infty)}$ is

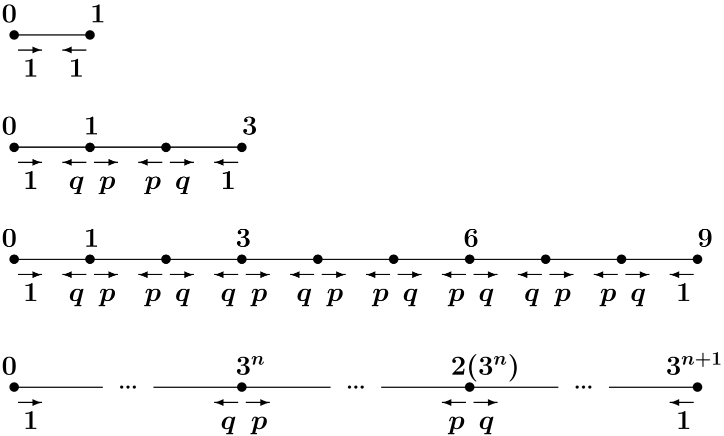
$$\bigcup_{n\geq 0} \sigma(L_0^{(j_n)}) = \bigcup_{n\geq 0} c_{j_n}^{-1}\{0\},$$

where the polynomials c_n are the characteristic polynomials of $L_0^{(n)}$, as defined in the previous proposition.

(3) Moreover, the set of eigenfunctions of $L^{(\infty)}$ with finite support is complete in ℓ^2.

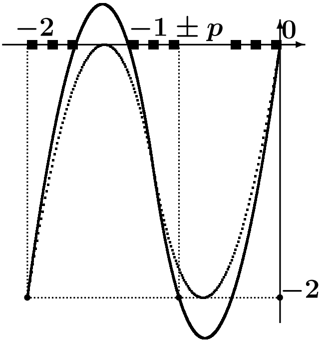
TECHNICAL DETAILS

Fix $p, q > 0$, $p + q = 1$, and define probabilistic Laplacians Δ_n on the segments $[0, 3^n]$ of $\mathbb{Z}_+$ inductively as a generator of the random walks:



and let $\Delta = \lim\limits_{n\to\infty} \Delta_n$ be the corresponding probabilistic Laplacian on $\mathbb{Z}_+$.

If $z \neq -1 \pm p$ and $R(z) = z(z^2+3z+2+pq)/pq$, then
$R(z) \in \sigma(\Delta_n) \iff z \in \sigma(\Delta_{n+1})$



Theorem (Joe P. Chen and T., JMP 2016). $\sigma(\Delta) = \mathcal{J}_R$, the Julia set of $R(z)$.

If $p = q$, then $\sigma(\Delta) = [-2, 0]$, spectrum is a.c.

If $p \neq q$, then $\sigma(\Delta)$ is a Cantor set of Lebesgue measure zero, spectrum is singularly continuous.

U.Andrews, J.P.Chen, G.Bonik, R.W.Martin, A.Teplyaev, **Wave equation on one-dimensional fractals with spectral decimation**. J. Fourier Anal. Appl. 23 (2017) 994–1027. `http://teplyaev.math.uconn.edu/fractalwave/`

Bellissard, Geronimo, Volberg, Yuditskii, **Are they limit periodic?** Complex analysis and dynamical systems II, 43–53, Contemp. Math., 382, Israel Math. Conf. Proc., Amer. Math. Soc., Providence, RI, 2005.
(Reviewed by Maxim S. Derevyagin)

Bellissard, **Renormalization group analysis and quasicrystals**. Ideas and methods in quantum and statistical physics (Oslo, 1988), 118–148, Cambridge Univ. Press, Cambridge, 1992.

Barnsley, Geronimo and Harrington, **Almost periodic Jacobi matrices associated with Julia sets for polynomials**. Comm. Math. Phys. 99 (1985), 303–317.

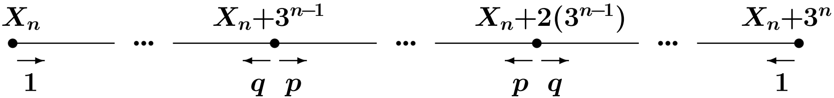
Bellissard, Bessis and Moussa, **Chaotic states of almost periodic Schrodinger operators**. Phys. Rev. Lett. 49 (1982), 701–704.

Bellissard and Simon, **Cantor spectrum for the almost Mathieu equation**
J. Funct. Anal. 48 (1982), 408–419.

There are uncountably many "random" self-similar Laplacians $\boldsymbol{\Delta}$ on $\mathbb{Z}$:
For a sequence $\mathcal{K} = \{k_j\}_{j=1}^{\infty}$, $k_j \in \{0, 1, 2\}$, let

$$X_n = -\sum_{j=1}^{n} k_j 3^j$$

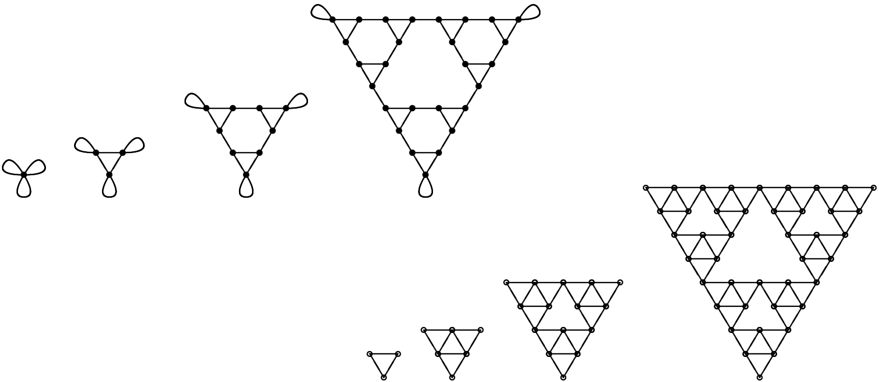
and $\boldsymbol{\Delta}_n$ is a probabilistic Laplacian on $[X_n, X_n+3^n]$:

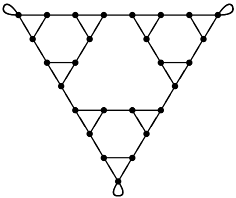


In the previous example $k_j = 0$ for all j.

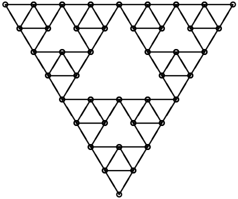
Theorem.
For any sequence $\mathcal{K}$ we have $\sigma(\boldsymbol{\Delta}) = \mathcal{J}_R$. The same is true for the Dirichlet Laplacian on $\mathbb{Z}_+$ (when $k_j \equiv 0$).

R. Grigorchuk and Z. Sunik, *Asymptotic aspects of Schreier graphs and Hanoi Towers groups.*





Sierpiński 3-graph
(Hanoi Towers-3 group)



Sierpiński 4-graph
(standard)

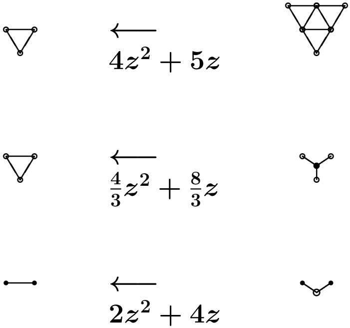
These three polynomials are conjugate:

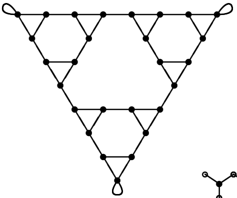
Sierpiński 3-graph (Hanoi Towers-3 group): $f(x) = x^2 - x - 3$
$f(3) = 3$, $f'(3) = 5$

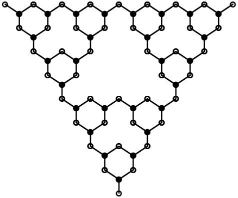
Sierpiński 4-graph, "adjacency matrix" Laplacian: $P(\lambda) = 5\lambda - \lambda^2$
$P(0) = 0$, $P'(0) = 5$

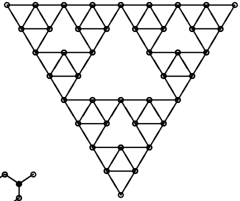
Sierpiński 4-graph, probabilistic Laplacian: $R(z) = 4z^2 + 5z$
$R(0) = 0$, $R'(0) = 5$

Theorem. Eigenvalues and eigenfunctions on the Sierpiński 3-graphs and Sierpiński 4-graphs are in one-to-one correspondence, with the exception of the eigenvalue $z = -\frac{3}{2}$ for the 4-graphs.









Sierpiński 3-graph
(Hanoi Towers-3 group)
$R(z) = 2z^2 + 4z$

Sierpiński 4-graph
(standard)
$R(z) = \frac{4}{3}z^2 + \frac{8}{3}z$

Let $\mathcal{H}$ and $\mathcal{H}_0$ be Hilbert spaces, and $U : \mathcal{H}_0 \to \mathcal{H}$ be an isometry.

Definition. We call an operator H **spectrally similar** to an operator H_0 with functions φ_0 and φ_1 if

$$U^*(H - z)^{-1}U = (\varphi_0(z)H_0 - \varphi_1(z))^{-1}$$

In particular, if $\varphi_0(z) \neq 0$ and $R(z) = \varphi_1(z)/\varphi_0(z)$, then

$$U^*(H - z)^{-1}U = \frac{1}{\varphi_0(z)}(H - R(z))^{-1}.$$

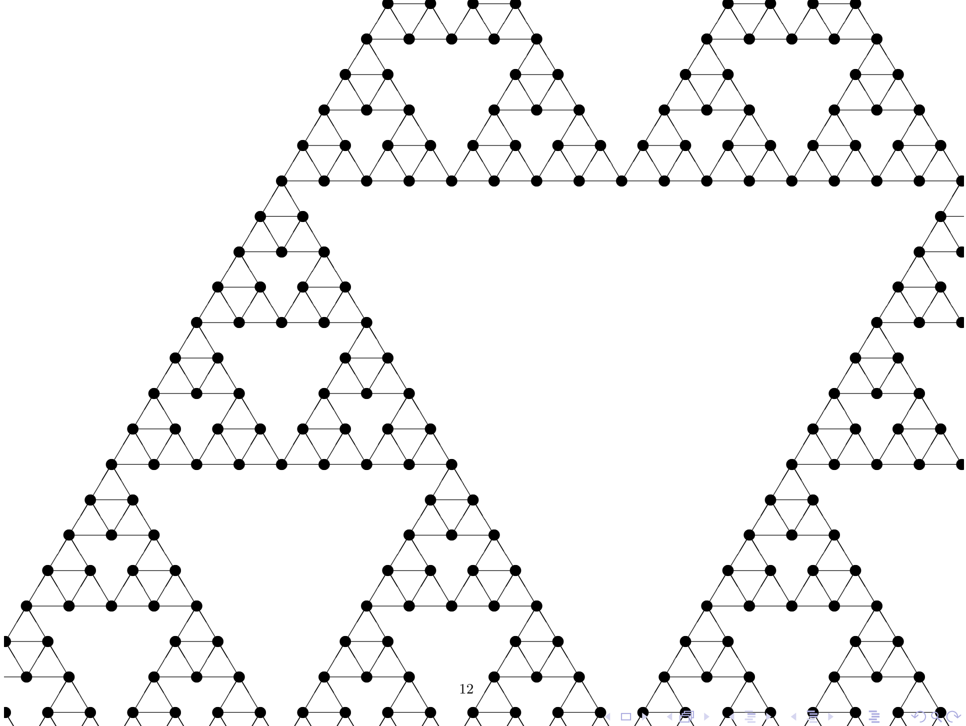
If $H = \begin{pmatrix} S & \bar{X} \\ X & Q \end{pmatrix}$ then

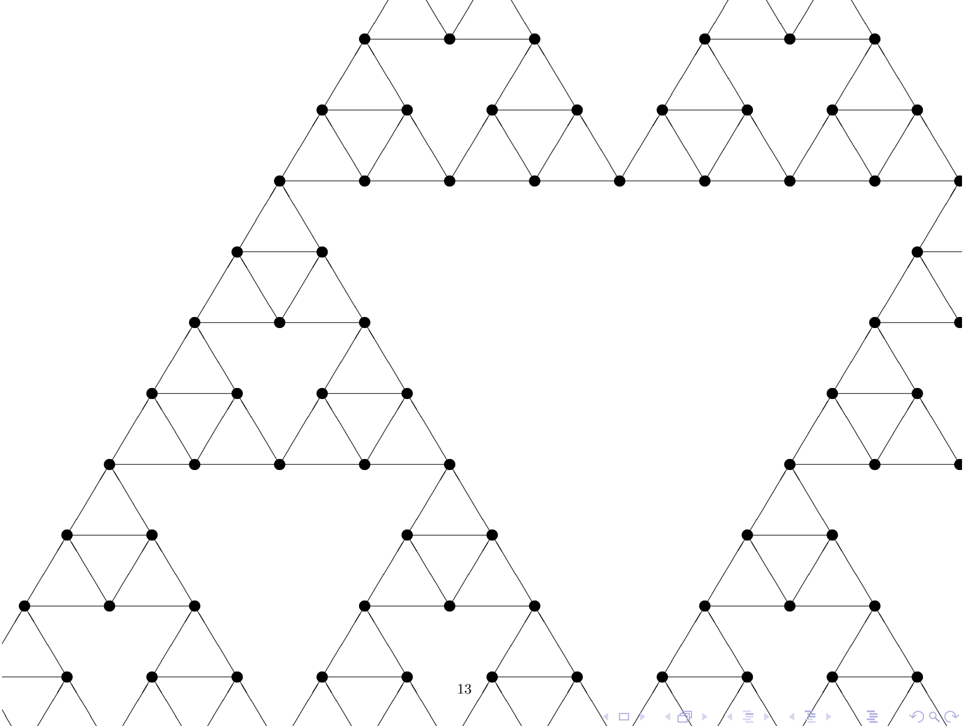
$$S - zI_0 - \bar{X}(Q - zI_1)^{-1}X = \varphi_0(z)H_0 - \varphi_1(z)I_0$$

Theorem (Malozemov and T.). If Δ is the graph Laplacian on a self-similar symmetric infinite graph, then

$$\mathfrak{J}_R \subseteq \sigma(\Delta_\infty) \subseteq \mathfrak{J}_R \cup \mathfrak{D}_\infty$$

where $\mathfrak{D}_\infty$ is a discrete set and $\mathfrak{J}_R$ is the Julia set of the rational function R.





R. Rammal and G. Toulouse, *Random walks on fractal structures and percolation clusters.* J. Physique Letters **44** (1983), L13–L22.

R. Rammal, *Spectrum of harmonic excitations on fractals.* J. Physique **45** (1984).

S. Alexander, *Some properties of the spectrum of the Sierpiński gasket in a magnetic field.* Phys. Rev. B **29** (1984).

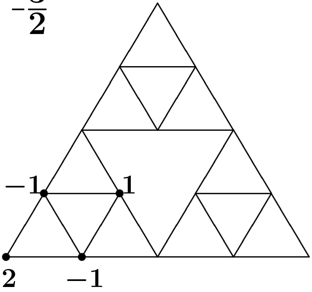
E. Domany, S. Alexander, D. Bensimon and L. Kadanoff, *Solutions to the Schrödinger equation on some fractal lattices.* Phys. Rev. B (3) **28** (1984).

Y. Gefen, A. Aharony and B. B. Mandelbrot, *Phase transitions on fractals. I. Quasilinear lattices. II. Sierpiński gaskets. III. Infinitely ramified lattices.* J. Phys. A **16** (1983–1984).

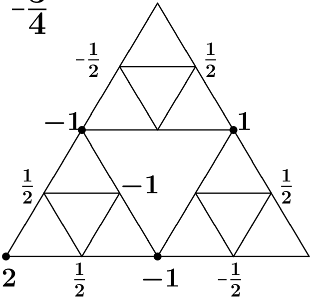
R. B. Stinchcombe, *Fractals, phase transitions and criticality.* Fractals in the natural sciences. Proc. Roy. Soc. London Ser. A **423** (1989), 17–33.

J. Béllissard, *Renormalization group analysis and quasicrystals,* Ideas and methods in quantum and statistical physics (Oslo, 1988). Cambridge Univ. Press, 1992.

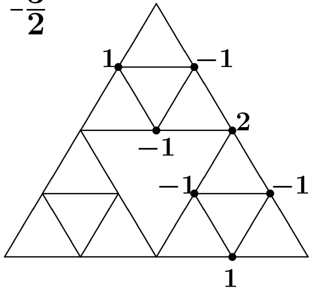
$z = -\frac{3}{2}$



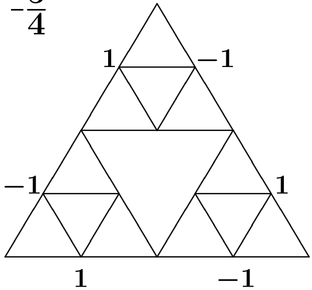
$z = -\frac{3}{4}$



$z = -\frac{3}{2}$



$z = -\frac{5}{4}$

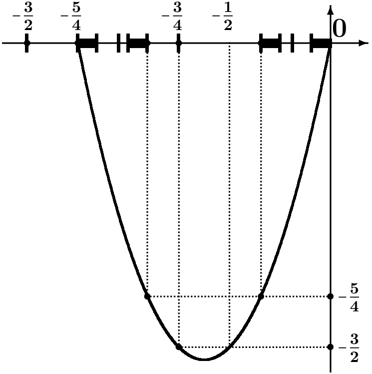


Let Δ be the probabilistic Laplacian (generator of a simple random walk) on the **Sierpiński lattice**. If $z \neq -\frac{3}{2}, -\frac{5}{4}, -\frac{1}{2}$, and $R(z) = z(4z+5)$, then

$$R(z) \in \sigma(\Delta) \Longleftrightarrow z \in \sigma(\Delta)$$

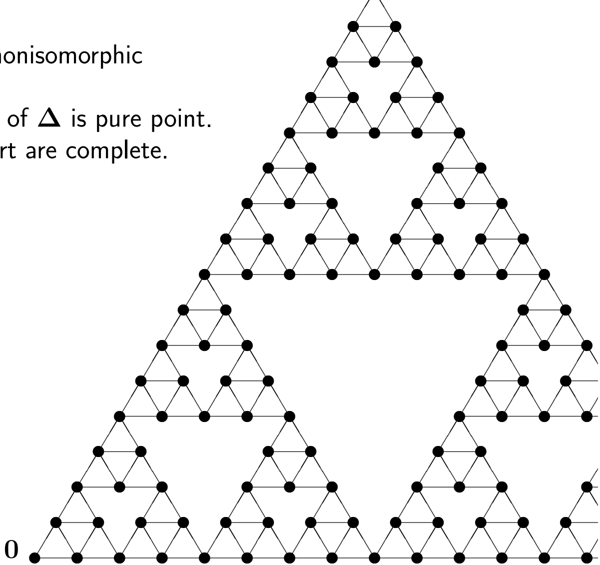
$$\sigma(\Delta) = \mathfrak{J}_R \bigcup \mathfrak{D}$$

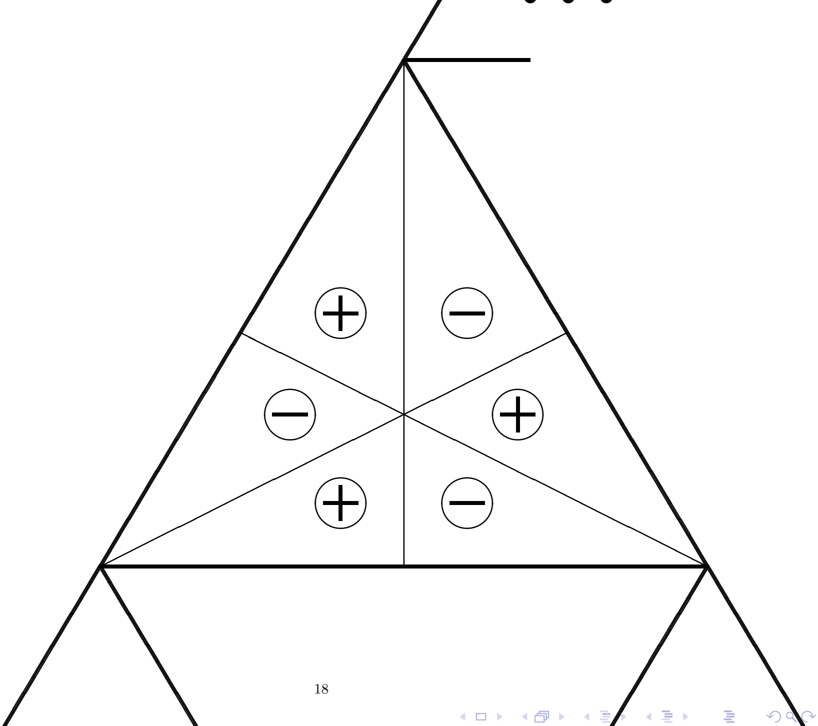
where $\mathfrak{D} \stackrel{\text{def}}{=} \{-\frac{3}{2}\} \bigcup \left(\bigcup_{m=0}^{\infty} R^{-m}\{-\frac{3}{4}\} \right)$
and $\mathfrak{J}_R$ is the Julia set of $R(z)$.



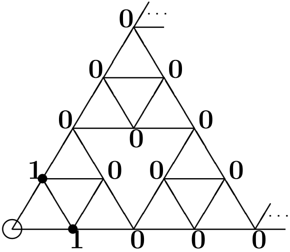
There are uncountably many nonisomorphic Sierpiński lattices.

Theorem (T). The spectrum of $\mathbf{\Delta}$ is pure point. Eigenfunctions with finite support are complete.





Let $\Delta^{(0)}$ be the Laplacian with zero (Dirichlet) boundary condition at ∂L. Then the compactly supported eigenfunctions of $\Delta^{(0)}$ are **not** complete (eigenvalues in $\mathcal{E}$ is not the whole spectrum).



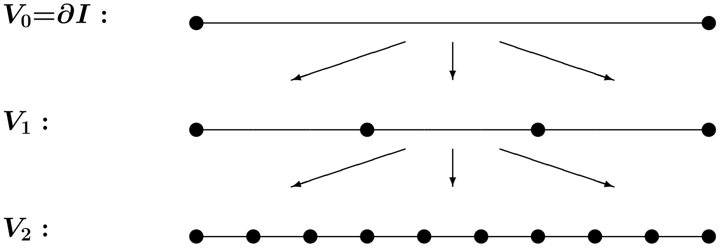
Let $\partial L^{(0)}$ be the set of two points adjacent to ∂L and $\omega_\Delta^{(0)}$ be the spectral measure of $\Delta^{(0)}$ associated with $\mathbb{1}_{\partial L^{(0)}}$. Then $\operatorname{supp}(\omega_\Delta^{(0)}) = \mathfrak{J}_R$ has Lebesgue measure zero and

$$\frac{d(\omega_\Delta^{(0)} \circ R_{1,2})}{d\omega_\Delta^{(0)}}(z) = \frac{(8z+5)(2z+3)}{(2z+1)(4z+5)}$$

Three **contractions** $F_1, F_2, F_3 : \mathbb{R}^1 \to \mathbb{R}^1$, $F_j(x) = \frac{1}{3}(x+p_j)$, with fixed points $p_j = 0, \frac{1}{2}, 1$. The interval $I=[0,1]$ is a unique compact set such that

$$I = \bigcup_{j=1,2,3} F_j(I)$$

The *boundary* of I is $\partial I = V_0 = \{0,1\}$ and the *discrete approximations* to I are $V_n = \bigcup_{j=1,2,3} F_j(V_{n-1}) = \left\{\frac{k}{3^n}\right\}_{k=0}^{3^n}$



Definition. The *discrete Dirichlet (energy) form* on V_n is

$$\mathcal{E}_n(f) = \sum_{\substack{x,y \in V_n \\ y \sim x}} (f(y) - f(x))^2$$

and the *Dirichlet (energy) form* on I is $\mathcal{E}(f) = \lim_{n\to\infty} 3^n \mathcal{E}_n(f) = \int_0^1 |f'(x)|^2 dx$

Definition. A function h is *harmonic* if it minimizes the energy given the boundary values.

Proposition. $3\mathcal{E}_{n+1}(f) \geqslant \mathcal{E}_n(f)$ and $3\mathcal{E}_{n+1}(h) = \mathcal{E}_n(h) = 3^{-n}\mathcal{E}(h)$ for a harmonic h.

Proposition. The Dirichlet (energy) form on I is *self-similar* in the sense that

$$\mathcal{E}(f) = 3 \sum_{j=1,2,3} \mathcal{E}(f \circ F_j)$$

Definition. The *discrete Laplacians* on V_n are

$$\Delta_n f(x) = \frac{1}{2} \sum_{\substack{y \in V_n \\ y \sim x}} f(y) - f(x), \quad x \in V_n \setminus V_0$$

and the Laplacian on I is $\Delta f(x) = \lim_{n \to \infty} 9^n \Delta_n f(x) = f''(x)$

Gauss–Green (integration by parts) formula:

$$\mathcal{E}(f) = -\int_0^1 f \Delta f dx + f f' \Big|_0^1$$

Spectral asymptotics: Let $\rho(\lambda)$ be the *eigenvalue counting function* of the Dirichlet or Neumann Laplacian Δ:

$$\rho(\lambda) = \#\{j : \lambda_j < \lambda\}.$$

Then

$$\lim_{\lambda \to \infty} \frac{\rho(\lambda)}{\lambda^{d_s/2}} = \frac{1}{\pi}$$

where $d_s = 1$ is the spectral dimension.

Definition. The *spectral zeta function* is $\zeta_\Delta(s) = \sum_{\lambda_j \neq 0} (-\lambda_j)^{-s/2}$
Its poles are the *complex spectral dimensions.*

Let $R(z)$ be a polynomial of degree N such that its Julia set $\mathfrak{J}_R \subset (-\infty, 0]$, $R(0) = 0$ and $c = R'(0) > 1$.

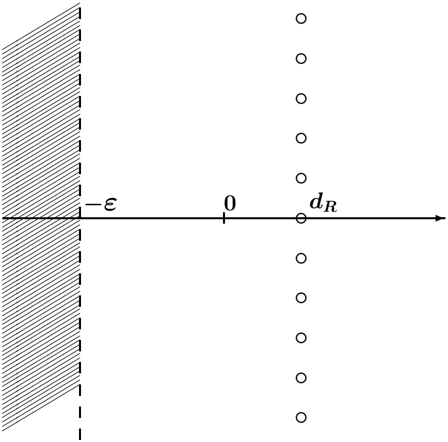
Definition. The *zeta function of* $R(z)$ for $\mathrm{Re}(s) > d_R = \frac{2\log N}{\log c}$ is

$$\zeta_R^{z_0}(s) = \lim_{n\to\infty} \sum_{z \in R^{-n}\{z_0\}} (-c^n z)^{-s/2} = \sum \lambda_j^{-s/2}$$

Theorem. $\zeta_R^{z_0}(s) = \dfrac{f_1(s)}{1-Nc^{-s/2}} + f_2^{z_0}(s)$, where $f_1(s)$ and $f_2^{z_0}(s)$ are analytic for $\mathrm{Re}(s)>0$. If $\mathfrak{J}_R$ is totally disconnected, then this meromorphic continuation extends to $\mathrm{Re}(s)>-\varepsilon$, where $\varepsilon>0$.

In the case of polynomials this theorem has been improved by Grabner et al.

$d_R \in$ the poles of $\zeta_R^{z_0} \subseteq \left\{\frac{2\log N + 4in\pi}{\log c} : n \in \mathbb{Z}\right\}$



Theorem. $\zeta_\Delta(s) = \zeta^0_R(s)$ where $R(z) = z(4z^2+12z+9)$.

The Riemann zeta function $\zeta(s)$ satisfies $\zeta(s) = \pi^s \zeta^0_R(s)$ The only complex spectral dimension is the pole at $s = 1$.

A sketch of the proof: If $z \neq -\frac{1}{2}, -\frac{3}{2}$, then

$$R(z) \in \sigma(\Delta_n) \iff z \in \sigma(\Delta_{n+1})$$

and so $\zeta_\Delta(s) = \zeta^0_R(s)$ since the eigenvalues λ_j of Δ are limits of the eigenvalues of $9^n \Delta_n$.

Also $\lambda_j = -\pi^2 j^2$ and so

$$\zeta_\Delta(s) = \sum_{j=1}^{\infty} \left(\pi^2 j^2\right)^{-s/2} = \pi^{-s}\zeta(s)$$

where $\zeta(s)$ is the Riemann zeta function. *Q.E.D.*

$$\zeta(s) = \pi^s \lim_{n\to\infty} \sum_{\substack{z\in R^{-n}\{0\} \\ z\neq 0}} \left(-9^n z\right)^{-s/2}$$

Definition. Δ_μ is μ*–Laplacian* if

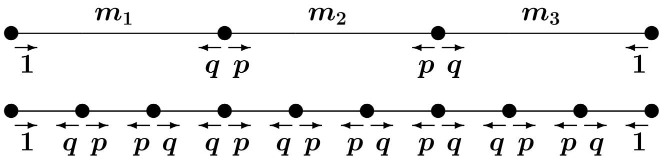
$$\mathcal{E}(f) = \int_0^1 |f'(x)|^2 dx = -\int_0^1 f \Delta_\mu f d\mu + f f'\Big|_0^1.$$

Definition. A probability measure μ is *self-similar* with weights m_1, m_2, m_3 if $\mu = \sum_{j=1,2,3} m_j \mu \circ F_j$.

Proposition. $\Delta_\mu f(x) = \frac{f''}{\mu} = \lim_{n\to\infty} \left(1+\frac{2}{pq}\right)^n \Delta_n f(x)$.

$$\Delta_n f(\tfrac{k}{3^n}) = \begin{cases} pf(\frac{k-1}{3^n}) + qf(\frac{k+1}{3^n}) - f(\frac{k}{3^n}) \\ qf(\frac{k-1}{3^n}) + pf(\frac{k+1}{3^n}) - f(\frac{k}{3^n}) \end{cases}$$

where $m_1 = m_3$, $p = \frac{m_2}{m_1+m_2}$, $q = \frac{m_1}{m_1+m_2}$, and



Spectral asymptotics: If $\rho(\lambda)$ is the eigenvalue counting function of the Dirichlet or Neumann Laplacian Δ_μ, then

$$0 < \liminf_{\lambda\to\infty} \frac{\rho(\lambda)}{\lambda^{d_s/2}} \leqslant \limsup_{\lambda\to\infty} \frac{\rho(\lambda)}{\lambda^{d_s/2}} < \infty$$

where the spectral dimension is

$$d_s = \frac{\log 9}{\log(1+\frac{2}{pq})} \leqslant 1.$$

All the inequalities are strict if and only if $p \neq q$.

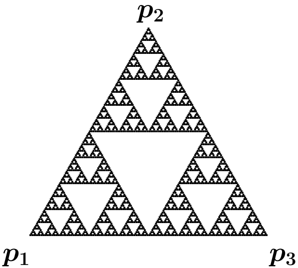
Proposition. $R(z) \in \sigma(\Delta_n) \iff z \in \sigma(\Delta_{n+1})$

where $z \neq -1 \pm p$ and $R(z) = z(z^2+3z+2+pq)/pq$.

Note that $R'(0) = 1 + \frac{2}{pq}$, and $d_s = d_R$.

Theorem. $\zeta_{\Delta_\mu}(s) = \zeta^0_R(s)$

Three **contractions** $F_1, F_2, F_3 : \mathbb{R}^2 \to \mathbb{R}^2$,
$F_j(x) = \frac{1}{2}(x+p_j)$, with fixed points p_1, p_2, p_3.



The **Sierpiński gasket** is a unique compact set S such that

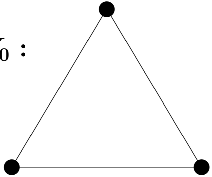
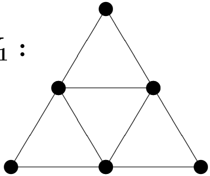
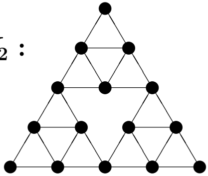
$$S = \bigcup_{j=1,2,3} F_j(S)$$

Definition. The *boundary* of S is

$$\partial S = V_0 = \{p_1, p_2, p_3\}$$

and *discrete approximations* to S are

$$V_n = \bigcup_{j=1,2,3} F_j(V_{n-1})$$

V_0 :  V_1 :  V_2 : 

Definition. The *discrete Dirichlet (energy) form* on V_n is

$$\mathcal{E}_n(f) = \sum_{\substack{x,y \in V_n \\ y \sim x}} (f(y) - f(x))^2$$

and the *Dirichlet (energy) form* on S is

$$\mathcal{E}(f) = \lim_{n \to \infty} \left(\tfrac{5}{3}\right)^n \mathcal{E}_n(f)$$

Definition. A function h is *harmonic* if it minimizes the energy given the boundary values.

Proposition. $\frac{5}{3}\mathcal{E}_{n+1}(f) \geqslant \mathcal{E}_n(f)$
$\frac{5}{3}\mathcal{E}_{n+1}(h) = \mathcal{E}_n(h) = \left(\frac{5}{3}\right)^{-n}\mathcal{E}(h)$ for a harmonic h.

Theorem (Kigami). $\mathcal{E}$ is a local regular Dirichlet form on S which is self-similar in the sense that

$$\mathcal{E}(f) = \frac{5}{3} \sum_{j=1,2,3} \mathcal{E}(f \circ F_j)$$

Definition. The *discrete Laplacians* on V_n are

$$\Delta_n f(x) = \frac{1}{4} \sum_{\substack{y \in V_n \\ y \sim x}} f(y) - f(x), \quad x \in V_n \setminus V_0$$

and the Laplacian on S is

$$\Delta_\mu f(x) = \lim_{n \to \infty} 5^n \Delta_n f(x)$$

if this limit exists and $\Delta_\mu f$ is continuous.

Gauss–Green (integration by parts) formula:

$$\mathcal{E}(f) = -\int_S f \Delta_\mu f d\mu + \sum_{p \in \partial S} f(p) \partial_n f(p)$$

where μ is the normalized Hausdorff measure, which is self-similar with weights $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$:

$$\mu = \frac{1}{3} \sum_{j = 1, 2, 3} \mu \circ F_j.$$

Spectral asymptotics: If $\rho(\lambda)$ is the eigenvalue counting function of the Dirichlet or Neumann Laplacian Δ_μ, then

$$0 < \liminf_{\lambda\to\infty} \frac{\rho(\lambda)}{\lambda^{d_s/2}} < \limsup_{\lambda\to\infty} \frac{\rho(\lambda)}{\lambda^{d_s/2}} < \infty$$

where the spectral dimension is

$$1 < d_s = \frac{\log 9}{\log 5} < 2.$$

Proposition. $R(z) \in \sigma(\Delta_n) \iff z \in \sigma(\Delta_{n+1})$ where $z \neq -\frac{1}{2}, -\frac{3}{4}, -\frac{5}{4}$ and $R(z) = z(5+4z)$.

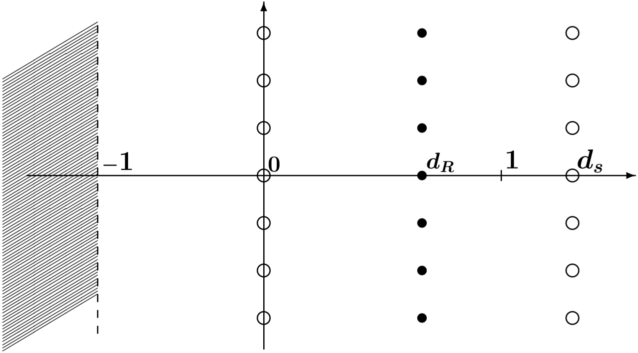
Theorem (Fukushima, Shima). Every eigenvalue of Δ_μ has a form

$$\lambda = 5^m \lim_{n\to\infty} 5^n R^{-n}(z_0)$$

where $R^{-n}(z_0)$ is a preimage of $z_0 = -\frac{3}{4}, -\frac{5}{4}$ under the n-th iteration power of the polynomial $R(z)$. The multiplicity of such an eigenvalue is $C_1 3^m + C_2$.

Theorem. Zeta function of the Laplacian on the Sierpiński gasket is

$$\zeta_{\Delta_\mu}(s) = \frac{1}{2}\zeta_R^{-\frac{3}{4}}(s)\left(\frac{1}{5^{s/2}-3}+\frac{3}{5^{s/2}-1}\right) + \frac{1}{2}\zeta_R^{-\frac{5}{4}}(s)\left(\frac{3\cdot 5^{-s/2}}{5^{s/2}-3}-\frac{5^{-s/2}}{5^{s/2}-1}\right)$$



$$d_s = \frac{\log 9}{\log 5}$$

$$d_R = \frac{\log 4}{\log 5}$$

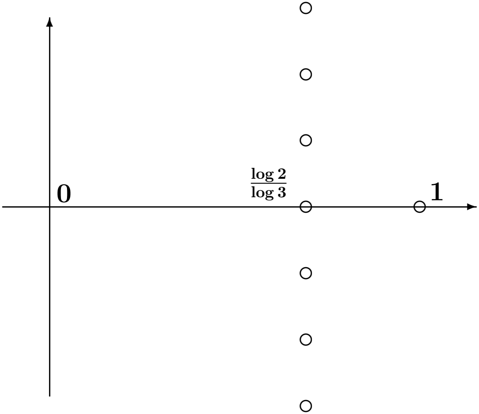
Definition. If $\mathcal{L}$ is a fractal string, that is, a disjoint collection of intervals of lengths l_j, then its *geometric zeta function* is $\zeta_{\mathcal{L}}(s) = \sum l_j^s$.

Theorem (Lapidus). If $A = -\frac{d^2}{dx^2}$ is a Neumann or Dirichlet Laplacian on $\mathcal{L}$, then $\zeta_A(s) = \pi^{-s}\zeta(s)\zeta_{\mathcal{L}}(s)$.

<u>**Example: Cantor self-similar fractal string.**</u>

If $\mathcal{L}$ is the complement of the middle third Cantor set in $[0,1]$, then the complex spectral dimensions are 1 and $\{\frac{\log 2 + 2in\pi}{\log 3} : n \in \mathbb{Z}\}$,

$$\zeta_{\mathcal{L}}(s) = \frac{1}{1-2\cdot 3^{-s}}, \quad \zeta_A(s) = \zeta(s)\frac{\pi^{-s}}{1-2\cdot 3^{-s}}$$


0
log 2
log 3
1

Definition. A post critically finite (p.c.f.) self-similar set F is a compact connected metric space with a finite boundary $\partial F \subset F$ and contractive injections $\psi_i : F \to F$ such that

$$F = \Psi(F) = \bigcup_{i=1}^{k} \psi_i(F)$$

and

$$\psi_v(F) \bigcap \psi_w(F) \subseteq \psi_v(\partial F) \bigcap \psi_w(\partial F),$$

for any two different words v and w of the same length. Here for a finite word $w \in \{1, \ldots, k\}^m$ we define $\psi_w = \psi_{w_1} \circ \ldots \circ \psi_{w_m}$.

We assume that ∂F is a minimal such subset of F. We call $\psi_w(F)$ an m-cell. ***The p.c.f. assumption is that every boundary point is contained in a single 1-cell.***

Theorem (Kigami, Lapidus). The spectral dimension of the Laplacian Δ_μ is the unique solution of the equation

$$\sum_{i=1}^{k} (r_i \mu_i)^{d_s/2} = 1$$

Conjecture. On every p.c.f. fractal F there exists a local regular Dirichlet form $\mathcal{E}$ which gives positive capacity to the boundary points and is self-similar in the sense that

$$\mathcal{E}(f) = \sum_{i=1}^{k} \rho_i \mathcal{E}(f \circ \psi_i)$$

for a set of positive refinement weights $\rho = \{\rho_i\}_{i=1}^{k}$.

Definition. The group G of acts on a finitely ramified fractal F if each $g \in G$ is a homeomorphism of F such that $g(V_n) = V_n$ for all $n \geqslant 0$.

Proposition. Suppose a group G of acts on a self-similar finitely ramified fractal F and G restricted to V_0 is the whole permutation group of V_0. Then there exists a unique, up to a constant, G-invariant self-similar resistance form $\mathcal{E}$ with equal energy renormalization weights ρ_i and

$$\mathcal{E}_0(f,f) = \sum_{x,y \in V_0} \big(f(x) - f(y)\big)^2.$$

Moreover, for any G-invariant self-similar measure μ the Laplacian Δ_μ has the spectral self-similarity property (a.k.a. spectral decimation).

end of the talk :-)

Thank you!