Spectral analysis on self-similar graphs, fractals, and groups

Alexander Teplyaev
University of Connecticut

** (part iv) **

July 11-12, 2022
CentraleSupélec
Université Paris-Saclay
A subset $\Lambda \subset \mathbb{R}^d$ is a **Delone set** if it is uniformly discrete:

$$\exists \varepsilon > 0 : |\vec{x} - \vec{y}| > \varepsilon \quad \forall \vec{x}, \vec{y} \in \Lambda$$

and relatively dense:

$$\exists R > 0 : \Lambda \cap B_R(\vec{x}) \neq \emptyset \quad \forall \vec{x} \in \mathbb{R}^d.$$ 

A Delone set has **finite local complexity** if $\forall R > 0 \exists$ finitely many clusters $P_1, \ldots, P_n_R$ such that for any $\vec{x} \in \mathbb{R}^d$ there is an $i$ such that the set $B_R(\vec{x}) \cap \Lambda$ is translation-equivalent to $P_i$.

A Delone set $\Lambda$ is **aperiodic** if $\Lambda - \vec{t} = \Lambda$ implies $\vec{t} = \vec{0}$. It is **repetitive** if for any cluster $P \subset \Lambda$ there exists $R_P > 0$ such that for any $\vec{x} \in \mathbb{R}^d$ the cluster $B_{R_P}(\vec{x}) \cap \Lambda$ contains a cluster which is translation-equivalent to $P$.

These sets have applications in crystallography ($\approx 1920$), coding theory, approximation algorithms, and the theory of quasicrystals.
Aperiodic tilings were discovered by mathematicians in the early 1960s, and, some twenty years later, they were found to apply to the study of natural quasicrystals (1982 Dan Shechtman, 2011 Nobel Prize in Chemistry).
Penrose tiling
pattern space of a Delone set

Let $\Lambda_0 \subset \mathbb{R}^d$ be a Delone set. The pattern space (hull) of $\Lambda_0$ is the closure of the set of translates of $\Lambda_0$ with respect to the metric $\varrho$, i.e.

$$\Omega_{\Lambda_0} = \overline{\{ \varphi_{\vec{t}}(\Lambda_0) : \vec{t} \in \mathbb{R}^d \}}.$$

**Definition**

Let $\Lambda_0 \subset \mathbb{R}^d$ be a Delone set and denote by $\varphi_{\vec{t}}(\Lambda_0) = \Lambda_0 - \vec{t}$ its translation by the vector $\vec{t} \in \mathbb{R}^d$. For any two translates $\Lambda_1$ and $\Lambda_2$ of $\Lambda_0$ define $\varrho(\Lambda_1, \Lambda_2) = \inf\{ \varepsilon > 0 : \exists \vec{s}, \vec{t} \in B_\varepsilon(\vec{0}) : B_{1/\varepsilon}(\vec{0}) \cap \varphi_{\vec{s}}(\Lambda_1) = B_{1/\varepsilon}(\vec{0}) \cap \varphi_{\vec{t}}(\Lambda_2) \} \wedge 2^{-1/2}$

**Assumption**

The action of $\mathbb{R}^d$ on $\Omega$ is uniquely ergodic: $\Omega$ is a compact metric space with the unique $\mathbb{R}^d$-invariant probability measure $\mu$. 
Topological solenoids
(similar topological features as the pattern space $\Omega$):

(i) If $\tilde{W} = (\tilde{W}_t)_{t \geq 0}$ is the standard Gaussian Brownian motion on $\mathbb{R}^d$, then for any $\Lambda \in \Omega$ the process $X_t^\Lambda := \varphi_{\tilde{W}_t}(\Lambda) = \Lambda - \tilde{W}_t$ is a conservative Feller diffusion on $(\Omega, \varrho)$.

(ii) The semigroup $P_t f(\Lambda) = \mathbb{E}[f(X_t^\Lambda)]$ is self-adjoint on $L^2_\mu$, Feller but not strong Feller.

Its associated Dirichlet form is regular, strongly local, irreducible, recurrent, and has Kusuoka-Hino dimension $d$.

(iii) The semigroup $(P_t)_{t > 0}$ does not admit heat kernels with respect to $\mu$. It does have Gaussian heat kernel with respect to the not-$\sigma$-finite (no Radon-Nykodim theorem) pushforward measure $\lambda_\Omega^d$

\begin{equation}
p_\Omega(t, \Lambda_1, \Lambda_2) = \begin{cases} 
p_{\mathbb{R}^d}(t, h_{\Lambda_1}^{-1}(\Lambda_2)) & \text{if } \Lambda_2 \in \text{orb}(\Lambda_1), \\
0 & \text{otherwise.} \end{cases} \end{equation}

(iv) There are no semi-bounded or $L^1$ harmonic functions ("Liouville-type")
no classical inequalities

Useful versions of the Poincare, Nash, Sobolev, Harnack inequalities DO NOT HOLD, except in orbit-wise sense.

The unitary Koopman operators $U_t$ on $L^2(\Omega, \mu)$ defined by $U_t f = f \circ \varphi_t$ commute with the heat semigroup

$$U_t P_t = P_t U_t$$

hence commute with the Laplacian $\Delta$, and all spectral operators, such as the unitary Schrödinger semigroup.

... hence we may have continuous spectrum (no eigenvalues) under some assumptions even though $\mu$ is a probability measure on the compact set $\Omega$.

Under special conditions $P_t$ is connected to the evolution of a Phason: “Phason is a quasiparticle existing in quasicrystals due to their specific, quasiperiodic lattice structure. Similar to phonon, phason is associated with atomic motion. However, whereas phonons are related to translation of atoms, phasons are associated with atomic rearrangements. As a result of these rearrangements, waves, describing the position of atoms in crystal, change phase, thus the term “phason” (from the wikipedia)”.
Phason evolution


The unitary Koopman operators $U_{\vec{t}}$ on $L^2(\Omega, \mu)$ defined by $U_{\vec{t}}f = f \circ \varphi_{\vec{t}}$
commute with the heat semigroup

$$U_{\vec{t}}P_t = P_t U_{\vec{t}}$$

hence commute with the Laplacian $\Delta$, and all spectral operators, including the unitary Schrödinger semigroup $e^{i\Delta t}$

$$U_{\vec{t}} e^{i\Delta t} = e^{i\Delta t} U_{\vec{t}}$$

Recent physics work on phason (“accounts for the freedom to choose the origin”): Topological Properties of Quasiperiodic Tilings (Yaroslav Don, Dor Gitelman, Eli Levy and Eric Akkermans Technion Department of Physics)
https://phsites.technion.ac.il/eric/talks/
We thank C. Schochet for useful discussions.

Consider a Thue-Morse. We link these two phases, thus establishing a "Bloch theorem" for specific types of quasiperiodic tilings.

**Substitution Rules and 1D Tilings**

Define a substitution rule by:

\[\sigma(\mathbf{a}) = \mathbf{a} + \mathbf{a} \quad \sigma(\mathbf{b}) = \mathbf{a} \]

Associate occurrence matrix:

\[M = \begin{pmatrix} 1 & 1 \end{pmatrix}\]

Consider only primitive matrices:

\[\alpha = \mathbf{a} \quad \beta = \mathbf{a} + \mathbf{b}\]

Define atomic density:

\[\rho(\mathbf{a}) = \sum \delta(\mathbf{a} - \mathbf{n})\]

Useful Tools

In periodic structures, topological numbers are described by Chern numbers. This does not happen in quasiperiodic tilings, since there exists no notion of a Brillouin zone that alternative tools exist to describe topological properties of quasiperiodic tilings. We now enumerate some of them.

- Tilting space \( T (\mathbb{R} \times \mathbb{R}) \) and its hull \( \mathbb{R}^T \)
- Cohomology \( H^2(\mathbb{R}^T; \mathbb{Z}) \)
- Simplicial cohomology \( H^2(\mathbb{R}; \mathbb{Z}) \)
- Bratteli graphs \([11, 12]\).
- K-theory, \(K(G)\) group and the abstract gap labeling theorem \([4, 13]\).
- The Bloch theorem described above can be given an interpretation for 1D CBP tilings (for an irrational slope \( \alpha \in \mathbb{Q} \)) by means of the "commutative diagram".

**Spectral Properties of Tilings**

The gaps in the integrated density of states are given by the gap labeling theorem \([4, 13]\).

\[N_\alpha = \rho_\alpha + \rho_{\alpha^*} \quad (\text{mod} 1) \quad \alpha, \alpha^* \in \mathbb{Z} \]

The gaps in the integrated density of states are given by the gap labeling theorem \([4, 13]\).

\[N_\alpha = \rho_\alpha + \rho_{\alpha^*} \quad (\text{mod} 1) \quad \alpha, \alpha^* \in \mathbb{Z} \]

Here we used the Fibonacci sequence \((s = 1/1)\) with \(d_1 = 233\) sites.

**Conclusion**

We have defined two types of phases—a structural and spectral one—whose windings unveil topological features of quasiperiodic tilings. We have found a relation between these two phases, which can be interpreted as a Bloch-like theorem. We have considered here a subset of tilings, which are known as Sturmian (CBP) words. Our results can be extended to a broader families of tilings in one dimension, and to tilings in higher dimensions \(D > 1\).

**References**

[8] The Bloch theorem described above can be given an interpretation for 1D CBP tilings (for an irrational slope \( \alpha \in \mathbb{Q} \)) by means of the "commutative diagram".

Here we used the Fibonacci sequence \((s = 1/1)\) with \(d_1 = 233\) sites.
Helmholtz, Hodge and de Rham

**Theorem**

Assume $d = 1$. Then the space $L^2(\Omega, \mu, \mathbb{R}^1)$ admits the orthogonal decomposition

$$L^2(\Omega, \mu, \mathbb{R}^1) = \text{Im} \nabla \oplus \mathbb{R}(dx).$$

In other words, the $L^2$-cohomology is 1-dimensional, which is surprising because the **de Rham cohomology is not one dimensional**.
