

# Spectral analysis on self-similar graphs, fractals, and groups

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University of Connecticut



July 11-12, 2022  
CentraleSupélec  
Université Paris-Saclay

# Abstract

The lectures will describe how fractal research appears naturally in many areas of pure and applied mathematics and other sciences and engineering. Fractal shape optimization allows for better wave absorption properties, which is a joint work with Anna Rozanova-Pierrat (CentraleSupélec) and Michael Hinz (Bielefeld). Moreover, spectral theory, geometry of graphs, and dynamical systems are used to analyze spectral properties of the random walk generator on finitely ramified self-similar graphs, fractals, and some self-similar groups. In particular, pure point or singular continuous spectrum appears naturally in such settings. The standard examples include the Sierpinski triangle, the Vicsek tree, and the Schreier graphs of the Hanoi and basilica self-similar groups studied by Grigorchuk, Zuk, Nekrashevych, Bartholdi, Lyubich, Dang et al. We will discuss spectral dimensions, self-similar random walks and their diffusion limits, and the role of symmetries and finite ramification in computing the spectrum explicitly.

# Plan

- Monday, July 11, 2022
- Tuesday, July 12, 2022

(1) Introduction: intrinsic and extrinsic fractals

(2) Wave propagation and absorption models with a Robin boundary condition in domains with a non-Lipschitz boundary.

*A long-term joint research project with*

- ▶ Anna Rozanova-Pierrat (CentraleSupélec)
- ▶ Michael Hinz (Bielefeld)

(3) Intrinsic spectral analysis on fractals and self-similar groups

(4) Aperiodic order

## early history of fractals

- in nature, **fractals are everywhere**
- in mathematics and logic, the first mention of fractals is Zeno's Paradox "Achilles and the tortoise" (c. 490–430 BC)

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$$

$$C = \frac{1}{3} C + \frac{1}{3} C + \frac{2}{3} C$$



- ▶ this is a weakly self-similar set, a zero-dimensional fractal

- the Cantor set: Henry John Stephen Smith (1874) and Georg Cantor (1883)

$$0 < \textit{dimension} = \frac{\log 2}{\log 3} < 1$$

- the **Koch snowflake** (1906)
- Gaston Julia (1893 – 1978), Pierre Fatou (1878 – 1929) and Benoit Mandelbrot (1924 – 2010)

**ANALYSE MATHÉMATIQUE.** — *Sur une courbe dont tout point est un point de ramification.* Note <sup>(1)</sup> de **M. W. SIERPINSKI**, présentée par M. Émile Picard.

Le but de cette Note est de donner un exemple d'une courbe cantorienne et jordanienne en même temps, dont tout point est un point de ramification. (Nous appelons *point de ramification* d'une courbe  $\mathfrak{e}$  un point  $p$  de cette courbe, s'il existe trois continus, sous-ensembles de  $\mathfrak{e}$ , ayant deux à deux le point  $p$  et seulement ce point commun.)

Soient  $T$  un triangle régulier donné;  $A, B, C$  respectivement ses sommets : gauche, supérieur et droit. En joignant les milieux des côtés du triangle  $T$ , nous obtenons quatre nouveaux triangles réguliers (*fig. 1*), dont trois,  $T_0, T_1, T_2$ , contenant respectivement les sommets  $A, B, C$ , sont situés parallèlement à  $T$  et le quatrième triangle  $U$  contient le centre du triangle  $T$ ; nous excluons tout l'intérieur du triangle  $U$ .

Les sommets des triangles  $T_0, T_1, T_2$  nous les désignerons respectivement :

---

(<sup>1</sup>) Séance du 1<sup>er</sup> février 1915.

$$S = U \frac{1}{3} (P_i + S)$$

triangles  $U_0, U_1, U_2$ , situés parallèlement à  $U$ , dont les intérieurs seront

Fig. 1.

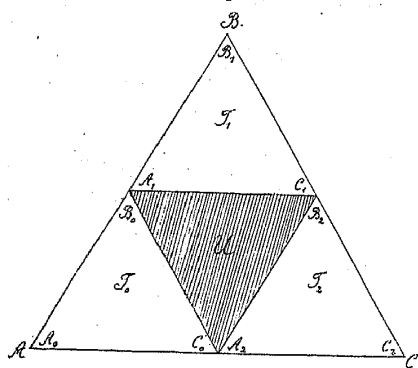
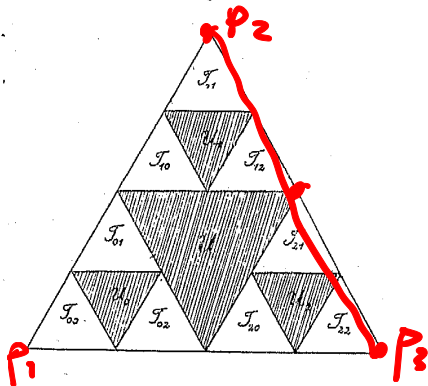


Fig. 2.



exclus (fig. 2). Avec chacun des triangles  $T_{\lambda, \lambda}$  procédons de même et ainsi

Fig. 3.

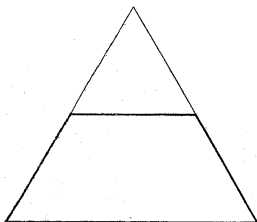
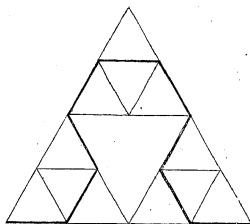


Fig. 4.



d'eux se rencontrent quatre segments différents, situés entièrement sur l'ensemble  $\mathcal{C}$ .

Donc, tous les points de la courbe  $\mathcal{C}$ , sauf peut-être les points A, B, C, sont ses points de ramification.

Pour obtenir une courbe dont tous les points sans exception sont ses

Fig. 5.

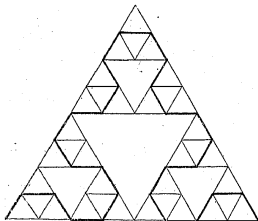
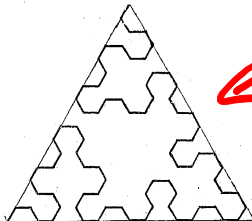


Fig. 6.



[0, 1]

points de ramification, il suffit de diviser un hexagone régulier en six triangles équilatéraux et dans chacun d'eux inscrire une courbe  $\mathcal{C}$ .

# Newton's law of universal gravitation (April 1686)

$$F = G \frac{m_1 m_2}{r^2}$$

Newton's laws of motion

$$F = ma$$

This is a **space-time** relation.

...

Maxwell's equations (1861) lead to the Einstein's Theory of Relativity (1905)

$$E = mc^2$$

Q: quantize ?

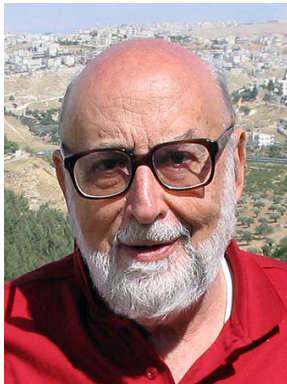


# François Englert

From Wikipedia, the free encyclopedia

**François Baron Englert** (French: [ɑ̃ɡlɛʁ]; born 6 November 1932) is a Belgian theoretical physicist and 2013 Nobel prize laureate (shared with Peter Higgs). He is Professor emeritus at the Université libre de Bruxelles (ULB) where he is member of the Service de Physique Théorique. He is also a Sackler Professor by Special Appointment in the School of Physics and Astronomy at Tel Aviv University and a member of the Institute for Quantum Studies at Chapman University in California. He was awarded the 2010 J. J. Sakurai Prize for Theoretical Particle Physics (with Gerry Guralnik, C. R. Hagen, Tom Kibble, Peter Higgs, and Robert Brout), the Wolf Prize in Physics in 2004 (with Brout and Higgs) and the High Energy and Particle Prize of the European Physical Society (with Brout and Higgs) in 1997 for the mechanism which unifies short and long range interactions by generating massive gauge vector bosons. He has made contributions in statistical physics, quantum field theory, cosmology, string theory and supergravity.<sup>[4]</sup> He is the recipient of the 2013 Prince of Asturias Award in technical and scientific research, together with Peter Higgs and the CERN

**François Englert**



François Englert in Israel, 2007

**METRIC SPACE-TIME AS FIXED POINT  
OF THE RENORMALIZATION GROUP EQUATIONS  
ON FRACTAL STRUCTURES**

**F. ENGLERT, J.-M. FRÈRE<sup>1</sup> and M. ROOMAN<sup>2</sup>**

*Physique Théorique, C.P. 225, Université Libre de Bruxelles, 1050 Brussels, Belgium*

Ph. SPINDEL

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Received 19 February 1986

We take a model of foamy space-time structure described by self-similar fractals. We study the propagation of a scalar field on such a background and we show that for almost any initial conditions the renormalization group equations lead to an effective highly symmetric metric at large scale.

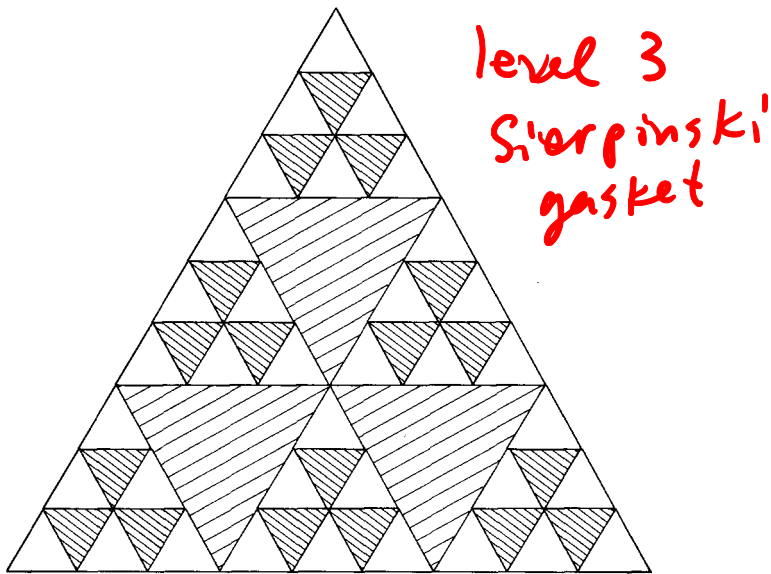


Fig. 1. The first two iterations of a 2-dimensional 3-fractal.

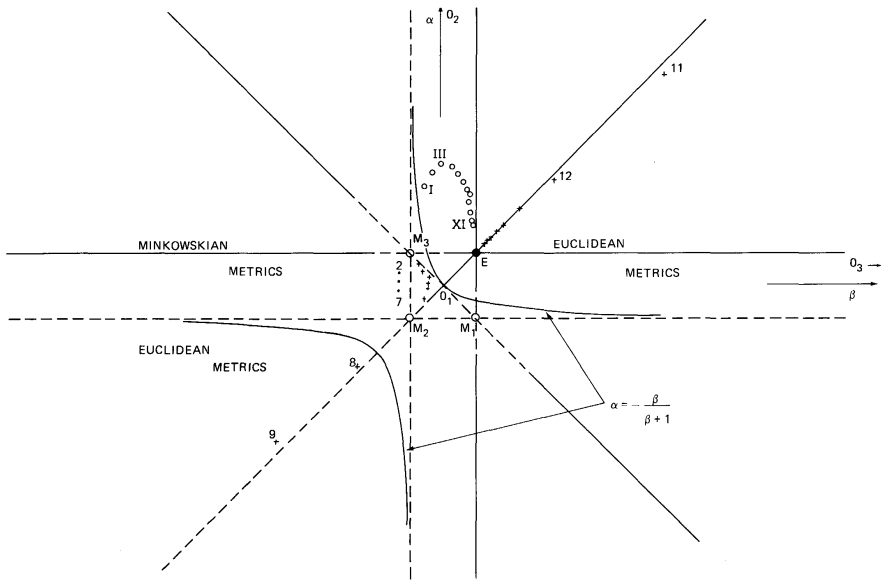


Fig. 5. The plane of 2-parameter homogeneous metrics on the Sierpinski gasket. The hyperbole  $\alpha = -\beta/(\beta + 1)$  separates the domain of euclidean metrics from minkowskian metrics and corresponds - except at the origin - to 1-dimensional metrics.  $M_1, M_2, M_3$  denote unstable minkowskian fixed geometries while  $E$  corresponds to the stable euclidean fixed point. The unstable fixed points  $O_1, O_2$  and  $O_3$  associated to 0-dimensional geometries are located at the origin and at infinity on the  $(\alpha, \beta)$  coordinates axis. The six straight lines are subsets invariant with respect to the recursion relation but repulsive in the region where they are dashed. The first points of two sequences of iterations are drawn. Note that for one of them the 10th point ( $\alpha = -56.4, \beta = -52.5$ ) is outside the frame of the figure.

self-affine  
fractal

harmonic  
coordinates  
on the  
Sierpinsky  
gasket

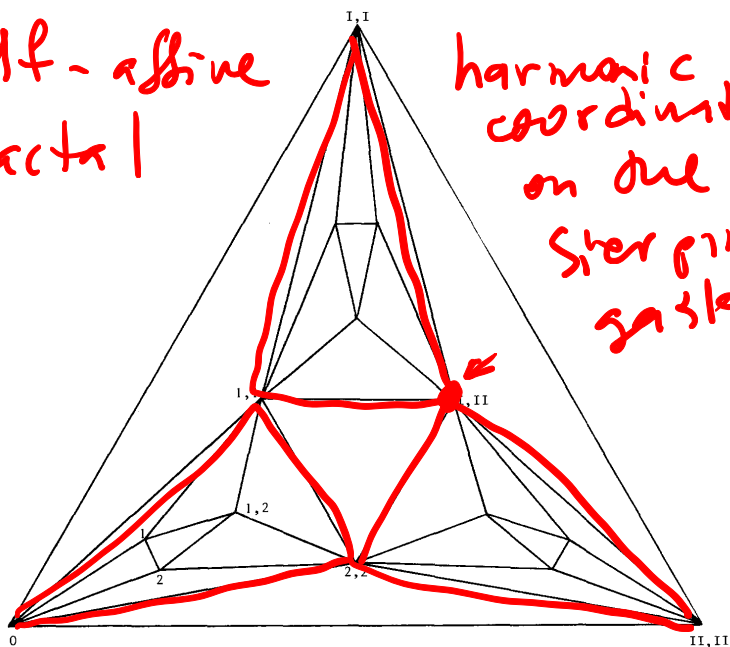
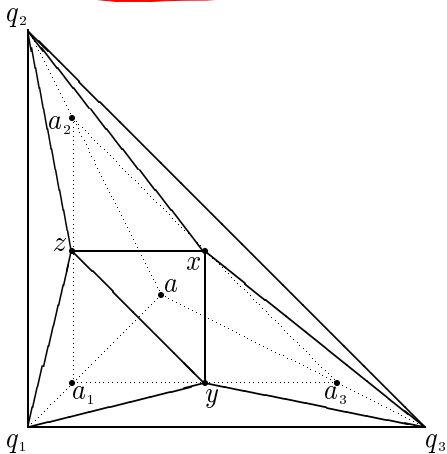


Fig. 10. A metrical representation of the two first iterations of a 2-dimensional 2-fractal corresponding to the euclidean fixed point. Vertices are labelled according to fig. 4.



**Figure 6.4.** Geometric interpretation of Proposition 6.1.

1786

## The Spectral Dimension of the Universe is Scale Dependent

J. Ambjørn,<sup>1,3,\*</sup> J. Jurkiewicz,<sup>2,†</sup> and R. Loll<sup>3,‡</sup>

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<sup>2</sup>Mark Kac Complex Systems Research Centre, Marian Smoluchowski Institute of Physics, Jagellonian University, Reymonta 4, PL 30-059 Krakow, Poland

<sup>3</sup>Institute for Theoretical Physics, Utrecht University, Leuvenlaan 4, NL-3584 CE Utrecht, The Netherlands

(Received 13 May 2005; published 20 October 2005)

We measure the spectral dimension of universes emerging from nonperturbative quantum gravity, defined through state sums of causal triangulated geometries. While four dimensional on large scales, the quantum universe appears two dimensional at short distances. We conclude that quantum gravity may be “self-renormalizing” at the Planck scale, by virtue of a mechanism of dynamical dimensional reduction.

DOI: 10.1103/PhysRevLett.95.171301

PACS numbers: 04.60.Gw, 04.60.Nc, 98.80.Qc

*Quantum gravity as an ultraviolet regulator?*—A shared hope of researchers in otherwise disparate approaches to quantum gravity is that the microstructure of space and time may provide a physical regulator for the ultraviolet infinities encountered in nonperturbative quantum field theory.

*tral dimension*, a diffeomorphism-invariant quantity obtained from studying diffusion on the quantum ensemble of geometries. On large scales and within measuring accuracy, it is equal to four, in agreement with earlier measurements of the large-scale dimensionality based on the

other hand, the “short-distance spectral dimension,” obtained by extrapolating Eq. (12) to  $\sigma \rightarrow 0$  is given by

$$D_S(\sigma = 0) = 1.80 \pm 0.25, \quad (15)$$

and thus is compatible with the integer value two.

## Random Geometry and Quantum Gravity

A thematic semestre at Institut Henri Poincaré

14 April, 2020 - 10 July, 2020

Organizers : John BARRETT, Nicolas CURIEN, Razvan GURAU,  
Renate LOLL, Gregory MIERMONT, Adrian TANASA

↑  
Physics

↑  
Math



## Fractal space-times under the microscope: a renormalization group view on Monte Carlo data

Martin Reuter and Frank Saueressig

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**ABSTRACT:** The emergence of fractal features in the microscopic structure of space-time is a common theme in many approaches to quantum gravity. In this work we carry out a detailed renormalization group study of the spectral dimension  $d_s$  and walk dimension  $d_w$  associated with the effective space-times of asymptotically safe Quantum Einstein Gravity (QEG). We discover three scaling regimes where these generalized dimensions are approximately constant for an extended range of length scales: a classical regime where  $d_s = d$ ,  $d_w = 2$ , a semi-classical regime where  $d_s = 2d/(2+d)$ ,  $d_w = 2+d$ , and the UV-fixed point regime where  $d_s = d/2$ ,  $d_w = 4$ . On the length scales covered by three-dimensional Monte Carlo simulations, the resulting spectral dimension is shown to be in very good agreement with the data. This comparison also provides a natural explanation for the apparent puzzle between the short distance behavior of the spectral dimension reported from Causal Dynamical Triangulations (CDT), Euclidean Dynamical Triangulations (EDT), and Asymptotic Safety.

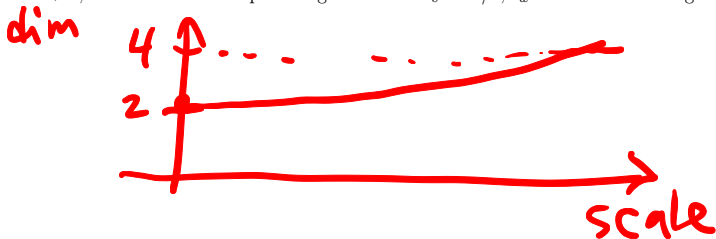
**KEYWORDS:** Models of Quantum Gravity, Renormalization Group, Lattice Models of Gravity, Nonperturbative Effects

JHEP12(2011)012

# Fractal space-times under the microscope: A Renormalization Group view on Monte Carlo data

Martin Reuter and Frank Saueressig

a classical regime where  $d_s = d, d_w = 2$ , a semi-classical regime where  $d_s = 2d/(2+d)$ ,  $d_w = 2+d$ , and the UV-fixed point regime where  $d_s = d/2, d_w = 4$ . On the length scales covered



# Toy model: Hanoi towers game

W Tours de Hanoï — Wikipédia

https://fr.wikipedia.org/wiki/Tours\_de\_Hanoï

Non connecté Discussion Contributions Créer un compte Se connecter

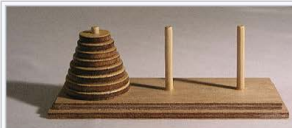
Article Discussion Lire Modifier Plus

Rechercher dans Wikipédia

**Tours de Hanoï**

*☞ Pour les articles homonymes, voir [Hanoï \(homonymie\)](#).*

**Les tours de Hanoï** (originellement, la **tour d'Hanoï**<sup>a</sup>) sont un **jeu de réflexion** imaginé par le **mathématicien** français **Édouard Lucas**, et consistant à déplacer des disques de diamètres différents d'une tour de « départ » à une tour d'« arrivée » en passant par une tour « intermédiaire » ,



Modèle d'une tour de Hanoï (avec huit disques).

Accueil  
Portails thématiques  
Article au hasard  
Contact  
Contribuer  
Débuter sur Wikipédia  
Aide  
Communauté  
Modifications récentes

The puzzle was invented by the French mathematician Édouard Lucas in 1883.

# Asymptotic aspects of Schreier graphs and Hanoi Towers groups

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Received 23 January, 2006; accepted after revision +++++

Presented by Étienne Ghys

## Abstract

We present relations between growth, growth of diameters and the rate of vanishing of the spectral gap in Schreier graphs of automaton groups. In particular, we introduce a series of examples, called Hanoi Towers groups since they model the well known Hanoi Towers Problem, that illustrate some of the possible types of behavior. *To cite this article: R. Grigorchuk, Z. Šuník, C. R. Acad. Sci. Paris, Ser. I 344 (2006)*

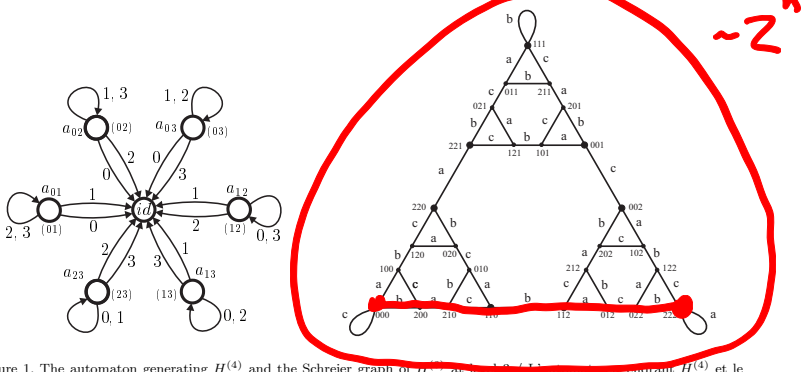
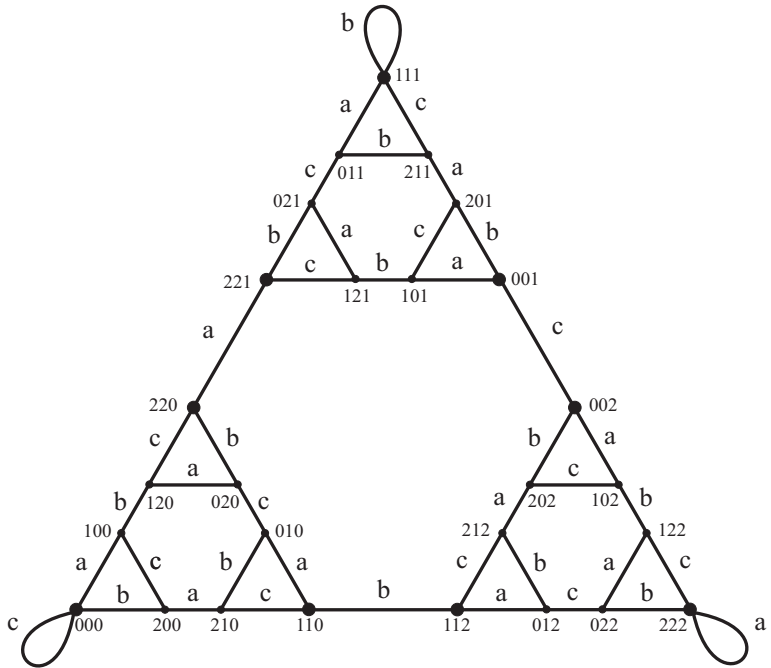


Figure 1. The automaton generating  $H^{(4)}$  and the Schreier graph of  $H^{(3)}$  at level 3 / L'automate engendrant  $H^{(4)}$  et le graphe de Schreier de  $H^{(3)}$  au niveau 3



1986

- R. Rammal and G. Toulouse, *Random walks on fractal structures and percolation clusters*. J. Physique Letters **44** (1983)
- R. Rammal, *Spectrum of harmonic excitations on fractals*. J. Physique **45** (1984)
- E. Domany, S. Alexander, D. Bensimon and L. Kadanoff, *Solutions to the Schrödinger equation on some fractal lattices*. Phys. Rev. B (3) **28** (1984)
- Y. Gefen, A. Aharony and B. B. Mandelbrot, *Phase transitions on fractals. I. Quasilinear lattices. II. Sierpiński gaskets. III. Infinitely ramified lattices*. J. Phys. A **16** (1983)**17** (1984)

# Main early mathematical results

Sheldon Goldstein, *Random walks and diffusions on fractals*. Percolation theory and ergodic theory of infinite particle systems (Minneapolis, Minn., 1984–1985), IMA Vol. Math. Appl., 8, Springer

Summary: we investigate the asymptotic motion of a random walker, which at time  $n$  is at  $\mathbf{X}(n)$ , on certain 'fractal lattices'. For the 'Sierpiński lattice' in dimension  $d$  we show that, as  $L \rightarrow \infty$ , the process  $\mathbf{Y}_L(t) \equiv \mathbf{X}([(d+3)^L t])/2^L$  converges in distribution to a diffusion on the Sierpiński gasket, a Cantor set of Lebesgue measure zero. The analysis is based on a simple 'renormalization group' type argument, involving self-similarity and 'decimation invariance'. In particular,

$$d = 2$$

$$|\mathbf{X}(n)| \sim n^\gamma,$$

where  $\gamma = (\ln 2) / \ln(d+3) \leq 2$ .



Shigeo Kusuoka, *A diffusion process on a fractal*. Probabilistic methods in mathematical physics (Katata/Kyoto, 1985), 1987.

$$\gamma = \frac{\ln 2}{\ln 5}$$

$$\dim = \frac{\log 3}{\log 2}$$

# Main classes of fractals considered

- **[0, 1]**
- Sierpiński gasket
- nested fractals
- p.c.f. self-similar sets, possibly with various symmetries
- finitely ramified self-similar sets, possibly with various symmetries
- infinitely ramified self-similar sets, with local symmetries, and with heat kernel estimates (such as the Generalized Sierpiński carpets)
- metric measure Dirichlet spaces, possibly with heat kernel estimates (MMD+HKE)



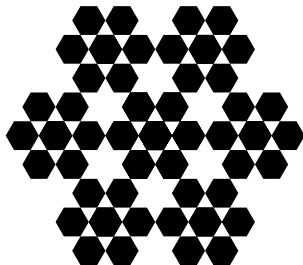
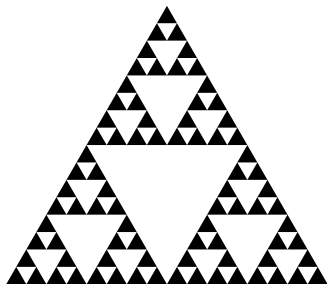


Figure: Sierpiński gasket and Lindstrøm snowflake (nested fractals), p.c.f., finitely ramified)

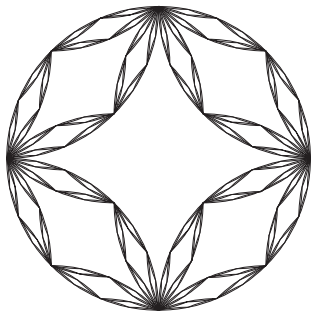


Figure: Diamond fractals, non-p.c.f., but finitely ramified

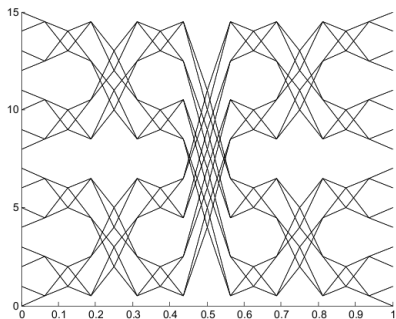


Figure: Laakso Spaces (Ben Steinhurst), infinitely ramified

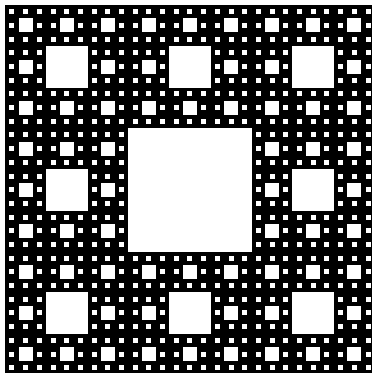


Figure: Sierpiński carpet, infinitely ramified

# Existence, uniqueness, heat kernel estimates: geometric renormalization for $F$ -invariant Dirichlet forms

## Brownian motion:

Thiele (1880) Bachelier (1900)

Einstein (1905) Smoluchowski (1906)

Wiener (1920'), Doob, Feller, Levy Kolmogorov (1930'),

Doebelin, Dynkin, Hunt, Ito ...

← Poincaré

Lebesgue  
Cantor

$$\text{distance} \sim \sqrt{\text{time}}$$

"Einstein space-time relation for Brownian motion"

Wiener process in  $\mathbb{R}^n$  satisfies  $\frac{1}{n} \mathbb{E} |W_t|^2 = t$  and has a  
Gaussian transition density:

$$p_t(x, y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x - y|^2}{4t}\right)$$

- De Giorgi-Nash-Moser estimates for elliptic and parabolic PDEs;
- Li-Yau (1986) type estimates on a geodesically complete Riemannian manifold with  $\text{Ricci} \geq 0$ :

$$p_t(x, y) \sim \frac{1}{V(x, \sqrt{t})} \exp\left(-c \frac{d(x, y)^2}{t}\right)$$

distance  $\sim \sqrt{\text{time}}$

Gaussian:

$$p_t(x, y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x - y|^2}{4t}\right)$$

$n = \dim$

Li-Yau Gaussian-type:

$$p_t(x, y) \sim \frac{1}{V(x, \sqrt{t})} \exp\left(-c \frac{d(x, y)^2}{t}\right)$$

Sub-Gaussian:

$$p_t(x, y) \sim \frac{1}{t^{d_H/d_w}} \exp\left(-c \left(\frac{d(x, y)^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right)$$

$\text{distance} \sim (\text{time})^{\frac{1}{d_w}}$

fractals  $d_H = \text{Hausdorff}$

Brownian motion on  $\mathbb{R}^d$ :  $\mathbb{E}|\mathbf{X}_t - \mathbf{X}_0| = ct^{1/2}$ .

Anomalous diffusion:  $\mathbb{E}|\mathbf{X}_t - \mathbf{X}_0| = o(t^{1/2})$ , or (in regular enough situations),

$$\mathbb{E}|\mathbf{X}_t - \mathbf{X}_0| \approx t^{1/d_w}$$

with  $d_w > 2$ .

Here  $d_w$  is the so-called **walk dimension** (should be called “**walk index**” perhaps).

This phenomena was first observed by mathematical physicists working in the transport properties of disordered media, such as (critical) percolation clusters.



$$p_t(x, y) \sim \frac{1}{t^{d_H/d_w}} \exp\left(-c \frac{d(x, y)^{\frac{d_w}{d_w-1}}}{t^{\frac{1}{d_w-1}}}\right)$$

$$\text{distance} \sim (\text{time})^{\frac{1}{d_w}}$$

1983

$d_H$  = Hausdorff dimension

$\frac{1}{\gamma} = d_w$  = "walk dimension" ( $\gamma$ =diffusion index)

$\frac{2d_H}{d_w} = d_S$  = "spectral dimension" (diffusion dimension)

First example: Sierpiński gasket; Kusuoka, Fukushima, Kigami, Barlow, Bass, Perkins (mid 1980'—)

$d_w = 2 \iff$  Gaussian  
 $d_w > 2$  connected fractals  
 $d_w < 2 \longrightarrow$  jumps

# Causal dynamical triangulation

video: Dynamical triangulation of the 2-torus

video: Causal dynamical triangulations

Barycentric subdivision

A. Brzoska, D.J. Kelleher, H. Panzo, A. Teplyaev, Dual graphs and modified Barlow–Bass resistance estimates for repeated barycentric subdivisions

# Intrinsic one-dimensional fractal

U.Andrews, J.P.Chen, G.Bonik, R.W.Martin, A.Teplyaev,  
Wave equation on one-dimensional fractals with spectral decimation.  
J. Fourier Anal. Appl. 23 (2017) 994–1027

wave equations animation

# Wave propagation and absorption models with a Robin boundary condition in domains with a non-Lipschitz boundary

Joint research with Anna ROZANOVA-PIERRAT (CentraleSupélec)

and

A. Dekkers (CS), M. Hinz (Bielefeld) et al.