BV and harmonic functions on fractals

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Fractals in Pure and Applied Sciences

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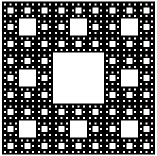
"Sapienza" Università di Roma



Plan of the talk:

Introduction: analysis on “**fractafolds**”*

- Physics motivation
- Group Theory and Complex Dynamics motivation
- Heat Kernel Estimates and Dirichlet Forms
- Spectral analysis
- Elements of differential geometry

BV spaces on fractals with Dirichlet forms

- Patricia Alonso-Ruiz, Fabrice Baudoin, Li Chen, Luke Rogers, Nages Shanmugalingam, T.
- Definitions of $\boldsymbol{BV}$ and $\boldsymbol{wBE(\kappa)}$

- *Strichartz: *A fractafold, a space that is locally modeled on a specified fractal, is the fractal equivalent of a manifold.*
    - *A “fractafold” is to a fractal what a manifold is to a Euclidean half-space.*

Let K be a fractal [F]. Then a *fractafold* $\mathcal{F}$ based on K is a connected Hausdorff topological space such that every point x in $\mathcal{F}$ has a neighborhood homeomorphic to a neighborhood in K. There is no generally agreed upon definition of "fractal", other than "I know one when I see one", but there are several well-defined classes of fractals, such as Kigami's *p.c.f. (post-critically finite) self-similar fractals* [Ki1]. We are interested in this class of fractals because one can do analysis on them: under certain additional hypotheses, one can construct a Laplacian Δ on K and study properties of the spectrum of Δ. (Of course it should be emphasized here that the Laplacian is not uniquely determined by the topology of K, but rather depends on certain additional geometric structures, just as the Laplacian on a manifold depends on the choice of a Riemannian metric.) One of the purposes of introducing fractafolds in this context is that we may easily extend the Laplacian from K to $\mathcal{F}$, and thereby obtain a larger class of objects on which to do analysis ([B], [Ki1], [Ki2], [S2]).

Physics motivation (Intro 1)

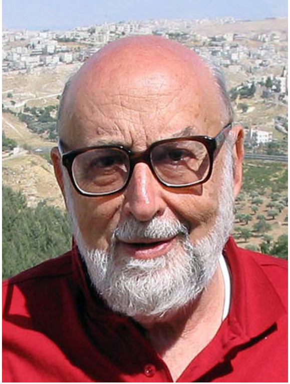
- R. Rammal and G. Toulouse, *Random walks on fractal structures and percolation clusters.* J. Physique Letters **44** (1983)
- R. Rammal, *Spectrum of harmonic excitations on fractals.* J. Physique **45** (1984)
- E. Domany, S. Alexander, D. Bensimon and L. Kadanoff, *Solutions to the Schrödinger equation on some fractal lattices.* Phys. Rev. B (3) **28** (1984)
- Y. Gefen, A. Aharony and B. B. Mandelbrot, *Phase transitions on fractals. I. Quasilinear lattices. II. Sierpiński gaskets. III. Infinitely ramified lattices.* J. Phys. A **16** (1983)**17** (1984)

François Englert

From Wikipedia, the free encyclopedia

François Baron Englert (French: [ɑ̃glɛʁ]; born 6 November 1932) is a Belgian theoretical physicist and 2013 Nobel prize laureate (shared with Peter Higgs). He is Professor emeritus at the Université libre de Bruxelles (ULB) where he is member of the Service de Physique Théorique. He is also a Sackler Professor by Special Appointment in the School of Physics and Astronomy at Tel Aviv University and a member of the Institute for Quantum Studies at Chapman University in California. He was awarded the 2010 J. J. Sakurai Prize for Theoretical Particle Physics (with Gerry Guralnik, C. R. Hagen, Tom Kibble, Peter Higgs, and Robert Brout), the Wolf Prize in Physics in 2004 (with Brout and Higgs) and the High Energy and Particle Prize of the European Physical Society (with Brout and Higgs) in 1997 for the mechanism which unifies short and long range interactions by generating massive gauge vector bosons. He has made contributions in statistical physics, quantum field theory, cosmology, string theory and supergravity.[4] He is the recipient of the 2013 Prince of Asturias Award in technical and scientific research,

François Englert



François Englert in Israel, 2007

Nuclear Physics B280 [FS 18] (1987) 147–180
North-Holland, Amsterdam

METRIC SPACE-TIME AS FIXED POINT OF THE RENORMALIZATION GROUP EQUATIONS ON FRACTAL STRUCTURES

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Ph. SPINDEL

Faculté des Sciences, Université de l'Etat à Mons, 7000 Mons, Belgium

Received 19 February 1986

We take a model of foamy space-time structure described by self-similar fractals. We study the propagation of a scalar field on such a background and we show that for almost any initial conditions the renormalization group equations lead to an effective highly symmetric metric at large scale.

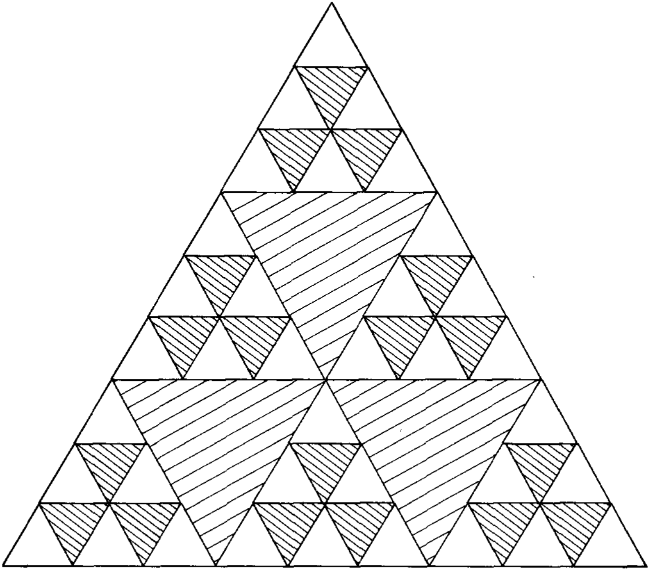

Fig. 1. The first two iterations of a 2-dimensional 3-fractal.

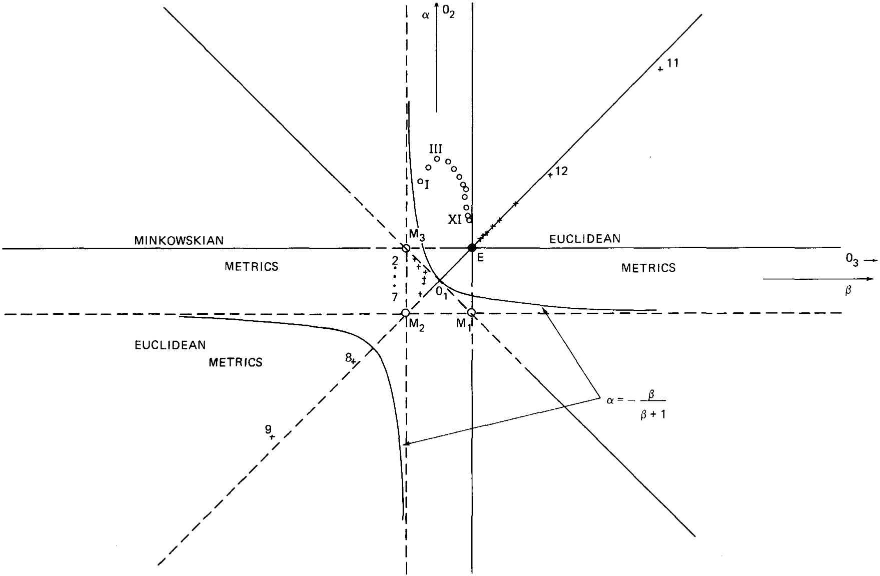

Fig. 5. The plane of 2-parameter homogeneous metrics on the Sierpinski gasket. The hyperbole $\alpha = -\beta/(\beta+1)$ separates the domain of euclidean metrics from minkowskian metrics and corresponds – except at the origin – to 1-dimensional metrics. M_1, M_2, M_3 denote unstable minkowskian fixed geometries while E corresponds to the stable euclidean fixed point. The unstable fixed points 0_1, 0_2 and 0_3 associated to 0-dimensional geometries are located at the origin and at infinity on the (α, β) coordinates axis. The six straight lines are subsets invariant with respect to the recursion relation but repulsive in the region where they are dashed. The first points of two sequences of iterations are drawn. Note that for one of them the 10th point ($\alpha = -56.4$, $\beta = -52.5$) is outside the frame of the figure.

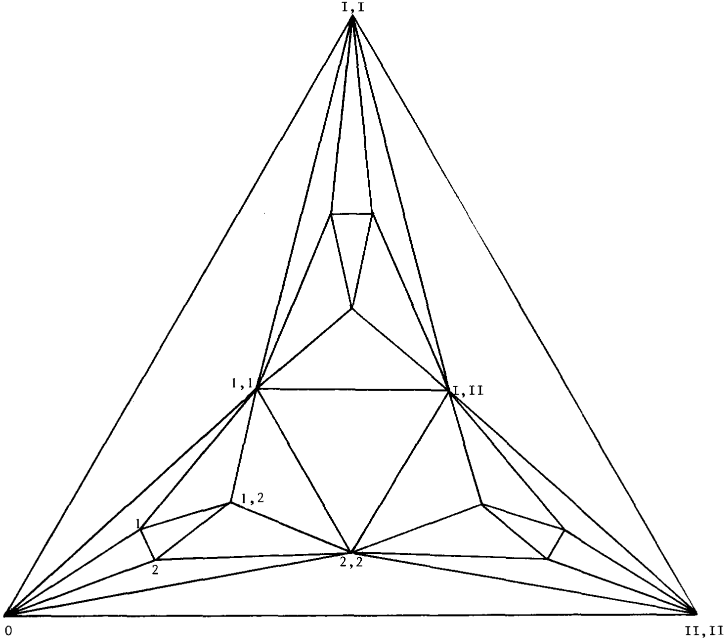

Fig. 10. A metrical representation of the two first iterations of a 2-dimensional 2-fractal corresponding to the euclidean fixed point. Vertices are labelled according to fig. 4.

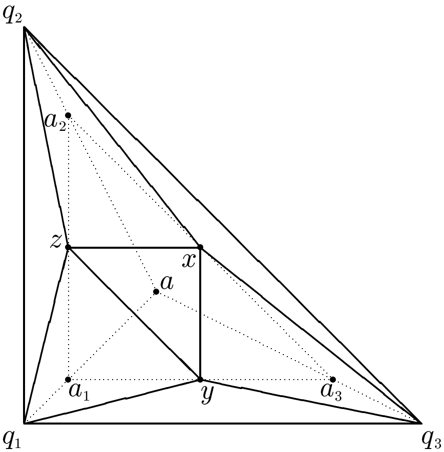

Figure 6.4. Geometric interpretation of Proposition 6.1.

PRL **95,** 171301 (2005) PHYSICAL REVIEW LETTERS week ending 21 OCTOBER 2005

The Spectral Dimension of the Universe is Scale Dependent

J. Ambjørn,[1,3,*] J. Jurkiewicz,[2,†] and R. Loll[3,‡]

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[2]*Mark Kac Complex Systems Research Centre, Marian Smoluchowski Institute of Physics, Jagellonian University, Reymonta 4, PL 30-059 Krakow, Poland*
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(Received 13 May 2005; published 20 October 2005)

We measure the spectral dimension of universes emerging from nonperturbative quantum gravity, defined through state sums of causal triangulated geometries. While four dimensional on large scales, the quantum universe appears two dimensional at short distances. We conclude that quantum gravity may be "self-renormalizing" at the Planck scale, by virtue of a mechanism of dynamical dimensional reduction.

DOI: 10.1103/PhysRevLett.95.171301 PACS numbers: 04.60.Gw, 04.60.Nc, 98.80.Qc

Quantum gravity as an ultraviolet regulator?—A shared hope of researchers in otherwise disparate approaches to quantum gravity is that the microstructure of space and time may provide a physical regulator for the ultraviolet infinities encountered in perturbative quantum field theory.

tral dimension, a diffeomorphism-invariant quantity obtained from studying diffusion on the quantum ensemble of geometries. On large scales and within measuring accuracy, it is equal to four, in agreement with earlier measurements of the large-scale dimensionality based on the other hand, the "short-distance spectral dimension," obtained by extrapolating Eq. (12) to $\sigma \to 0$ is given by

$$D_S(\sigma = 0) = 1.80 \pm 0.25, \qquad (15)$$

and thus is compatible with the integer value two.

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Fractal space-times under the microscope: a renormalization group view on Monte Carlo data

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Abstract: The emergence of fractal features in the microscopic structure of space-time is a common theme in many approaches to quantum gravity. In this work we carry out a detailed renormalization group study of the spectral dimension d_s and walk dimension d_w associated with the effective space-times of asymptotically safe Quantum Einstein Gravity (QEG). We discover three scaling regimes where these generalized dimensions are approximately constant for an extended range of length scales: a classical regime where $d_s = d, d_w = 2$, a semi-classical regime where $d_s = 2d/(2+d), d_w = 2+d$, and the UV-fixed point regime where $d_s = d/2, d_w = 4$. On the length scales covered by three-dimensional Monte Carlo simulations, the resulting spectral dimension is shown to be in very good agreement with the data. This comparison also provides a natural explanation for the apparent puzzle between the short distance behavior of the spectral dimension reported from Causal Dynamical Triangulations (CDT), Euclidean Dynamical Triangulations (EDT), and Asymptotic Safety.

Keywords: Models of Quantum Gravity, Renormalization Group, Lattice Models of Gravity, Nonperturbative Effects

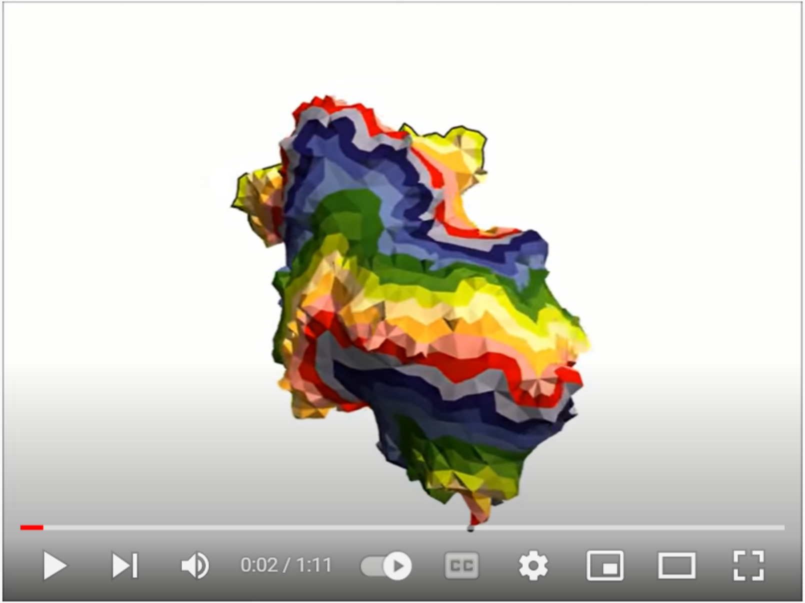
Fractal space-times under the microscope: A Renormalization Group view on Monte Carlo data (Martin Reuter, Frank Saueressig):

Three scaling regimes of the effective space-times of asymptotically safe Quantum Einstein Gravity (QEG):

1. a classical regime $d_s = d, d_w = 2$,
2. a semi-classical regime $d_s = 2d/(2+d),\ d_w = 2+d$,
3. the UV-fixed point regime $d_s = d/2,\ d_w = 4$.

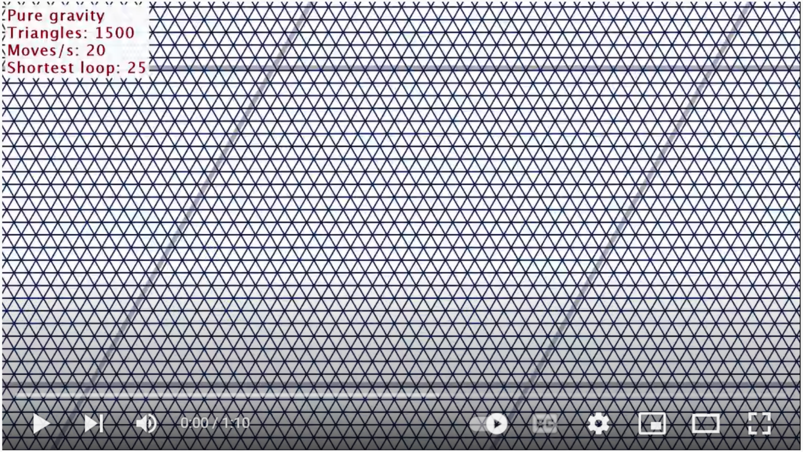
On the length scales covered by three-dimensional Monte Carlo simulations, the resulting spectral dimension is in very good agreement with the data and provides a natural explanation for the apparent puzzle between the short distance behavior of the spectral dimension reported from Causal Dynamical Triangulations (CDT), Euclidean Dynamical Triangulations (EDT), and Asymptotic Safety.

- Mathav Murugan: $d_w = d_f$ consistent with $d_s = 2d_f/d_w = 2$
- Growth and percolation on the uniform infinite planar triangulation by Omer Angel (GAFA 2003)
- Anomalous diffusion of random walk on random planar maps by Ewain Gwynne and Tom Hutchcroft (PTRF 2020)



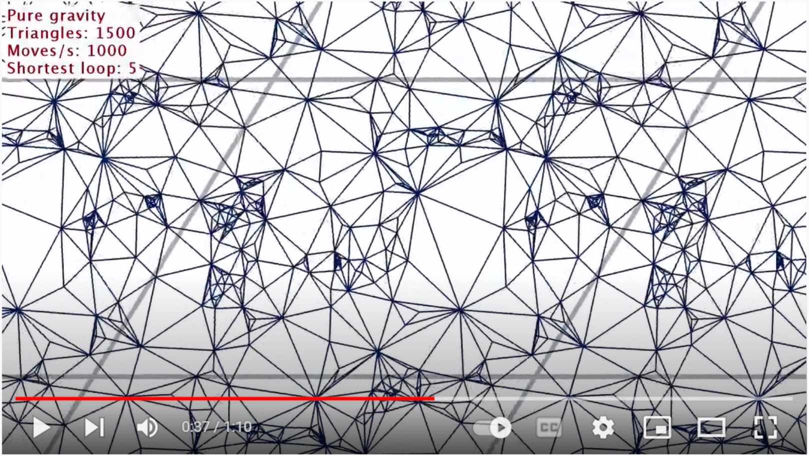
Causal dynamical triangulations

25,971 views Jan 26, 2013 Causal dynamical triangulation (CDT) is a lattice model



Dynamical triangulation of the 2-torus

1,435 views **Sep 7, 2013** This video illustrates a Monte Carlo simulation for two-dimensional quantum gravity on a torus. Starting with a regular triangulation of the torus repeatedly a so-called flip move is performed on a randomly chosen edge. The triangulations obtained after a large



Dynamical triangulation of the 2-torus

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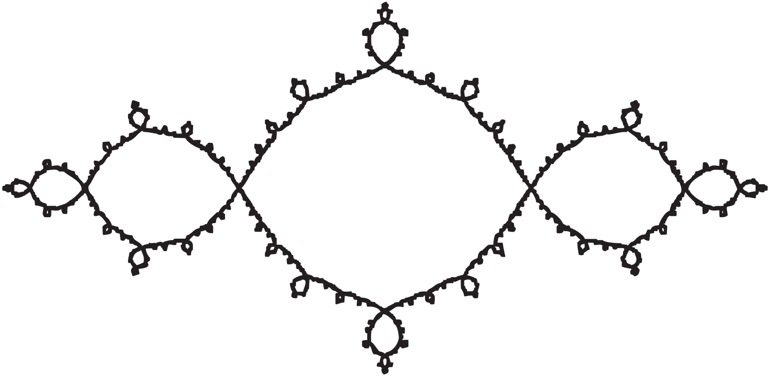
Group Theory and Complex Dynamics (Intro 2)



The basilica Julia set, the Julia set of $z^2 - 1$ and the limit set of the basilica group of exponential growth (Grigorchuk, Żuk, Bartholdi, Virág, Nekrashevych, Kaimanovich, Nagnibeda et al.).

Asymptotic aspects of Schreier graphs and Hanoi Towers groups

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Department of Mathematics, Texas A&M University, MS-3368, College Station, TX, 77843-3368, USA

Received 23 January, 2006; accepted after revision +++++

Presented by Étienne Ghys

Abstract

We present relations between growth, growth of diameters and the rate of vanishing of the spectral gap in Schreier graphs of automaton groups. In particular, we introduce a series of examples, called Hanoi Towers groups since they model the well known Hanoi Towers Problem, that illustrate some of the possible types of behavior. *To cite this article: R. Grigorchuk, Z. Šunić, C. R. Acad. Sci. Paris, Ser. I 344 (2006).*

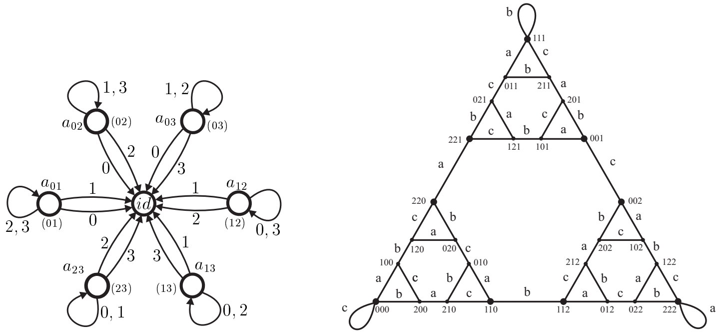

Figure 1. The automaton generating $H^{(4)}$ and the Schreier graph of $H^{(3)}$ at level 3 / L'automate engendrant $H^{(4)}$ et le graphe de Schreier de $H^{(3)}$ au niveau 3

Tours de Hanoï — Wikipédia

https://fr.wikipedia.org/wiki/Tours_de_Hanoï

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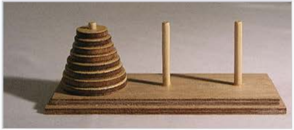
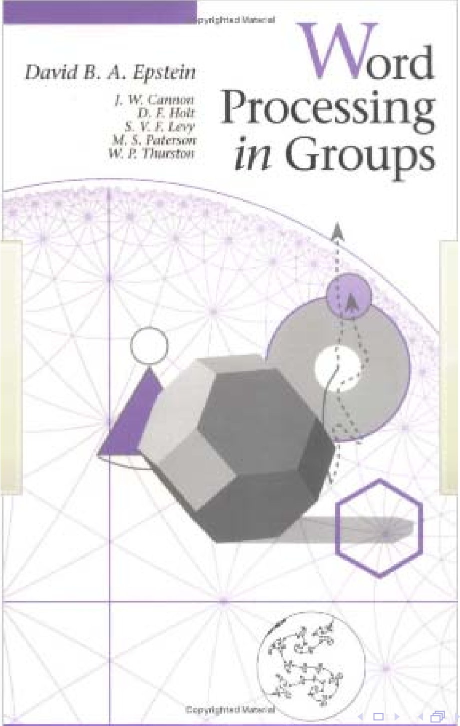
Tours de Hanoï

Pour les articles homonymes, voir Hanoï (homonymie).

Les tours de Hanoï (originellement, **la tour d'Hanoï**[a]) sont un jeu de réflexion imaginé par le mathématicien français Édouard Lucas, et consistant à déplacer des disques de diamètres différents d'une tour de « départ » à une tour d'« arrivée » en passant par une tour « intermédiaire »,



Modèle d'une tour de Hanoï (avec huit disques).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Heat Kernel Estimates and Dirichlet Forms (Intro 3)

$$p_t(x,y) \sim \frac{1}{t^{d_f/d_w}} \exp\left(-c\frac{d(x,y)^{\frac{d_w}{d_w-1}}}{t^{\frac{1}{d_w-1}}}\right)$$

$$distance \sim (time)^{\frac{1}{d_w}}$$

$$d_f = \text{Hausdorff dimension}$$

$$\frac{1}{\gamma} = d_w = \text{“walk dimension” } (\gamma\text{=diffusion index})$$

$$\frac{2d_f}{d_w} = d_s = \text{“spectral dimension” (diffusion dimension)}$$

First example: Sierpiński gasket; Kusuoka, Fukushima, Kigami, Barlow, Bass, Perkins (mid 1980'—)

Stability Theorem (Barlow, Bass, Kumagai (2006))

Under natural assumptions on the MMD (geodesic Metric Measure space with a regular symmetric conservative Dirichlet form), the sub-Gaussian ***heat kernel estimates are stable under rough isometries****, i.e. under maps that preserve distance and energy up to scalar factors.*

Gromov-Hausdorff + energy

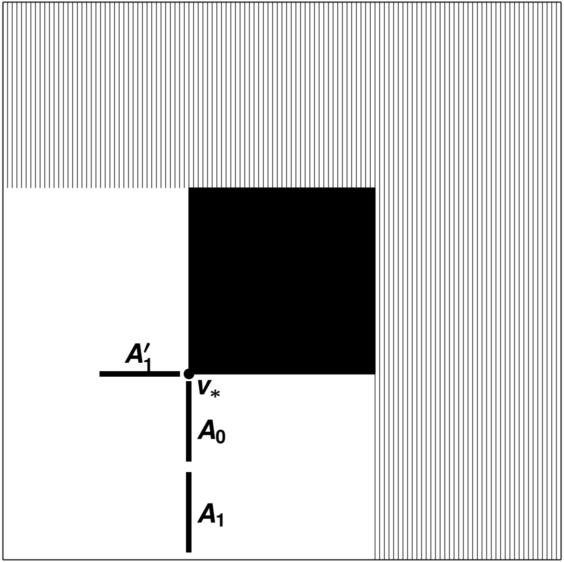
Theorem. (Barlow, Bass, Kumagai, T. (1989–2010).) On any generalized Sierpiński carpet there exists a unique, up to a scalar multiple, local regular Dirichlet form that is invariant under the local isometries.

Therefore there is a unique symmetric Markov process and **a unique Laplacian**.

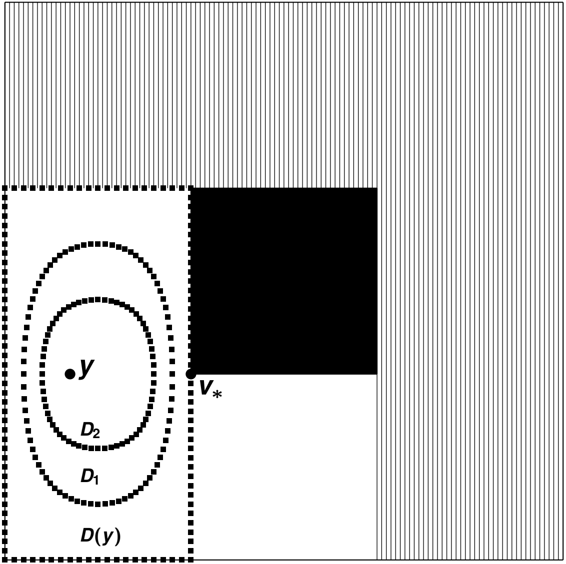
Moreover, the Markov process is strong Feller and its transition density satisfies sub-Gaussian heat kernel estimates.

Main difficulties: If it is not a cube in $\mathbb{R}^n$, then

- $d_S < d_f$, $d_w > 2$
- the energy measure and the Hausdorff measure are mutually singular;
- the domain of the Laplacian is not an algebra;
- if $d(x, y)$ is the shortest path metric, then $d(x, \cdot)$ is not in the domain of the Dirichlet form (not of finite energy) and so methods of Differential geometry seem to be not applicable;
- Lipschitz functions are not of finite energy;
- in fact, we can not compute any functions of finite energy;
- Fourier and complex analysis methods seem to be not applicable.



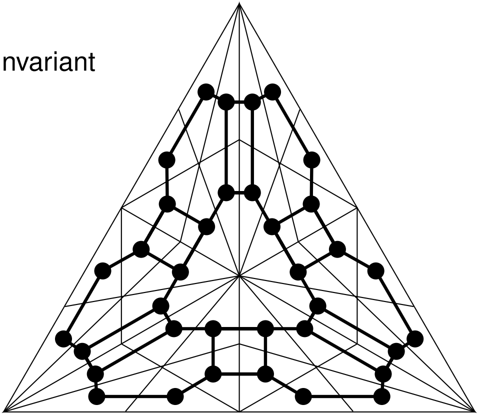
The half-face $\boldsymbol{A_1}$ corresponds to a “slide move”,
and the half-face $\boldsymbol{A_1'}$ corresponds to a “corner move”,
analogues of the “corner” and “knight's” moves in [BB89].


y
v_*
D_2
D_1
$D(y)$

However, it seems the uniqueness can fail in some natural settings, such as repeated barycentric subdivisions.

Theorem (Kelleher, Panzo, Brzoska, T.). The dual triangular and edge graphs have reciprocal resistance scaling factors $\rho = 1/\rho^T$ with $5/4 < \rho < 3/2$.

Conjecture. The reflection-invariant Dirichlet form is not unique.



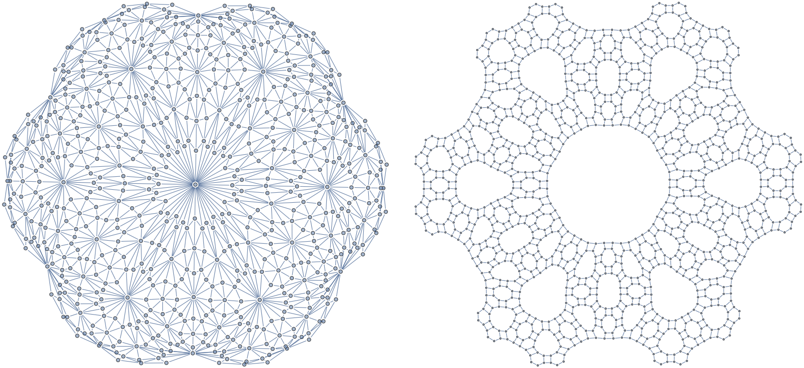


FIGURE 3. On the left: the graph G_4^T for barycentric subdivision of a 2-simplex. On the right: the adjacency (dual) graph G_4.

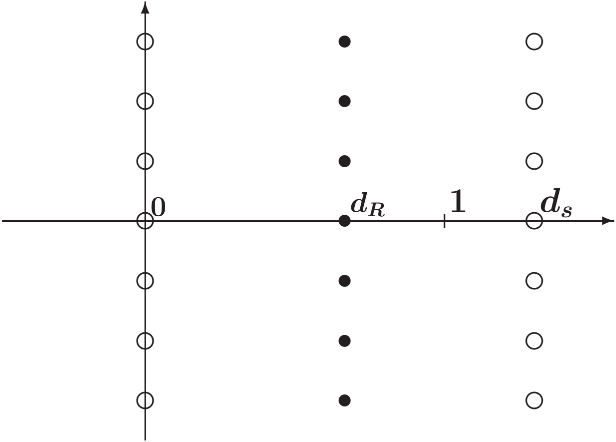
Spectral analysis (Intro 4)

Theorem. (Derfel, Grabner, Vogl; T.; Kajino (2007–2011)) For a large class of **finitely ramified symmetric fractals**, which includes the Sierpiński gaskets, and may include the Sierpiński carpets, the spectral zeta function

$$\zeta(s) = \sum \lambda_j^{s/2}$$

has a meromorphic continuation from the half-pain $Re(s) > d_S$ to $\mathbb{C}$. Moreover, all the poles and residues are computable from the geometric data of the fractal. Here λ_j are the eigenvalues if the unique symmetric Laplacian.

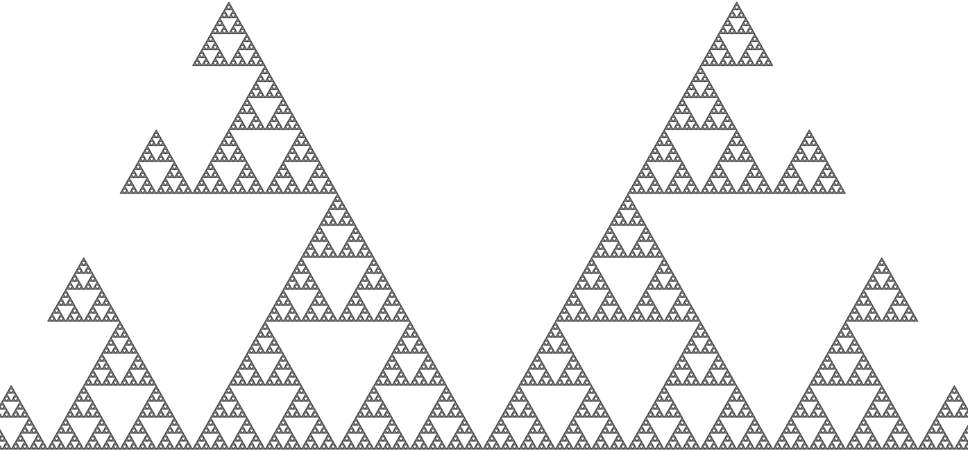
- Example: $\zeta(s)$ is the Riemann zeta function up to a trivial factor in the case when our fractal is $[0, 1]$.
- In more complicated situations, such as the Sierpiński gasket, there are infinitely many non-real poles, which can be called complex spectral dimensions, and are related to oscillations in the spectrum.



$$d_s = \frac{\log 9}{\log 5}$$

$$d_R = \frac{\log 4}{\log 5}$$

Poles (white circles) of the spectral zeta function of the Sierpiński gasket.



A part of an infinite Sierpiński gasket.

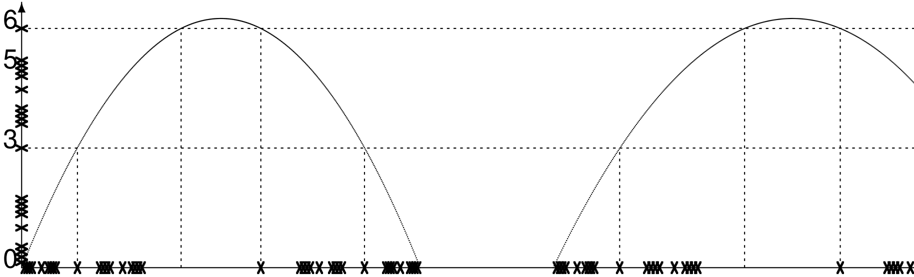
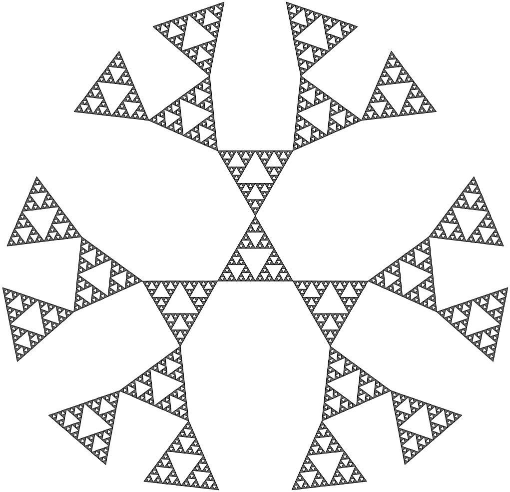
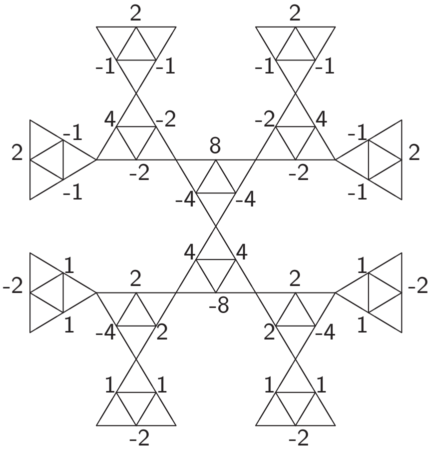


Figure: An illustration to the computation of the spectrum on the infinite Sierpiński gasket. The curved lines show the graph of the function $\mathfrak{R}(\cdot)$.

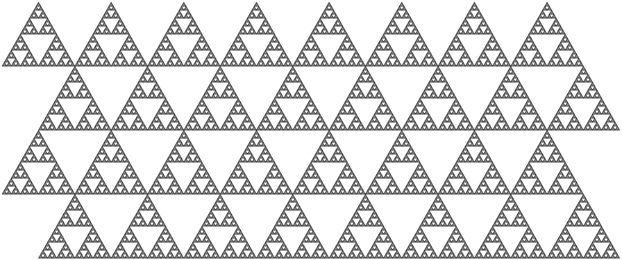
Theorem. (T. 1998, Quint 2009) On the Barlow-Perkins infinite Sierpiński fractafold the spectrum of the Laplacian consists of a **dense set of eigenvalues** $\mathfrak{R}^{-1}(\Sigma_0)$ **of infinite multiplicity** and a **singularly continuous component of spectral multiplicity one supported on** $\mathfrak{R}^{-1}(\mathcal{J}_R)$.

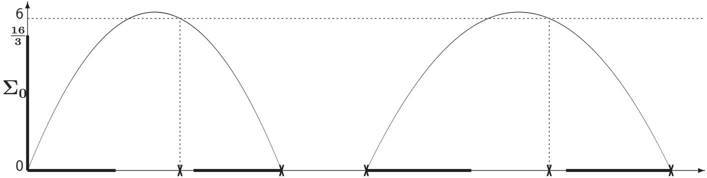


The Tree Fractafold.



An eigenfunction on the Tree Fractafold.



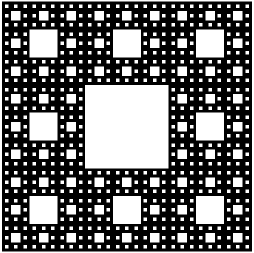


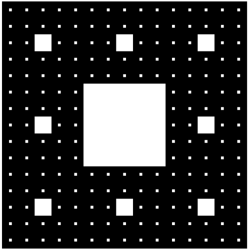
Theorem. (Strichartz, T. 2010) The Laplacian on the periodic triangular lattice finitely ramified Sierpiński fractal field consists of absolutely continuous spectrum and pure point spectrum. The **absolutely continuous spectrum is $\mathfrak{R}^{-1}[0, \frac{16}{3}]$.** The **pure point spectrum** consists of two infinite series of eigenvalues of infinite multiplicity. The spectral resolution is given in the main theorem.

Elements of differerential geometry (Intro 5)

- J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces,* Geom. Funct. Anal. (1999)
- J. Heinonen, *Lectures on analysis on metric spaces.* Universitext 2001. *Nonsmooth calculus,* Bull. Amer. Math. Soc. (2007)
- J. Heinonen, P. Koskela, N. Shanmugalingam, J. Tyson, *Sobolev classes of Banach space-valued functions and quasiconformal mappings*. J. Anal. Math. 85 (2001)
- M. Bonk, L. Capogna, and X. Zhou, Green functions in metric measure spaces, Preprint, November 2022

Can we define harmonic differential forms, a de Rham complex, and obtain a version of de Rham's theorem?





Standard self-similar carpet $\boldsymbol{S_a}$ with $\mathbf{a} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$

Non self-similar carpet $\boldsymbol{S_a}$ with $\mathbf{a} = (\frac{1}{3}, \frac{1}{5}, \frac{1}{7} \ldots)$

Proposition

(Mackay/Tyson/Wildrick'13) If $\mathbf{a} \in l^2$ *then* $\boldsymbol{S_a}$ *has positive two dim Lebesgue measure and "all classical-type Sobolev inequalities".*

Theorem (Hinz/T. '15)

1-dim Hodge-Helmholtz composition holds (despite that $\dim_H = 2$*).*

Theorem (Hinz/T. '17)

If $\mathbf{a} \in l^2$, $\lim_{n\to\infty} \frac{a_1 \cdots a_{n-1}}{a_n} = 0$ *then* $\mathrm{dom}(\mathrm{curl}^*) = \{0\}$ *and* (curl, C^1) *is not closable.*

BV and Besov spaces on fractals with Dirichlet forms (Patricia Alonso-Ruiz, Fabrice Baudoin, Li Chen, Luke Rogers, Nages Shanmugalingam, T.)

References:

Besov class via heat semigroup on Dirichlet spaces

I: Sobolev type inequalities
arXiv:1811.04267 *J. Funct. Analysis (2020)*

II: BV functions and Gaussian heat kernel estimates
arXiv:1811.11010 *Calc. Var. PDE (2020)*

III: BV functions and sub-Gaussian heat kernel estimates
arXiv:1903.10078 *Calc. Var. PDE (2021)*

BV functions and fractional Laplacians on Dirichlet spaces
arXiv:1910.13330 *revised December 2022*

+ recent papers by Alonso-Ruiz, Fabrice Baudoin, Li Chen

sub-Gaussian Heat Kernel Estimates (sGHKE)

$$p_t(x,y) \sim \frac{1}{t^{d_f/d_w}} \exp\left(-c \frac{d(x,y)^{\frac{d_w}{d_w-1}}}{t^{\frac{1}{d_w-1}}}\right)$$

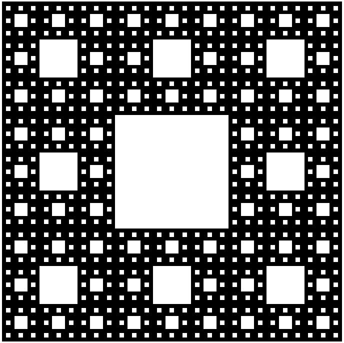
$$\textit{distance} \sim (\textit{time})^{\frac{1}{d_w}}$$

$$d_f = \text{Hausdorff dimension}$$

$$\frac{1}{\gamma} = d_w = \text{“walk dimension” } (\gamma\text{=diffusion index})$$

$$\frac{2d_f}{d_w} = d_S = \text{“spectral dimension” (diffusion dimension)}$$

First example: Sierpiński gasket; Kusuoka, Fukushima, Kigami, Barlow, Bass, Perkins (mid 1980'—)



$$1 = d_t = d_{mart} < d_{tH} = \frac{\ln 2}{\ln 3} + 1 < d_S < d_f = \frac{\ln 8}{\ln 3} < 2 < d_w$$

For Sierpinski carpets there exists a unique Dirichlet form and diffusion process due to [Barlow and Bass 1998, 1999] (see also [B-B-Kumagai-T 2010])

Open questions:

On the Sierpinski carpet,

$$\kappa = d_W - d_f + d_{tH} - 1 = d_W - d_f + \frac{\log 2}{\log 3}$$

would give the best Hölder exponent for harmonic functions?
[*Strongly supported by numerical results: L.Rogers et al*]

Note that $(d_W - d_f)$ –Hölder continuity is classical:
Martin Barlow. Diffusions on fractals. In Lectures on probability theory and statistics (Saint-Flour, 1995), volume 1690 of Lecture Notes in Math., pages 1–121. Springer, Berlin, 1998.
Martin Barlow. Heat kernels and sets with fractal structure. In Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002), volume 338 of Contemp. Math., pages 11–40. Amer. Math. Soc., Providence, RI, 2003

Here $d_{tH} = \frac{\ln 2}{\ln 3} + 1$ is the ***topological-Hausdorff dimension*** of the Sierpinski carpet defined in Theorem 5.4 in:

[**R.Balka, Z.Buczolich, M.Elekes.** ***A new fractal dimension: the topological Hausdorff dimension***. Adv. Math. 2015.]

Roughly speaking,

$$d_{tH} :=$$

$$1 + \inf\{\textit{Hausdorff dim. of boundaries of a base of open sets}\}$$

Barlow (Proceedings of SMS Montreal, 2011):

Given a regular fractal F, since L and M are given by the construction, one can calculate d_f easily. The constant ρ which gives d_w is somehow deeper, and seems to require some analysis on the set or its approximations. Loosely one can say that d_f is a 'geometric' constant, while d_w is an 'analytic' constant. One may guess that in some sense ρ or β are in general inaccessible by any purely geometric argument. (An exception is for trees, where one has $d_w = 1 + d_f$.)

BV and weak Bakry-Émery non-negative curvature

Definition

$BV(X) := KS^{\lambda_1^\#,1}(X) = \mathbf{B}^{1,\alpha_1^\#}(X)$ with $\alpha_1^\# = \frac{\lambda_1^\#}{d_W}$ the L^1–Besov critical exponent, and for $f \in BV(X)$

$$\mathrm{Var}(f) := \liminf_{r \to 0^+} \iint_{\Delta_r} \frac{|f(y) - f(x)|}{r^{\lambda_1^\#} \mu(B(x,r))} \, d\mu(y) \, d\mu(x).$$

Definition

We say that $(X, \mu, \mathcal{E}, \mathcal{F})$ satisfies the weak-Bakry-Émery non-negative curvature condition $wBE(\kappa)$ if there exist a constant $C > 0$ and a parameter $0 < \kappa < d_W$ such that for every $t > 0$, $g \in L^\infty(X, \mu)$ and $x, y \in X$,

$$|P_t g(x) - P_t g(y)| \leq C \frac{d(x,y)^\kappa}{t^{\kappa/d_W}} \|g\|_{L^\infty(X,\mu)}. \tag{1}$$

- If (X, d, μ) satisfies $wBE(\kappa)$ with $\kappa = \frac{d_W}{2}$, then the form $\mathcal{E}$ admits a carré du champ operator, which means that $d_w = 2$ by [**Kajino-Murugan 2019** Ann. Probab. 48 November 2020]

- For nested fractals $\lambda_1^{\#} = \lambda_1^* = d_W \alpha_1^* = d_f$

- For the Sierpinski carpet we conjecture $\lambda_1^{\#} = \lambda_1^* = d_f - d_{tH} + 1$, where $d_{tH} = \dfrac{\ln 2}{\ln 3} + 1$ is the topological-Hausdorff dimension of the Sierpinski carpet

Open question (Martin Barlow): Are there two fractals with the same values of d_f, d_w but different critical exponents α_1^*?

Preliminary answer is positive if we compare the SC and

Fitzsimmons PJ, Hambly BM, Kumagai T. Transition density estimates for Brownian motion on **affine nested fractals**. Comm. Math. Phys. 1994

based on: **Jun Kigami 1992:** Hausdorff dimensions of self-similar sets and shortest path metrics. J. Math. Society of Japan. 1995

Open question (Martin Barlow) follow-up:

Can we construct

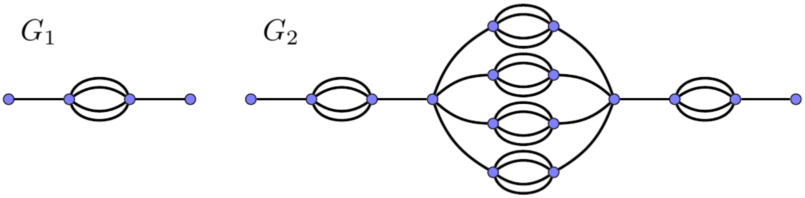
- **a p.c.f. self-similar fractal, as in Kigami 1993**
- with two-sided sub-Gaussian Heat Kernel Estimates
- for any pair of real numbers d_f, d_s

$$d_f \geqslant 1, \quad \frac{2d_f}{d_f+1} \leqslant d_s \leqslant d_f$$

$d_s = \frac{2d_R}{d_R+1}$ if $d_R < \infty$. *See also:* **Martin Barlow.** ***Which values of the volume growth and escape time exponent are possible for a graph?*** Rev. Mat. Iberoamericana, 2004. **Ben Hambly.** On the asymptotics of the eigenvalue counting function for **random recursive Sierpinski gaskets**. Probab. Theory Rel. Fields 2000. Brownian motion on a **random recursive Sierpinski gasket**. Ann. Probab. 1997. **Brownian motion on a homogeneous random fractal**. Probab. Theory Rel. Fields 1992. **Jun Kigami, Michel Lapidus**. Weyl's problem for the spectral distribution of Laplacians on pcf self-similar fractals. Comm. Math. Phys. 1993.

1. *self-similar fractals, Jun Kigami 1989-2009,*
2. *Patrick Fitzsimmons, Ben Hambly, Takashi Kumagai 1994*
3. *Laakso 2000 ...*
4. *Martin Barlow and Steven Evans 2004*
5. *Jeff Cheeger and Bruce Kleiner 2015*
6. *Patricia Alonso-Ruiz 2018, 2021*
7. *Gamal Mograby, Luke Rogers et al 2023*

.. with not-back-on-the-envelop calculations ... or with a computer assisted proof ... a part of a study of **projective limit fractals**, *in particular,* **affine p.c.f. bubble-diamond fractals**



See also Lang and Plaut 2001.

Further examples of spaces to which our theory applies can be constructed by taking products of nested fractals where the condition $wBE(d_W - d_f)$ is valid.

The n-fold product X^n supports a heat kernel obtained by tensoring, the walk dimension remains d_W on the product and $wBE(d_W - d_f)$ is still true and the Hausdorff dimension is now nd_f.

Theorem. If X is a nested fractal, then for every $n \in \mathbb{N}$, the space $BV(X^n) = \mathbf{B}^{1,d_f/d_W}(X^n)$ is dense in $L^1(X^n, \mu^{\otimes n})$ and our wBE Assumption is satisfied.

For **nested fractals** we do have $\kappa = d_W - d_f > 0$.
A set has finite perimeter if and only if it has finite boundary, $P(E) \sim \#(\partial E)$.

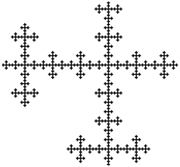
Theorem (research in progress)

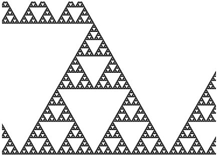
$f \in BV$ iff ∇f is a "vector valued Radon measure".

This is understood in the distributional sense (Hinz, Rogers, Strichartz et al)

Corollary (research in progress)

1. *on the Vicsek set, any BV function is $\mathbb{R}^1$-BV along each geodesic path.*
2. *on the Sierpiński gasket, any* **BV function is discontinuous**.





For Sierpinski carpets,

$$\alpha_1^* \geq (d_f - d_{tH} + 1)/d_W, \qquad (2)$$

However the Barlow-Bass theory only yields $wBE(\kappa)$ for $\kappa = d_W - d_f$, not for

$$\kappa = d_W - d_f + d_{tH} - 1.$$

We believe equality holds in (2) for α_1^* and post an open question about the weak Bakry-Émery estimate at criticality.

Note that, if $1 < d_S = 2\frac{d_f}{d_W} < 2$, proving $wBE(\kappa)$ for $\kappa > d_W - d_f$ would involve improving the Hölder continuity estimates for harmonic functions in [BB89,BB99,Ba98], strongly supported by numerical calculations in [L.Rogers et al].

Conjecture: for generalized Sierpinski carpets

$$\alpha_1^* = (d_f - d_{tH} + 1)/d_W$$

and the condition $wBE(\kappa)$ is valid for some $\kappa > (d_W - d_f)_+$.

Why do we care?

Many reasons, including

- Martin Barlow, Thierry Coulhon, Alexander Grigor'yan. Manifolds and graphs with slow heat kernel decay. Invent. Math. 144 (2001), no. 3, 609–649.
- Joint Spectra and related Topics in Complex Dynamics and Representation Theory: BIRS Banff 23w5033 May 21–26, 2023
- Quantum gravity and other topics in physics
- Applied mathematics

Gamal Mograby, Kasso Okoudjou, Luke Rogers et. al

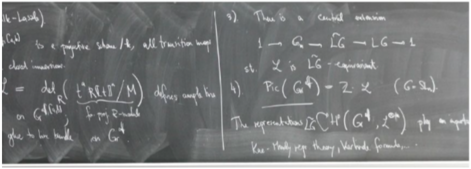
Maxim Derevyagin, Gerald Dunne, Gamal Mograby

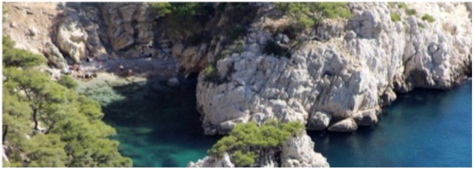
Gabriel Claret, Michael Hinz, Anna Rozanova-Pierrat et. al

Simone Creo, Michael Hinz, Maria Rosaria Lancia

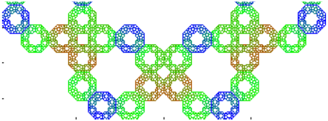
Luminy: Analysis on fractals and networks, and applications March 18-22, 2024

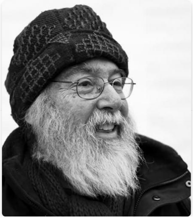
CIRM Luminy – Event N^o 2950





7th Cornell Conference on Analysis, Probability, and Mathematical Physics on Fractals: June 4–8, 2022





In Memory of Professor Robert Strichartz

We will be dedicating the entire conference to Professor Strichartz. A special session will be scheduled during the conference for all to attend and reflect on their thoughts and memories of Bob. Bob is appreciated and recognized for his organizing of the Fractals Conference Community in 2002. He will be profoundly missed by family, friends, colleagues, and most of all, the students he mentored and influenced throughout his career.

A message from the Cornell Department of Mathematics Chair, Tara Holm:

Dear friends,

I am sad to share that our colleague and friend Professor Robert Strichartz died yesterday, 19 December 2021, after a long illness. He was 78.

8th Cornell Conference on Analysis, Probability, and Mathematical Physics on Fractals: June 2025

Everybody is invited !

End of the talk ... Thank you! :-)