## **Regularity Properties of Gaussian Random Fields**

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For a continuous random field  $X = \{X(t), t \in \mathbb{R}^N\}$ , there are four types of regularity questions:

- (i) Local regularity at a point  $t^0 \in \mathbb{R}^N$ . In the Gaussian case:
  - law of the iterated logarithm (LIL)
  - Chung's law of the iterated logarithm
- (ii) Global regularity on a compact interval , e.g.,  $T = [0, 1]^N$ :
  - Uniform modulus of continuity
  - Modulus of non-diffenerability

For local oscillation of X(t) near a fixed point  $t^0 \in \mathbb{R}^N$ , we study the following question: Are there functions  $\varphi_1, \varphi_2 : \mathbb{R}^N \to \mathbb{R}_+$  and constants  $\kappa_1, \kappa_2 \in (0, \infty)$  such that

$$\limsup_{r \to 0} \max_{|h| \le r} \frac{|X(t^0 + h) - X(t^0)|}{\varphi_1(h)} = \kappa_1, \qquad \text{a.s.},$$

and

$$\liminf_{r \to 0} \max_{|h| \le r} \frac{|X(t^0 + h) - X(t^0)|}{\varphi_2(h)} = \kappa_2, \qquad \text{a.s.?}$$

Many authors have studied these questions for Gaussian random fields, usually under the extra condition of stationary increments. See, e.g., Marcus and Rosen (2006) for Gaussian processes, Li and Shao (2001), Meerschaert, Wang and X. (2013) for Gaussian random fields.

We will use the following setting from Dalang, Mueller and X. (2017), which can handle Gaussian random fields with non-stationary increments and anisotropy. It is more convenient for applications to the solutions of linear SPDEs.

## **Condition** (A1)

Consider a compact interval  $T \subset \mathbb{R}^N$ . There exists a Gaussian random field  $\{v(A, t) : A \in \mathscr{B}(\mathbb{R}_+), t \in T\}$  such that (a) For every fixed  $t \in T$ ,  $A \mapsto v(A, t)$  is a real-valued Gaussian noise,  $v(\mathbb{R}_+, t) = X(t)$ , and  $v(A, \cdot)$  and  $v(B, \cdot)$  are independent whenever A and B are disjoint.

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#### **Condition** (A1) (continued)

(b) There are constants  $a_0 > 0$  and  $\gamma_j > 0$ , j = 1, ..., N such that for all  $a_0 \le a \le b \le \infty$  and  $s = (s_1, ..., s_N)$ ,  $t = (t_1, ..., t_N) \in T$ ,

$$\left\| v([a,b),s) - X(s) - v([a,b),t) + X(t) \right\|_{L^{2}}$$

$$\leq C \Big( \sum_{j=1}^{N} a^{\gamma_{j}} |s_{j} - t_{j}| + b^{-1} \Big),$$
(1)

where  $||Y||_{L^2} = [\mathbb{E}(Y^2)]^{1/2}$  for a random variable *Y* and

$$\left\| v([0,a_0),s) - v([0,a_0),t) \right\|_{L^2} \le C \sum_{j=1}^N |s_j - t_j|.$$
 (2)

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The parameters  $\gamma_j$  (j = 1, ..., N) are important for characterizing sample path properties of X(t).

Let

$$H_j = (\gamma_j + 1)^{-1}$$
 and  $Q = \sum_{j=1}^N H_j^{-1}$ .

Define the metric  $\rho(s, t)$  on  $\mathbb{R}^N$  by

$$\rho(s,t) = \sum_{j=1}^{N} |s_j - t_j|^{H_j}.$$

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In order to see that (A1) is satisfied by the solution of an SPED, one needs to construct the random field v(A, x).

As an example, consider the solution of the linear onedimensional heat equation driven by space-time white noise. In this case,  $\mathbb{R}^N$  is replaced by  $\mathbb{R}_+ \times \mathbb{R}$ , and X(t) is u(t, x). Dalang, Mueller and X. (2017) defined

$$v(A,t,x) = \iint_{\max(|\tau|^{\frac{1}{4}},|\xi|^{\frac{1}{2}})\in A} e^{-i\xi x} \frac{e^{-i\tau t} - e^{-t\xi^2}}{|\xi|^2 - i\tau} W(d\tau,d\xi),$$

and verified that (A1) is satisfied with  $\gamma_1 = 3$ ,  $\gamma_2 = 1$ . Thus,  $H_1 = 1/4$  and  $H_2 = 1/2$ . The following lemmas are needed for applying general Gaussian methods. For example, Lemma 1.1 can be applied to derive an upper bound for the uniform modulus of continuity for  $\{X(t), t \in T\}$ .

## Lemma 1.1 [Dalang, Mueller, X. (2017)]

Under (A1), there is a constant  $c \in (0, \infty)$  such that  $\rho(s, t)$ 

$$d_X(s,t) \le c \,\rho(s,t), \quad \forall \, s,t \in T, \tag{3}$$

where  $d_X(s, t) = ||X(s) - X(t)||_{L^2}$  is the canonical metric.

Condition (A1) indicates that X(t) can be approximated by v([a, b], t). The following lemma quantifies the approximation error.

## Lemma 1.2 [Dalang, Mueller, X. (2017)]

Assume that (A1) holds. Consider b > a > 1 and r > 0 small. Set

$$A = \sum_{j=1}^{N} a^{H_j^{-1} - 1} r^{H_j^{-1}} + b^{-1}.$$

There are constants  $A_0$ , K and c such that for  $A \le A_0 r$  and

$$u \ge KA \log^{1/2} \left(\frac{r}{A}\right),\tag{4}$$

## Lemma 1.2 (Continued)

we have for all  $t^0 \in T$ ,

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$$\mathbb{P}\left\{\sup_{t\in S(t^{0},r)} |X(t) - X(t^{0}) - (v([a,b],t) - v([a,b],t^{0}))| \ge u\right\}$$

$$\le \exp\left(-\frac{u^{2}}{cA^{2}}\right),$$
where  $S(t^{0},r) = \{t\in T: \rho(t,t^{0}) \le r\}.$ 
(5)

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# Besides (A1), we also need the following "non-degeneracy" condition.

## **Condition** (A2)

## $||X(t)||_{L^2} \ge c > 0$ for all $t \in T$ and

$$\mathbb{E}[(X(s) - X(t))^2] \ge K\rho(s, t)^2$$
 for all  $s, t \in T$ .

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## Theorem 1.1 [Lee and X. 2021]

Let  $X = \{X(t), t \in \mathbb{R}^N\}$  be a centered Gaussian field that satisfies (A1) and (A2). Then for every  $t^0 \in T$ , there is a constant  $\kappa_1 = \kappa_1(t^0) \in (0, \infty)$  such that

$$\limsup_{|h| \downarrow 0} \sup_{s \in [-h,h]} \frac{|X(t^0 + s) - X(t^0)|}{\varphi_1(s)} = \kappa_1, \qquad \text{a.s.}, \tag{6}$$

where

$$arphi_1(s) = 
ho(0,s) igg[ \log \log \Big(1 + rac{1}{\prod_{j=1}^N |s_j|^{H_j}} \Big) igg]^rac{1}{2}, \qquad orall s \in \mathbb{R}^N.$$

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## 2. Chung's law of the iterated logarithm

For studying Chung's LIL at  $t^0 \in T$ , we need the following assumption on the small ball probability of *X*.

## **Condition** (A3)

There is a constant *c* such that for all  $t^0 \in T$ , r > 0 and  $0 < \varepsilon < r$ ,

$$\mathbb{P}\Big\{\max_{\rho(s,t^0)\leq r}|X(s)-X(t^0)|\leq \varepsilon\Big\}\leq \exp\Big(-c\Big(\frac{r}{\varepsilon}\Big)^{\mathcal{Q}}\Big).$$

A similar lower bound for  $\mathbb{P}\left\{\max_{\rho(s,t^0)\leq r} |X(s)| \leq \varepsilon\right\}$  is given in Lemma 2.2 below, which can be proved by applying the following general result due to Talagrand (1993) [cf. p. 257, Ledoux (1996)].

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### Lemma 2.1 [Talagrand (1993)]

Let  $\{Y(t), t \in S\}$  be an  $\mathbb{R}$ -valued centered Gaussian process indexed by a bounded set *S*. If there is a decreasing function  $\psi : (0, \delta] \to (0, \infty)$  such that  $N(S, d_Y, \varepsilon) \leq \psi(\varepsilon)$  for all  $\varepsilon \in (0, \delta]$  and there are constants  $c_4 \geq c_3 > 1$  such that

$$c_3\psi(\varepsilon) \le \psi(\varepsilon/2) \le c_4\psi(\varepsilon)$$
 (7)

for all  $\varepsilon \in (0, \delta]$ , then there is a constant *K* depending only on  $c_3$  and  $c_4$  such that for all  $u \in (0, \delta)$ ,

$$\mathbb{P}\left(\sup_{s,t\in\mathcal{S}}|Y(s)-Y(t)|\leq u\right)\geq\exp\big(-K\psi(u)\big).$$
(8)

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#### Lemma 2.2

Under (A1), there is a constant  $c' \in (0, \infty)$  such that for every  $t^0 \in T$ , r > 0 and  $0 < \varepsilon < r$ ,

$$\mathbb{P}\Big\{\max_{\rho(s,t^0)\leq r}|X(s)-X(t^0)|\leq \varepsilon\Big\}\geq \exp\Big(-c'\Big(\frac{r}{\varepsilon}\Big)^{\mathcal{Q}}\Big).$$
(9)

*Proof.* Let  $S = \{s \in T : \rho(s, t^0) \le r\}$ . It follows from Lemma 2.1 that for all  $\varepsilon \in (0, r)$ ,

$$N(S, d_X, \varepsilon) \leq c \prod_{i=1}^N \left(\frac{r}{\varepsilon}\right)^{\frac{1}{H_i}} = c \left(\frac{r}{\varepsilon}\right)^{\mathcal{Q}} := \psi(\varepsilon).$$

Clearly  $\psi(\varepsilon)$  satisfies the condition (7) in Lemma 2.1. Hence the lower bound in (9) follows from (8). The following is Chung's law of the iterated logarithm.

### Theorem 2.1 [Lee and X. 2021]

Let  $X = \{X(t), t \in \mathbb{R}^N\}$  be a centered Gaussian random field that satisfies (A1) and (A3). Then for every  $t^0 \in T$ , there is a constant  $\kappa_2 = \kappa_2(t^0) \in (0, \infty)$  such that

$$\liminf_{r \to 0} \frac{\max_{s:\rho(s,t^0) \le r} |X(t^0 + s) - X(t^0)|}{r(\log \log 1/r)^{-1/Q}} = \kappa_2, \quad \text{a.s.}, \quad (10)$$
  
where  $Q = \sum_{j=1}^N H_j^{-1}.$ 

Chung's LIL describes the smallest local oscillation of X(t), which is useful for studying hitting probabilities and fractal properties of *X*.

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**Proof of Theorem 2.1** Assumption (A1) implies a 0-1 law for the limit in the left hand-side of (10). We need to prove that  $\kappa_2 \in (0, \infty)$ . It is sufficient to prove that for some constants  $c_5, c_6 \in (0, \infty)$ ,

$$\liminf_{r \to 0} \frac{\max_{s:\rho(s,t^0) \le r} |X(t^0 + s) - X(t^0)|}{r(\log \log 1/r)^{-1/Q}} \ge c_5, \qquad \text{a.s.}, \qquad (11)$$

and

$$\liminf_{r \to 0} \frac{\max_{s:\rho(s,t^0) \le r} |X(t^0 + s) - X(t^0)|}{r(\log \log 1/r)^{-1/Q}} \le c_6, \qquad \text{a.s.}$$
(12)

In fact, (11) and (12) imply that  $\kappa_2 \in [c_5, c_6]$ .

*Proof of (11).* For any integer  $n \ge 1$ , let  $r_n = e^{-n}$ . Let  $\eta > 0$  be a constant and consider the event

$$A_n = \left\{ \max_{\rho(s,t^0) \le r_n} |X(s) - X(t^0)| \le \eta r_n (\log \log 1/r_n)^{-1/Q} \right\}.$$

By (A3) we have

$$\mathbb{P}(A_n) \leq \exp\left(-\frac{c}{\eta^Q}\log n\right) = n^{-c/\eta^Q},$$

which is summable if  $\eta > 0$  is chosen small enough. Hence, (11) follows from the Borel-Cantelli lemma.

*Proof of (12).* For every integer  $n \ge 1$ , we take  $r_n = e^{-(n+n^2)}$  and  $d_n = e^{n^2}$ . Then it follows that

$$r_n d_n = e^{-n}$$
 and  $r_n d_{n+1} > e^n$ .

It's sufficient to prove that there exists a finite constant  $c_7$  such that

$$\liminf_{n \to \infty} \frac{\max_{\rho(s, t^0) \le r_n} |X(t^0 + s) - X(t^0)|}{r_n (\log \log 1/r_n)^{-1/Q}} \le c_7 \qquad \text{a.s.}$$
(13)

For proving (13) we will use (A1) to decompose X in a way similar to that in the proof of Theorem 1.1.

Define two Gaussian fields  $X_n$  and  $X_n$  by

$$X_n(s) = v([d_n, d_{n+1}), s)$$
 and  $\widetilde{X}_n(s) = X(s) - X_n(s)$ .

Then the Gaussian fields  $\{X_n(s), s \in \mathbb{R}^N\}$   $(n = 1, 2, \dots)$  are independent and for every  $n \ge 1$ ,  $X_n$  and  $\widetilde{X}_n$  are independent as well.

Denote  $\gamma(r) = r(\log \log 1/r)^{-1/Q}$ . We make the following two claims:

(i). There is a constant  $\eta > 0$  such that

$$\sum_{n=1}^{\infty} \mathbb{P}\left\{\max_{\rho(s,t^0) \le r_n} |X_n(t^0 + s) - X_n(t^0)| \le \eta \gamma(r_n)\right\} = \infty.$$
(14)

(ii). For every  $\eta_1 > 0$ ,

$$\sum_{n=1}^{\infty} \mathbb{P}\left\{\max_{\rho(s,t^0) \le r_n} |\widetilde{X}_n(t^0 + s) - X_n(t^0)| > \eta_1 \gamma(r_n)\right\} < \infty.$$
(15)

Since the events in (14) are independent, we see that (13)follows from (14), (15) and the Borel-Cantelli Lemma.

It remains to verify the claims (i) and (ii) above.

## By Lemma 2.2 and Anderson's inequality [see Anderson (1955)], we have

$$\begin{split} & \mathbb{P}\Big\{\max_{\rho(s,t^0)\leq r_n}|X_n(t^0+s)-X_n(t^0)|\leq \eta\gamma(r_n)\Big\}\\ &\geq \mathbb{P}\Big\{\max_{\rho(s,t^0)\leq r_n}|X(t^0+s)-X(t^0)|\leq \eta\gamma(r_n)\Big\}\\ &\geq \exp\Big(-\frac{c'}{\eta^2}\log(n+n^2)\Big)\\ &= (n+n^2)^{-c'/\eta^2}. \end{split}$$

Hence (i) holds for  $\eta > (2c')^{1/Q}$ .

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To prove (ii), we let  $S = \{s \in T : \rho(s, t^0) \le r_n\}$  and consider on *S* the metric

$$\widetilde{d}(s,t) = \left\|\widetilde{X}_n(t^0+s) - \widetilde{X}_n(t^0+t)\right\|_{L^2}.$$

By Lemma 1.1 we have  $\tilde{d}(s,t) \leq c \sum_{i=1}^{N} |s_i - t_i|^{H_i}$  for all  $s, t \in T$  and hence

$$N(S, \tilde{d}, \varepsilon) \leq c \left(\frac{r_n}{\varepsilon}\right)^Q$$

Now we estimate the  $\tilde{d}$ -diameter  $\tilde{D}$  of S. By (1) in (A1),

$$\tilde{d}(s,t) \leq C\Big(\sum_{j=1}^{N} d_n^{H_j^{-1}-1} |s_j - t_j| + d_{n+1}^{-1}\Big) \leq C e^{-n^2 - (\overline{H}^{-1} \wedge 2)n}$$

Thus 
$$\widetilde{D} \leq C e^{-n^2 - (\overline{H}^{-1} \wedge 2)n}$$
.

Notice that  $\widetilde{D} \leq r_n e^{-((\overline{H}^{-1} \wedge 2) - 1)n}$ . The Dudley's integral is

$$\int_{0}^{\widetilde{D}} \sqrt{\log N(S, \widetilde{d}, \varepsilon)} \, d\varepsilon \leq \int_{0}^{\widetilde{D}} \sqrt{\log(\frac{r_n}{\varepsilon})^{\mathcal{Q}}} \, d\varepsilon$$
$$\leq Cr_n \sqrt{n} \, e^{-((\overline{H}^{-1} \wedge 2) - 1)n}.$$

Hence for any  $\eta_1 > 0$ , it follows from by Lemma 1.3 that for all *n* large,

$$\mathbb{P}\left\{\max_{\rho(s,t^{0})\leq r_{n}}|\widetilde{X}_{n}(t^{0}+s)-X_{n}(t^{0})|>\eta_{1}\gamma(r_{n})\right\}$$
  
$$\leq \exp\left(-K\frac{\eta_{1}^{2}\gamma(r_{n})^{2}}{\widetilde{D}^{2}}\right)$$
  
$$\leq \exp\left(-K\eta_{1}^{2}(\log n)^{-2/Q}e^{((\overline{H}^{-1}\wedge 2)-1)n}\right).$$

Therefore (ii) holds. The proof of Theorem 2.1 is finished.

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In order to prove an exact uniform modulus of continuity, we will make use of Condition (A1) and the following:

## Condition (A4) [sectorial local nondeterminism]

There exists a constant c > 0 such that for all  $n \ge 1$  and  $u, t^1, \ldots, t^n \in T$ ,

$$\operatorname{Var}(X(u) | X(t^{1}), \dots, X(t^{n})) \ge c \sum_{j=1}^{N} \min_{1 \le k \le n} |u_{j} - t_{j}^{k}|^{2H_{j}}.$$
(16)

Condition (A4) and the following (A4') are properties of strong local nondeterminism for Gaussian random fields with certain anisotropy.

## Condition (A4') [strong local nondeterminism]

There exists a constant c > 0 such that  $\forall n \ge 1$  and  $u, t^1, \ldots, t^n \in T$ ,

$$\operatorname{Var}(X(u) | X(t^1), \dots, X(t^n)) \ge c \min_{1 \le k \le n} \rho(u, t^k)^2.$$
(17)

- The concept of local nondeterminism (LND) of a Gaussian process was first introduced by Berman (1973) for studying local times of Gaussian processes.
- Pitt (1978) extended Berman's definition to the setting of random fields.
- Cuzick and DuPreez (1982) introduced strong local φnonderterminism for Gaussian processes and showed its usefulness in studying local times.
- The "sectorial local nondeterminism" was proved by Khoshnevisan and X. (2007) for the Brownian sheet; and extended to fractional Brownian sheets by Wu and X. (2007).
- X. (2009), Luan and X. (2012) proved "strong local nondeterminism" for a large class of Gaussian fields with stationary increments.

# Theorem 3.1 [Meerschaert, Wang and X. (2013), Lee and X. (2021)]

If a centered Gaussian field  $X = \{X(t), t \in \mathbb{R}^N\}$  satisfies

$$\mathbb{E}\left[\left(X(s) - X(t)\right)^2\right] \le c\rho(s,t)^2 \quad \text{for all } s, t \in T \qquad (18)$$

and (A4). then

$$\lim_{r \to 0} \sup_{t,s \in T, \rho(s,t) \le r} \frac{|X(s) - X(t)|}{\rho(s,t)\sqrt{\log(1 + \rho(s,t)^{-1})}} = \kappa_3,$$
(19)

where  $\kappa_3 > 0$  is a constant.

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Due to the monotonicity in r, the limit in the left-hand side of (19) exists a.s. We only need to prove that the limit is a positive and finite constant. This is done in three parts:

(a). 
$$\lim_{r \to 0} \sup_{s,t \in T, \rho(s,t) \le r} \frac{|X(s) - X(t)|}{\rho(s,t) \sqrt{\log(1 + \rho(s,t)^{-1})}} \le c_8 < \infty, \quad \text{a.s.}$$

**(b).**  $\lim_{r\to 0} \sup_{s,t\in T, \rho(s,t)\leq r} \frac{|X(s)-X(t)|}{\rho(s,t)\sqrt{\log(1+\rho(s,t)^{-1})}} \geq c_9 > 0, \quad \text{a.s.}$ 

(c). Eq. (19) follows from (a), (b) and a zero-one law.

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