

An Introduction to Random Fields

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An introduction on random fields

A random field $X = \{X(t), t \in T\}$ is a family of random variables with values in state space S , where T is the parameter set.

We consider $T \subseteq \mathbb{R}^N$ and $S = \mathbb{R}^d$ ($d \geq 1$). Then X is called an (N, d) random field.

Random fields arise naturally in

- turbulence (A. N. Kolmogorov, 1941)
- oceanography
- spatial statistics, spatio-temporal geostatistics
- image and signal processing
- ...

Examples:

$X(t, \mathbf{x}) =$ the height of an ocean surface above certain nominal plane at time $t \geq 0$ and location $\mathbf{x} \in \mathbb{R}^2$.

$X(t, \mathbf{x}) =$ wind speed at time $t \geq 0$ and location $\mathbf{x} \in \mathbb{R}^3$.

$X(t, \mathbf{x}) =$ the levels of d pollutants (e.g., ozone, $\text{PM}_{2.5}$, nitric oxide, carbon monoxide, etc) measured at location $\mathbf{x} \in \mathbb{R}^3$ and time $t \geq 0$.

Topics on random fields

- 1 How to construct random fields?
- 2 How to characterize and analyze random fields?
- 3 How to estimate parameters in random fields?
- 4 How to use random fields to make predictions?

In this talk, we provide a brief introduction to (1) and (2).

1. Stationary random fields and their spectral representations

A real-valued random field $\{X(t), t \in \mathbb{R}^N\}$ is called **second-order stationary** if $\mathbb{E}(X(t)) \equiv m$, where m is a constant, and the covariance function depends on $s - t$ only:

$$\mathbb{E}[(X(s) - m)(X(t) - m)] = C(s - t), \quad \forall s, t \in \mathbb{R}^N.$$

It is known that **C is positive definite**. That is, for all $n \geq 1$, $t^j \in \mathbb{R}^N$ and all complex numbers $a_j \in \mathbb{C}$ ($j = 1, \dots, n$), we have

$$\sum_{i=1}^n \sum_{j=1}^n a_i \overline{a_j} C(t^i - t^j) \geq 0.$$

Spectral representations

Bochner's Theorem (1932) says that a bounded and continuous function C is positive definite if and only if there is a finite Borel measure μ such that

$$C(t) = \int_{\mathbb{R}^N} e^{i\langle t, x \rangle} d\mu(x), \quad \forall t \in \mathbb{R}^N.$$

Spectral representation theorem

If $X = \{X(t), t \in \mathbb{R}^N\}$ is a centered, stationary Gaussian random field with values in \mathbb{R} whose covariance function is the Fourier transform of μ , then there is a complex-valued Gaussian random measure \tilde{W} on $\mathcal{B}(\mathbb{R}^N)$ such that $\mathbb{E}(\tilde{W}(A)) = 0$,

$$\mathbb{E}(\tilde{W}(A)\overline{\tilde{W}(B)}) = \mu(A \cap B) \quad \text{and} \quad \tilde{W}(-A) = \overline{\tilde{W}(A)}$$

and X has the following Wiener integral representation:

$$X(t) = \int_{\mathbb{R}^N} e^{i\langle t, x \rangle} d\tilde{W}(x).$$

The finite measure μ is called the spectral measure of X .

The Matérn class

An important class of isotropic stationary random fields are those with the Matérn covariance function

$$C(t) = \frac{1}{\Gamma(\nu)2^{\nu-1}} \left(\sqrt{2\nu} \frac{|t|}{\rho} \right)^\nu K_\nu \left(\sqrt{2\nu} \frac{|t|}{\rho} \right),$$

where Γ is the Gamma function, K_ν is the modified Bessel function of the second kind, and ρ and ν are non-negative parameters.

Since the covariance function $C(t)$ depends only on the Euclidean norm $|t|$, the corresponding Gaussian field X is called **isotropic**.

By the inverse Fourier transform, one can show that the spectral measure of X has the following density function:

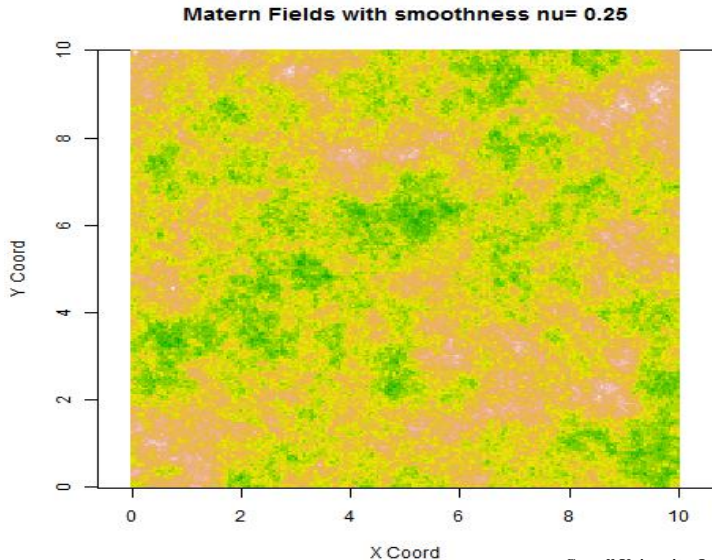
$$f(\lambda) = \frac{1}{(2\pi)^N} \frac{1}{(|\lambda|^2 + \frac{\rho^2}{2\nu})^{\nu + \frac{N}{2}}}, \quad \forall \lambda \in \mathbb{R}^N.$$

Whittle (1954) showed that the Gaussian random field X can be obtained as the solution to the following fractional SPDE

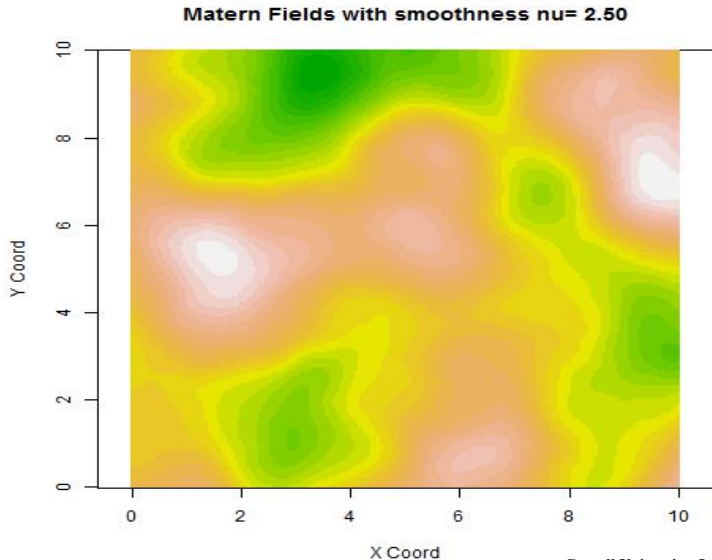
$$\left(\Delta + \frac{\rho^2}{2\nu}\right)^{\frac{\nu}{2} + \frac{N}{4}} X(t) = \dot{W}(t),$$

where $\Delta = \frac{\partial^2}{dt_1^2} + \cdots + \frac{\partial^2}{dt_N^2}$ is the N -dimensional Laplacian, and $\dot{W}(t)$ is the white noise.

A smooth Gaussian field: $N = 2, \nu = 0.25$



A smooth Gaussian field: $N = 2, \nu = 2.5$



Recent extensions:

- Gaussian random fields on the spheres, balls, and other manifolds (Marinucci and Peccati, 2011; Malyarenko, 2013; ...)
- Gaussian random fields on fractals (Baudoin and La-caux, 2000; Baudoin and Chen, 2022)
- Multivariate (stationary) Gaussian random fields

2. Gaussian fields with stationary increments

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian random field with stationary increments and $X(0) = 0$. Yaglom (1954) showed that, if $R(s, t) = \mathbb{E}[X(s)X(t)]$ is continuous, then $R(s, t)$ can be written as

$$R(s, t) = \langle s, At \rangle + \int_{\mathbb{R}^N} (e^{i\langle s, \lambda \rangle} - 1)(e^{-i\langle t, \lambda \rangle} - 1) \Delta(d\lambda),$$

where A is a nonnegative definite real $N \times N$ matrix and $\Delta(d\lambda)$ is a Borel measure which satisfies

$$\int_{\mathbb{R}^N} (1 \wedge |\lambda|^2) \Delta(d\lambda) < \infty. \quad (1)$$

The measure Δ is called the *spectral measure* of X .

We assume that $A = 0$. Then

$$\mathbb{E}[(X(s) - X(t))^2] = 2 \int_{\mathbb{R}^N} (1 - \cos\langle s - t, \lambda \rangle) \Delta(d\lambda);$$

and X has the stochastic integral representation:

$$X(t) \stackrel{d}{=} \int_{\mathbb{R}^N} (e^{i\langle t, \lambda \rangle} - 1) \widetilde{W}(d\lambda),$$

where $\stackrel{d}{=}$ denotes equality of all finite-dimensional distributions, $\widetilde{W}(d\lambda)$ is a centered complex-valued Gaussian random measure with Δ as its control measure.

Gaussian fields with stationary increments can be constructed by choosing spectral measures Δ .

Two examples

Example 1 If Δ has a density function

$$f_H(\lambda) = c(H, N) |\lambda|^{-(2H+N)},$$

where $H \in (0, 1)$ and $c(H, N) > 0$, then X is **fractional Brownian motion with index H** .

It can be verified that (for proper choice of $c(H, N)$),

$$\begin{aligned}\mathbb{E}[(X(s) - X(t))^2] &= 2c(H, N) \int_{\mathbb{R}^N} \frac{1 - \cos\langle s - t, \lambda \rangle}{|\lambda|^{2H+N}} d\lambda \\ &= |s - t|^{2H}.\end{aligned}$$

For the last identity, see, e.g., Schoenberg (1939).

- FBm X has stationary increments: for any $b \in \mathbb{R}^N$,

$$\left\{ X(t+b) - X(b), t \in \mathbb{R}^N \right\} \stackrel{d}{=} \left\{ X(t), t \in \mathbb{R}^N \right\},$$

where $\stackrel{d}{=}$ means equality in finite dimensional distributions.

- FBm X is **H -self-similar**: for every constant $c > 0$,

$$\left\{ X(ct), t \in \mathbb{R}^N \right\} \stackrel{d}{=} \left\{ c^H X(t), t \in \mathbb{R}^N \right\}.$$

(Statistical) self-similarity is naturally related to fractals.

Example 2 A large class of Gaussian fields can be obtained by letting spectral density functions satisfy (1) and

$$f(\lambda) \asymp \frac{1}{\left(\sum_{j=1}^N |\lambda_j|^{\beta_j}\right)^\gamma}, \quad \forall \lambda \in \mathbb{R}^N, |\lambda| \geq 1, \quad (2)$$

where $(\beta_1, \dots, \beta_N) \in (0, \infty)^N$ and $\gamma > \sum_{j=1}^N \frac{1}{\beta_j}$.

More conveniently, we re-write (2) as

$$f(\lambda) \asymp \frac{1}{\left(\sum_{j=1}^N |\lambda_j|^{H_j}\right)^{Q+2}}, \quad \forall \lambda \in \mathbb{R}^N, |\lambda| \geq 1, \quad (3)$$

where $H_j = \frac{\beta_j}{2} \left(\gamma - \sum_{i=1}^N \frac{1}{\beta_i} \right)$ and $Q = \sum_{j=1}^N H_j^{-1}$.

3 More examples of non-stationary Gaussian random fields

3.1 The Brownian sheet and fractional Brownian sheets

The Brownian sheet $W = \{W(t), t \in \mathbb{R}_+^N\}$ is a centered (N, d) -Gaussian field whose covariance function is

$$\mathbb{E}[W_i(s)W_j(t)] = \delta_{ij} \prod_{k=1}^N s_k \wedge t_k.$$

- When $N = 1$, W is Brownian motion in \mathbb{R}^d .
- W is $N/2$ -self-similar, but it **does not** have **stationary increments**.
- It gives rise to the Gaussian white noise \dot{W} , which can be used as a stochastic integrator.

Fractional Brownian sheet $W^{\vec{H}} = \{W^{\vec{H}}(t), t \in \mathbb{R}^N\}$ is a mean zero Gaussian field in \mathbb{R} with covariance function

$$\mathbb{E} \left[W^{\vec{H}}(s) W^{\vec{H}}(t) \right] = \prod_{j=1}^N \frac{1}{2} \left(|s_j|^{2H_j} + |t_j|^{2H_j} - |s_j - t_j|^{2H_j} \right),$$

where $\vec{H} = (H_1, \dots, H_N) \in (0, 1)^N$.

For all constants $c > 0$,

$$\left\{ W^{\vec{H}}(c^E t), t \in \mathbb{R}^N \right\} \stackrel{d}{=} \left\{ c W^{\vec{H}}(t), t \in \mathbb{R}^N \right\},$$

where $E = (a_{ij})$ is the $N \times N$ diagonal matrix with $a_{ii} = 1/(NH_i)$ for all $1 \leq i \leq N$ and $a_{ij} = 0$ if $i \neq j$. This is referred to as an “operator-scaling” property.

3.2 Linear stochastic heat equation

Consider the linear stochastic heat equation

$$\begin{aligned}\frac{\partial u}{\partial t}(t, x) &= \frac{1}{2} \Delta u(t, x) + \sigma \dot{W}, \quad t \geq 0, x \in \mathbb{R}^k, \\ u(0, x) &\equiv 0,\end{aligned}\tag{4}$$

where Δ is the Laplacian operator in the spatial variables, σ is a constant or a deterministic function, and \dot{W} is a Gaussian noise that is white in time and has a spatially homogeneous covariance [Dalang (1999)] given by the Riesz kernel with exponent $\beta \in (0, k \wedge 2)$, i.e.

$$\mathbb{E}(\dot{W}(t, x) \dot{W}(s, y)) = \delta(t - s) |x - y|^{-\beta}.$$

If $k = 1 = \beta$, then \dot{W} is the space-time Gaussian white noise considered by Walsh (1986).

It follows from Walsh (1986) and Dalang (1999) that the mild solution of (4) is the mean zero **Gaussian random field** $u = \{u(t, x), t \geq 0, x \in \mathbb{R}\}$ defined by

$$u(t, x) = \int_0^t \int_{\mathbb{R}} \tilde{G}_{t-r}(x - y) \sigma W(dr dy), \quad t \geq 0, x \in \mathbb{R},$$

where $\tilde{G}_t(x)$ is the Green kernel given by

$$\tilde{G}_t(x) = (2\pi t)^{-1/2} \exp\left(-\frac{|x|^2}{2t}\right), \quad \forall t > 0, x \in \mathbb{R}^k.$$

3.3 Linear stochastic wave equation

The linear stochastic wave equation

$$\begin{cases} \frac{\partial^2}{\partial t^2} u(t, x) = \Delta u(t, x) + \dot{W}(t, x), & t \geq 0, x \in \mathbb{R}^k, \\ u(0, x) = \frac{\partial}{\partial t} u(0, x) = 0, \end{cases} \quad (5)$$

where \dot{W} is a Gaussian noise as in the previous example with exponent $\beta \in (0, k \wedge 2)$.

The existence of real-valued process solution to (5) was studied by Walsh (1986) for the space-time white noise and by Dalang (1999) in the more general setting.

We recall briefly some known results.

Let G be the fundamental solution of the wave equation.
Then

$$G(t, x) = \frac{1}{2} \mathbf{1}_{\{|x| < t\}} \quad \text{if } k = 1;$$

$$G(t, x) = c_k \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{(k-2)/2} (t^2 - |x|^2)_+^{-1/2}, \quad \text{if } k \geq 2 \text{ is even};$$

$$G(t, x) = c_k \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{(k-3)/2} \frac{\sigma_t^k(dx)}{t}, \quad \text{if } k \geq 3 \text{ is odd},$$

where σ_t^k is the uniform surface measure on the sphere $\{x \in \mathbb{R}^k : |x| = t\}$.

Note that for $k \geq 3$, G is not a function but a distribution.

For any dimension $k \geq 1$, the Fourier transform of G in variable x is given by

$$\mathcal{F}(G(t, \cdot))(\xi) = \frac{\sin(t|\xi|)}{|\xi|}, \quad t \geq 0, \xi \in \mathbb{R}^k. \quad (6)$$

Dalang (1999) proved that the real-valued process solution of equation (5) is given by

$$u(t, x) = \int_0^t \int_{\mathbb{R}^k} G(t-s, x-y) W(ds dy), \quad (7)$$

The range of β has been chosen so that the stochastic integral exists.

The solution $u = \{u(t, x), t \geq 0, x \in \mathbb{R}^k\}$ is a centered Gaussian random field.

Recall from Theorem 2 of Dalang (1999) that

$$\mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^k} H(s, y) W(ds dy)\right)^2\right] = c \int_0^t ds \int_{\mathbb{R}^k} \frac{d\xi}{|\xi|^{k-\beta}} |\mathcal{F}(H(s, \cdot))(\xi)|^2 \quad (8)$$

provided that $s \mapsto H(s, \cdot)$ is a deterministic function with values in the space of nonnegative distributions with rapid decrease and

$$\int_0^t ds \int_{\mathbb{R}^k} \frac{d\xi}{|\xi|^{k-\beta}} |\mathcal{F}(H(s, \cdot))(\xi)|^2 < \infty.$$

Eq. (8) is a basic tool for studying the Gaussian random field $u = \{u(t, x), t \geq 0, x \in \mathbb{R}^k\}$.

3.4 Non-linear stochastic heat & wave eqs

Many authors have studied the following nonlinear SPDE:

$$\begin{cases} \mathcal{L}u = b(u) + \sigma(u)\dot{W}, & t \geq 0, x \in \mathbb{R}^k, \\ u(0, x) = \frac{\partial}{\partial t}u(0, x) = 0, \end{cases} \quad (9)$$

where \mathcal{L} is a partial differential operator, σ and b are non-random functions that satisfy some regularity conditions [e.g., σ and b are Lipschitz continuous.]

For example, $\mathcal{L}u = \frac{\partial u}{\partial t} - \frac{1}{2}\Delta u$ and $\mathcal{L}u = \frac{\partial^2 u}{\partial t^2} - \Delta u$ give the stochastic heat and wave equation, respectively.

The solutions, when they exist, are in general **non-Gaussian** random fields. We refer to Dalang (1999), Khoshnevisan (2014) for more information.

4. Multivariate Gaussian random fields

Consider a multivariate random field $\mathbf{X} = \{\mathbf{X}(t), t \in \mathbb{R}^N\}$ taking values in \mathbb{R}^d defined by

$$\mathbf{X}(t) = (X_1(t), \dots, X_d(t)), \quad t \in \mathbb{R}^N. \quad (10)$$

Their key features are:

- the components X_1, \dots, X_d are dependent.
- X_1, \dots, X_d may have different smoothness properties.

For any $i, j = 1, \dots, d$, define

$$C_{ij}(s, t) := \mathbb{E}[X_i(s)X_j(t)]. \quad (11)$$

They are called the cross-covariance functions of \mathbf{X} .

(i) The multivariate Matérn random fields

Gneiting, Kleiber and Schlather (2010) introduced a class of multivariate stationary Matérn models $\{X(t), t \in \mathbb{R}^N\}$ in (10) with marginal and cross-covariance functions of the form

$$C_{ij}(s, t) = M(s - t | \nu_{ij}, a_{ij}),$$

where

$$M(h | \nu, a) := \frac{2^{1-\nu}}{\Gamma(\nu)} (a|h|)^\nu K_\nu(a|h|).$$

and provided conditions for such matrix-valued functions to form legitimate cross-covariance functions.

See also Apanansovich, Genton and Sun (2012), Kleiber and Nychka (2013).

The bivariate Matérn fields

Let $\mathbf{X}(t) = (X_1(t), X_2(t))'$ be an \mathbb{R}^2 -valued Gaussian field whose covariance matrix is determined by

$$C(h) = \begin{pmatrix} c_{11}(h) & c_{12}(h) \\ c_{21}(h) & c_{22}(h) \end{pmatrix}, \quad (12)$$

where $c_{ij}(h) := \mathbb{E}[X_i(s+h)X_j(s)]$ are specified by

$$\begin{aligned} c_{11}(h) &= \sigma_1^2 M(h|\nu_1, a_1), \\ c_{22}(h) &= \sigma_2^2 M(h|\nu_2, a_2), \\ c_{12}(h) &= c_{21}(h) = \rho\sigma_1\sigma_2 M(h|\nu_{12}, a_{12}) \end{aligned} \quad (13)$$

with $a_1, a_2, a_{12}, \sigma_1, \sigma_2 > 0$ and $\rho \in (-1, 1)$.

Gneiting, et al. (2010) gave NSC for (12) to be valid. In particular, if $\rho \neq 0$, one must have

$$\frac{\nu_1 + \nu_2}{2} \leq \nu_{12}.$$

The parameters ν_1 and ν_2 control the smoothness of the sample function $t \mapsto \mathbf{X}(t)$.

For example, if $\nu_1 > 1$, then a.s. the sample function $t \mapsto X_1(t)$ is continuously differentiable. This can be proved using the spectral density.

Zhou and X. (2017, 2018) studied extreme values and estimation problems for a class of bivariate random fields that includes the bivariate Matérn fields.

(ii). Multivariate random fields with stationary increments

An \mathbb{R}^d -valued Gaussian random field $\mathbf{X} = \{\mathbf{X}(t), t \in \mathbb{R}^N\}$ is said to have stationary increments if $\forall t_0 \in \mathbb{R}^N$,

$$\{\mathbf{X}(t + t_0) - \mathbf{X}(t_0), t \in \mathbb{R}^N\} \stackrel{d}{=} \{\mathbf{X}(t) - \mathbf{X}(0), t \in \mathbb{R}^N\}.$$

A general framework for multivariate random fields with stationary increments was provided by Yaglom (1957).

As an example, we consider a spacial class of *operator fractional Brownian motions*.

Operator fractional Brownian motions

Let D be a linear operator on \mathbb{R}^d (or a $d \times d$ real matrix). The operator norm of D is defined by

$$\|D\| = \max_{|x|=1} |Dx|.$$

Denote the eigenvalues of D by

$$\lambda_k = \alpha_k + i\beta_k, \quad (k = 1, \dots, d).$$

We assume that

$$0 < \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_d < 1. \quad (14)$$

For any $c > 0$, we define the linear operator c^D by

$$c^D = \sum_{k=0}^{\infty} \frac{(\ln c)^k}{k!} D^k.$$

(a). *Moving average representation*

One can define **ofBm** $\mathbf{X} = \{\mathbf{X}(t), t \in \mathbb{R}\}$ in \mathbb{R}^d by using the stochastic integration method:

$$\mathbf{X}(t) = \int_{\mathbb{R}} [(t-r)_+^{D-\frac{1}{2}I} - (-r)_+^{D-\frac{1}{2}I}] W(dr), \quad (15)$$

where W is d -dimensional Brownian motion, is an operator fractional Brownian motion with exponent D .

It has the following properties:

- **stationary increments.**
- **(operator self-similarity)** For every constant $c > 0$,

$$\{\mathbf{X}(ct), t \in \mathbb{R}\} \stackrel{d}{=} \{c^D \mathbf{X}(t), t \in \mathbb{R}\}.$$

(b). *Harmonizable representation*

The Gaussian random field $Y = \{Y(t), t \in \mathbb{R}^N\}$ in \mathbb{R}^d defined by

$$Y(t) = \int_{\mathbb{R}^N} \frac{e^{i\langle t, r \rangle} - 1}{|r|^{D + \frac{N}{2}I}} \widetilde{\mathcal{W}}(dr), \quad (16)$$

where $\widetilde{\mathcal{W}}$ is a complex-valued Gaussian random measure on \mathbb{R}^d with Lebesgue control measure and i.i.d. components, is also an operator fractional Brownian motion with exponent D .

In order to verify that the stochastic integrals in (15) and (16) are well defined, it is sufficient to verify respectively that

$$\int_{\mathbb{R}} \left\| (t-r)_+^{D-\frac{1}{2}I} - (-r)_+^{D-\frac{1}{2}I} \right\|^2 dr < \infty,$$

and

$$\int_{\mathbb{R}^N} (1 - \cos \langle t, r \rangle) \left\| |r|^{-D-\frac{N}{2}I} \right\|^2 dr < \infty.$$

This is where condition (14) is needed.

(iii). Operator-scaling and operator-self-similar random fields

Li and X. (2011) constructed a large class of more general, namely, operator-scaling and operator-self-similar random fields with stationary increments.

Several authors have studied properties of these random fields. See, for example,

- Ercan Sönmez (2017, 2018, 2020).
- Kremer and Scheffler (2019) for further development and recent results.
- Shen, Stilian, and Hsing (2020).

(iv). Systems of stochastic partial differential equations

There has been active recent research on this topic, which we do not discuss here.

(v). Matrix-valued Gaussian random fields

Let $\xi = \{\xi(t) : t \in \mathbb{R}_+^N\}$ be a centered Gaussian random field and let $\{\xi_{i,j} : i, j \in \mathbb{N}\}$ be a family of independent copies of ξ .

Consider the symmetric $d \times d$ matrix-valued process $X = \{X_{i,j}(t); t \in \mathbb{R}_+^N, 1 \leq i, j \leq d\}$ defined by

$$X_{i,j}(t) = \begin{cases} \xi_{i,j}(t), & i < j; \\ \sqrt{2}\xi_{i,i}(t), & i = j; \\ \xi_{j,i}(t), & i > j. \end{cases} \quad (17)$$

One may study statistical and sample path properties of the eigenvalues of X .

Thank you!