Anti-attracting Maps and Eigenforms on Fractals

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Fixed Point Theorems

THEOREM 1. (Brouwer fixed point Theorem) If $f : A \to A$ is continuous and A is a non-empty compact and convex subset of $\mathbb{R}^n$, then f has a fixed point.

When A is not compact, for example is open, simple examples show that Theorem 1 does not hold. This happens if the boundary is

repelling

THEOREM 2 If A is an open convex bounded subset of $\mathbb{R}^n$ and $f : A \to A$ is continuous, then f has a fixed point provided ∂A is repelling for f. Roughly speaking, this means that every point of ∂A has a neighborhood U such that f maps $U \cap A$ towards the interior of A.

Proof (Hint) f maps continuously some non-empty compact and convex subset of A into itself.

Anti-attracting Maps

Let Z be an affine subset of $\mathbb{R}^n$. Let $\mathrm{Int}(A)$ denote the interior of a subset A of Z where the interior is meant to be with respect to the euclidean topology on Z. For every $\widetilde{x}, x \in Z$, let

$$\mathrm{Ext}_{\widetilde{x}}(x) = \left\{\widetilde{x} + t(x - \widetilde{x}) : t > 1\right\}.$$

THEOREM 3. Let Z be an affine subset of $\mathbb{R}^n$, let K be a non-empty compact and convex subset of Z. Let θ be a continuous map from ∂K to $\mathrm{Int}(K)$ and let $\phi : K \to Z$ be a continuous map such that $\phi(x) \notin \mathrm{Ext}_{\theta(x)}(x)$ for every $x \in \partial K$.
Then ϕ has a fixed point on K on K.

Proof. (Hint) If θ is constant, say $\theta(x) = \bar{x}$, it suffices to take a fixed point of the map $p \circ f$ from K into K. Here

$$p(x) = \begin{cases} x & \text{if } x \in K \\ [x, \bar{x}] \cap \partial K & \text{otherwise} \end{cases}$$

The general case is slightly more complicated.

Anti-attracting Maps

Let Z and K be as in Theorem 3, let ϕ be a continuous map from $\mathrm{Int}(K)$ into itself, and let θ be a continuous map from ∂K to $\mathrm{Int}(K)$.
We say that $\bar{x} \in \partial K$ is *anti-attracting for* (ϕ, θ) if there exists a neighborhood $U_{\bar{x}}$ of $\bar{x}$ in Z such that for every $x \in U_{\bar{x}} \cap \mathrm{Int}(K)$ and every $x' \in U_{\bar{x}} \cap \partial K$ we have $\phi(x) \notin \mathrm{Ext}_{\theta(x')}(x)$. We say that ϕ is θ-anti-attracting if every $\bar{x} \in \partial K$ is anti-attracting for (ϕ, θ).

THEOREM 4 Let Z and K be as in Theorem 3. Let θ be a continuous map from ∂K into $\mathrm{Int}(K)$, and let ϕ be a θ-anti-attracting map from $\mathrm{Int}(K)$ into itself.
Then ϕ has a fixed point on $\mathrm{Int}(K)$

Fractals

Here, we discuss analysis on finitely ramified self-similar fractals.
A self-similar fractal $\mathcal{F}$ is defined by a set of finitely many contractive (i.e., having factor < 1) similarities $\psi_1, ..., \psi_k$ in $\mathbb{R}^\nu$.
Then, $\mathcal{F}$ is the unique non-empty compact set $\mathcal{F}$ in $\mathbb{R}^\nu$ such that

$$\mathcal{F} = \bigcup_{i=1}^{k} \psi_i(\mathcal{F}), \qquad (*)$$

Finitely ramified means, more or less that the "copies" $\psi_i(\mathcal{F})$ of the fractal intersect only at finitely many points.

figure

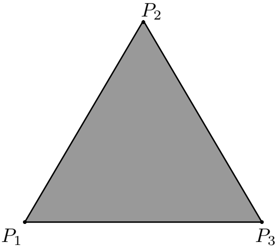

Figure 1. The Gasket, 0-step

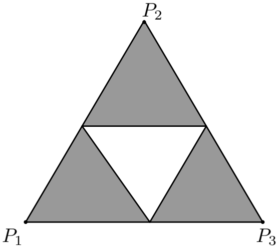

Figure 2. The Gasket, 1-step

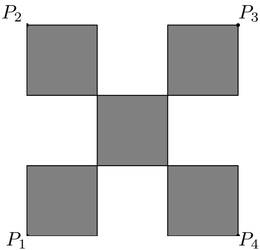

Figure 3. The Vicsek Set

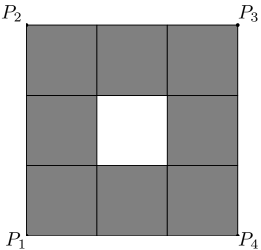

Figure 4. The Carpet

Examples of self-similar fractals are the (Sierpinski) Gasket, the Vicsek set, and the (Lindstrøm) Snowflake. An example of an infinitely ramified fractal is the Sierpinski Carpet.

We also require that the fractal is connected, so excluding fractals as the Cantor Set.

I will present now a more abstract and general setting. This setting represents a subclass, widely considered, of the set of the P.C.F. self-similar sets, a class of finitely ramified fractals introduced by J. Kigami. (Similar to *anchestor* introduced by J. Kigami)

Our setting consists of:

A finite set $\Psi = \{\psi_1, ..., \psi_k\}$ of one-to-one maps defined on a finite set $V^{(0)} = \{P_1, ..., P_N\}$ into some set, and put

$$V^{(1)} = \bigcup_{\psi\in\Psi} \psi(V^{(0)}) .$$

We call 1-cells the sets $V_i := \psi_i(V^{(0)})$ for $i = 1, ..., k$, and put

$$\mathcal{V} := \{i = 1, ..., k\}, \quad \mathcal{U} = \{1, ..., N\}.$$

In $\mathcal{V}$ we consider the graph $\mathcal{G}$ whose edges are $\{i_1, i_2\}$ such that $V_{i_1} \cap V_{i_2} \neq \emptyset$. We require: for each $j = 1, ..., N$

$$\psi_j(P_j) = P_j, \quad P_j \notin \psi_i(V^{(0)}) \quad \forall\, i \neq j\, \forall\, j = 1, ..., N, \qquad (a)$$

$$(V^{(1)}, \mathcal{G}) \text{ is a connected graph.} \qquad (b)$$

Let

$$J := \big\{\{j_1, j_2\} : j_1, j_2 = 1, ..., N, j_1 \neq j_2\big\}.$$

$\mathcal{D}(V^{(0)})$ or $\mathcal{D}$ will denote the set of the functionals E (Dirichlet forms on $V^{(0)}$) from $\mathbb{R}^{V^{(0)}}$ into $\mathbb{R}$ of the form

$$E(u) = \sum_{\{j_1,j_2\}\in J} c_{\{j_1,j_2\}}(E)\big(u(P_{j_1}) - u(P_{j_2})\big)^2$$

where the *coefficients* $c_{\{j_1,j_2\}}(E) = E_{\{j_1,j_2\}}$ are required to satisfy:

$$c_{\{j_1,j_2\}}(E) = E_{\{j_1,j_2\}} \geq 0.$$

By definition, $E \in \widetilde{\mathcal{D}}$ if $E \in \mathcal{D}$ and moreover $E(u) = 0$ if and only if u is constant.

This amounts to the fact that every two points in $\mathcal{U}$ can be connected by a path $(j_1, ..., j_n)$ such that $c_{\{j_h, j_{h+1}\}}(E) > 0$ for $h = 1, ..., n-1$.

Note that every $E \in \mathcal{D}$ is uniquely determined by its coefficients. In fact,

$$E_{\{j_1, j_2\}} = \frac{1}{4}\Big(E\big(\chi_{\{P_{j_1}\}} - \chi_{\{P_{j_2}\}}\big) - E\big(\chi_{\{P_{j_1}\}} + \chi_{\{P_{j_2}\}}\big)\Big).$$

Thus, E can be identified to a point in $\mathbb{R}^J$, namely

$$\big(E_{\{j_1, j_2\}} : \{j_1, j_2\} \in J\big).$$

2. Renormalization

For every $u \in \mathbb{R}^{V^{(0)}}$, every $E \in \widetilde{\mathcal{D}}$ and every $r \in W :=]0, +\infty[^{\mathcal{V}}$ ($r_i := r(i)$), put

$$\Lambda_r(E)(u) = \inf\left\{S_{1;r}(E)(v), v \in \mathcal{L}(u)\right\},$$

where by definition,

$$S_{1;r}(E)(v) := \sum_{i=1}^{k} r_i E(v \circ \psi_i),$$

$$\mathcal{L}(u) := \left\{v \in \mathbb{R}^{V^{(1)}} : v = u \text{ on } V^{(0)}\right\}.$$

It is well known that the infimum is attained at a unique function $v := H_{1,E;r}(u)$.

When $r \in W$, an element E of $\widetilde{\mathcal{D}}$ is said to be an *r-eigenform* with *eigenvalue* $\rho > 0$ if

$$\Lambda_r(E) = \rho E.$$

As this amounts to $\Lambda_{\frac{r}{\rho}}(E) = E$, we could also require $\rho = 1$. Note also that it is well-known that if there exist two r-eigenforms on the same fractal, then they have the same eigenvalue.
In other words, the r-eigenform can be not unique, but the eigenvalue is unique.

The r-eigenforms are important since they are in one-to-one correspondence with the r-self-similar energies

on all of the fractal

satisfying some nice properties which I will not specify here. Namely, the correspondence associates $\mathcal{E}$ to $E \in \widetilde{\mathcal{D}}$ where for every $v \in \mathbb{R}^K$ we define

$$\mathcal{E}(v) = \lim_{n\to+\infty} \frac{1}{\rho^n} \sum_{i_1,\dots i_n=1}^{k} r_{i_1}\cdots r_{i_n} E\big(v \circ \psi_{i_1} \circ \cdots \circ \psi_{i_n}\big).$$

The r-eigenforms, or more precisely the equivalence classes of

r-eigenforms where two elements E_1 and E_2 are equivalent if E_2 is a positive multiple of E_1, are also in one-to-one correspondence with the self-similar diffusions and with the harmonic structures on the fractal.

Natural problems in this context are the **Existence** and the **Uniqueness** of the r-eigenforms. Uniqueness will mean uniqueness up to a multiplicative constant. In fact, a positive multiple of an r-eigenform is an r-eigenform as well.

We give here some references to the problem of existence of an r-eigenform on a fractal.

[1] T. Lindstrøm, Brownian Motion on Nested Fractals. Mem. Amer. Math. Soc. No. 420 (1990).

[2] C. Sabot, Existence and Uniqueness of Diffusions on Finitely Ramified Self-Similar Fractals. Ann. Sci. École Norm. Sup. (4) 30 no. 5, 605-673 (1997).

[3] V. Metz, The short-cut test. Journal of Functional Analysis 220, 118-156 (2005).

[4] R. Peirone, Fixed points of anti-attracting maps and eigenforms on fractals, Mathematische Nachrichten, **294** (8),1578-1594, (2021)

[5] R. Peirone, A P.C.F. self-similar set with no self-similar energy, J. Fractal Geometry, 6, Issue 4, 2019, pp. 393-404

[6] R. Peirone. Existence of Self-similar Energies on Finitely Ramified Fractals, Journal d'Analyse Mathematique, vol. 123, issue 1 (2014), 35-94.

I now describe the results of the references.
T. Lindstrøm proved in [1] that there exists an 1-eigenform (that is, an eigenform with all weights equal to 1,) on the nested fractals C. Sabot in [2] proved a rather general criterion, V. Metz in [3] improved the results in [2]. This talk is based on the results of [4], where a new proof of the main result of [3] is given using Theorem 4. See [4] for more details.

A natural question which was been a well-known open question was whether an every P.C.F. self-similar set there exists an r-eigenform for a *suitable* $r \in W$. In [5] a counterexample is provided. On the other hand, in [6], a weak form of the conjecture is proved. Namely, on every fractal of the type considered here, there exists an r-eigenform but with respect to *a suitable set of maps defining the fractal*, not to the given set of maps $\{\psi_i : i = 1, ..., k\}$.

The problem of existence of an r-eigenform is usually a difficult problem.
The point is that the eigeform is required to be in $\widetilde{\mathcal{D}}$. In fact, In our setting an r-eigenform in $\mathcal{D}$ always exists for every set of weights r.

Idea of proof: the set

$$\big\{E \in \mathcal{D} : \Lambda_r(E) \geq cE,\ |E| = 1\big\}$$

is non-empty, compact and convex with a suitable definition of the norm $|\ |$, so $\frac{\Lambda_r}{|\Lambda_r|}$ has a fixed point.

For $E \in \mathcal{D}$ let

$$|E| := \sum_{d \in J} E_d$$

Note that Λ_r *is* 1*-homogeneous*, that is $\Lambda_r(aE) = a\Lambda_r(E)$ when a is positive.
Consider the map $\widehat{\Lambda}_r$ defined as

$$\widehat{\Lambda}_r(E) := \frac{\Lambda_r(E)}{\big|\Lambda_r(E)\big|}.$$

Let $\mathcal{D}_{\mathcal{N}} = \big\{E \in \mathcal{D} : |E| = 1\big\}$, $\widetilde{\mathcal{D}}_{\mathcal{N}} = \big\{E \in \widetilde{\mathcal{D}} : |E| = 1\big\}$.
Now, the existence of an r-eigenform amounts to the existence of a fixed point of $\widehat{\Lambda}$ on $\widetilde{\mathcal{D}}_{\mathcal{N}}$.

Anti-attracting forms on Fractals.

Recall that a form in $\mathcal{D}$ can be seen as an element of $\mathbb{R}^J$. Recall J is the set of different $\{j_1, j_2\}$ in $\{1, ..., N\}$, namey

$$E \text{ can be identified to } \big(E_{\{j_1,j_2\}} : \{j_1, j_2\} \in J\big).$$

So, we define specific sets in $\mathbb{R}^J$ which will play the roles of Z and K, Moreover, we will investigate the notion of an anti-attracting form with respect to a map obtained normalizing Λ_r. Let us define

$$\widetilde{L}(x) := \sum_{d \in J} x_d \quad \forall\, x \in \mathbb{R}^J,$$

$$|x| := \sum_{d \in J} |x_d| \quad \forall\, x \in \mathbb{R}^J,$$

$$Z := \big\{x \in \mathbb{R}^J : \widetilde{L}(x) = 1\big\},$$

$$\mathcal{D}_{\mathcal{N}} =: \big\{E \in \mathcal{D} : |E| = 1\big\} = \big\{E \in Z : E_d \geq 0 \quad \forall\, d \in J\big\}.$$

So, Z is an affine set in $\mathbb{R}^J$ and $\mathcal{D}_{\mathcal{N}}$ is a non-empty compact and convex subset of Z. We easily characterize $\mathrm{Int}(\mathcal{D}_{\mathcal{N}})$. In fact we have $\mathrm{Int}(\mathcal{D}_{\mathcal{N}}) = \mathcal{D}_{\mathcal{N}}^{(1)}$ where

$$\mathcal{D}_{\mathcal{N}}^{(1)} := \{E \in \mathcal{D}_{\mathcal{N}} : E_d > 0 \; \forall \, d \in J\} \subseteq \widetilde{\mathcal{D}}.$$

It is known that if $E \in \widetilde{\mathcal{D}}$ satisfies $E_d > 0$ for every $d \in J$, so does $\Lambda_r(E)$. Thus $\widehat{\Lambda}_r$ maps continuously $\mathcal{D}_{\mathcal{N}}^{(1)}$ into itself. However, in general $\widehat{\Lambda}_r$ cannot be extended continuously on all of $\mathcal{D}_{\mathcal{N}}$. In fact, we could have $\Lambda_r(E) = 0$ for some $E \in \mathcal{D} \setminus \widetilde{\mathcal{D}}$.

We so need a nice decomposition of $\partial\mathcal{D}_{\mathcal{N}}$. Let

$$\mathcal{D}_{\mathcal{N}}^{(2)} := \mathcal{D}_{\mathcal{N}} \cap \widetilde{\mathcal{D}} \setminus \mathcal{D}_{\mathcal{N}}^{(1)},$$

$$\mathcal{D}_{\mathcal{N}}^{(3)} = \{E \in \mathcal{D}_{\mathcal{N}} \setminus \widetilde{\mathcal{D}} : \Lambda_r(E) \neq 0\},$$

$$\mathcal{D}_{\mathcal{N}}^{(4)} = \{E \in \mathcal{D}_{\mathcal{N}} \setminus \widetilde{\mathcal{D}} : \Lambda_r(E) = 0\},$$

where $r \in W$. In fact, it can be proved that the formula $\Lambda_r(E) = 0$ is independent of $r \in W$, but this is not important for our considerations since we fix a given $r \in W$. We easily have

$$\partial\mathcal{D}_{\mathcal{N}} = \mathcal{D}_{\mathcal{N}}^{(2)} \cup \mathcal{D}_{\mathcal{N}}^{(3)} \cup \mathcal{D}_{\mathcal{N}}^{(4)}.$$

Cearly, $\widehat{\Lambda}_r$ maps continuously $\mathcal{D}_{\mathcal{N}}^{(1)} \cup \mathcal{D}_{\mathcal{N}}^{(2)} \cup \mathcal{D}_{\mathcal{N}}^{(3)}$ into $\mathcal{D}_{\mathcal{N}}$.
We say that E is a *degenerate* r-eigenform
if $\widehat{\Lambda}(E) = \rho E$ for some $\rho > 0$ but $E \in \mathcal{D} \setminus \widetilde{\mathcal{D}}$.

When $E \in \mathcal{D}_{\mathcal{N}}^{(1)} \cup \mathcal{D}_{\mathcal{N}}^{(2)} \cup \mathcal{D}_{\mathcal{N}}^{(3)}$, *then* E *is a (possibly degenerate)* r*-eigenform if and only if it is a fixed point of* $\widehat{\Lambda}_r$.

We try to apply Theorem 4 in this setting. Thus, we try to characterize what $E \in \partial\mathcal{D}_\mathcal{N}$ are anti-attracting for $(\widehat{\Lambda}_r, \theta)$ when θ is a continuous map from $\partial\mathcal{D}_\mathcal{N}$ to $\mathcal{D}_\mathcal{N}^{(1)}$.

Lemma

Let $r \in W$ and let θ be a continuous map from $\partial\mathcal{D}_\mathcal{N}$ to $\mathcal{D}_\mathcal{N}^{(1)}$. Then every $\bar{E} \in \mathcal{D}_\mathcal{N}^{(4)}$ is anti-attracting for $(\widehat{\Lambda}_r, \theta)$.

Lemma

Let $r \in W$ and let θ be a continuous map from $\partial\mathcal{D}_\mathcal{N}$ to $\mathcal{D}_\mathcal{N}^{(1)}$. Then every $\bar{E} \in \mathcal{D}_\mathcal{N}^{(2)} \cup \mathcal{D}_\mathcal{N}^{(3)}$ such that $\widehat{\Lambda}_r(\bar{E}) \neq \bar{E}$ is anti-attracting for $(\widehat{\Lambda}_r, \theta)$.

Thus, the only (possibly) non anti-attracting forms are the degenerate eigenforms in $\mathcal{D}_\mathcal{N}^{(3)}$

NOTE: if $\widehat{\Lambda}_r(\bar{E}) = \bar{E}$ for some $E \in \mathcal{D}_\mathcal{N}^{(2)} \subseteq \widetilde{\mathcal{D}}$, we clearly have existence of an r-eigenform.

Renormalization along a degenerate eigenform

If $E \in \mathcal{D}$ let Ker$E := E^{-1}(0)$ Recall that if $\bar{E} \in \mathcal{D} \setminus \widetilde{\mathcal{D}}$, $\bar{E} \neq 0$, then Ker$(\bar{E})$ strictly contains the set of the constant functions.
Moreover, we denote by $\mathcal{D}_E$ the set $\{E' \in \mathcal{D} : \mathrm{Ker}(E') = \mathrm{Ker}(E)\}$. The sets of the form $\mathcal{D}_E$, $E \in \mathcal{D}$ will be called *P-parts*. Note that $\mathcal{D}_E$ is the unique P-part containing E.
Clearly, we have only finitely many P-parts.
The term "P-part " is due to Metz. Sabot used the equivalent notion of *G*-relation.
The following notion is due to Sabot and its aim is to approximate Λ_r near $\bar{E} \in \mathcal{D} \setminus \widetilde{\mathcal{D}}$ by minimizing along functions in Ker$(\bar{E})$. Namely, we define $\Lambda_{r,\bar{E}}(E) : \mathrm{Ker}(\bar{E}) \to \mathbb{R}$ as

$$\Lambda_{r,\bar{E}}(E)(u) = \inf\left\{S_{1,r}(E)(v) : v \in \mathcal{L}(u), v \circ \psi_i \in \mathrm{Ker}(\bar{E})\ \forall\, i = 1, ..., k\right\}$$

$$\forall\, u \in \mathrm{Ker}(\bar{E}).$$

Of course, $\Lambda_{r,\bar{E}}(E)$ is independent of $\bar{E}$ in a given *P*-part.

Note that trivially

$$\Lambda_r(E)(u) \le \Lambda_{r,\bar{E}}(E)(u).$$

Lemma (Sabot) Let $r \in W$ and let $\widetilde{E} \in \mathcal{D}_{\mathcal{N}}^{(1)}$. If $\bar{E} \in \mathcal{D}_{\mathcal{N}}^{(3)}$ and $\widehat{\Lambda}_r(\bar{E}) = \bar{E}$, then for every $\alpha < 1$ there exists a neighborhood U of $\bar{E}$ such that for every $E \in U \cap \mathcal{D}_{\mathcal{N}}^{(1)}$ and every $u \in \mathrm{Ker}(\bar{E})$ one has

$$\Lambda_r(E)(u) \ge \alpha\Lambda_{r,\bar{E}}(E)(u).$$

In other words

$$\frac{\Lambda_{r,\bar{E}}(E)}{\Lambda_r(E)} \underset{E \to \bar{E}}{\longrightarrow} 1.$$

We say that the *degenerate* r-eigenform $\bar{E} \in \mathcal{D}_{\mathcal{N}}^{(3)}$ with eigenvalue ρ is $\widetilde{E}$*-repelling* if

$$\exists\, \rho' > \rho : \; \Lambda_{r,\bar{E}}(\widetilde{E})(u) \geq \rho' \widetilde{E}(u) \; \forall\, u \in \mathrm{Ker}(\bar{E}).$$

We also say that the degenerate r-eigenform is repelling if it is $\widetilde{E}$-repellng for some $\widetilde{E} \in \mathcal{D}_{\mathcal{N}}^{(1)}$.

Lemma Let $r \in W$ and let θ be a continuous map from $\partial\mathcal{D}_{\mathcal{N}}$ to $\mathcal{D}_{\mathcal{N}}^{(1)}$. Suppose a degenerate r-eigenform $\bar{E} \in \mathcal{D}_{\mathcal{N}}^{(3)}$ is $\theta(\bar{E})$-repelling. Then $\bar{E}$ is anti-attracting for $(\widehat{\Lambda}_r, \theta)$.

Theorem

Suppose that there exists a continuous map $\theta : \partial\mathcal{D}_{\mathcal{N}} \to \mathcal{D}_{\mathcal{N}}^{(1)}$ such that every degenerate r-eigenform $\bar{E} \in \mathcal{D}_{\mathcal{N}}^{(3)}$ is $\theta(\bar{E})$-repelling. Then there exists an r-eigenform.

We now want to prove that, in the hypothesis of previous theorem, we can remove the continuity of θ. In other words, if every r-degenerate eigenform $\bar{E}$ in $\mathcal{D}_{\mathcal{N}}^{(3)}$ is repelling, that is, is repelling with respect to some $\theta(\bar{E})$, we can choose such a function θ to be continuous.

Theorem

Suppose that every degenerate r-eigenform $\bar{E} \in \mathcal{D}_{\mathcal{N}}^{(3)}$ is repelling. Then there exists a continuous map $\theta : \partial\mathcal{D}_{\mathcal{N}} \to \mathcal{D}_{\mathcal{N}}^{(1)}$ such that every degenerate r-eigenform $\bar{E} \in \mathcal{D}_{\mathcal{N}}^{(3)}$ is $\theta(\bar{E})$-repelling, thus there exists an r-eigenform.

Proof. (Hint) For a given $\bar{E} \in \mathcal{D}_{\mathcal{N}}^{(3)}$, there exists $\theta(\bar{E}) \in \mathcal{D}_{\mathcal{N}}^{(1)}$ such that $\bar{E}$ is $\theta(\bar{E})$-repelling. Then, extend θ using Tietze's Theorem.

Consequence (Metz)

A P-part is *non-trivial* if is different from $\widetilde{\mathcal{D}}$ and $\{0\}$.

A P-part $\mathcal{P}$ is Λ-*invariant* if $\Lambda_r(E) \in \mathcal{P}$ for a given $E \in \mathcal{P}$. It is easy to verify that this definition is independent both of E and of r.

Now, enumerate the nontrivial Λ-invariant P-parts containing a degenerate r-eigenform as $P_1, ..., P_s$. For every $i = 1, ..., s$, let $\bar{E}_i$ be a degenerate r-eigenform with eigenvalue ρ_i. Let

$$\gamma_i = \sup_{E \in \widetilde{\mathcal{D}}} \inf \left\{ \frac{\Lambda_{r,\bar{E}_i}(E)(u)}{E(u)} : u \in \mathrm{Ker}(\bar{E}_i), u \text{ non-constant} \right\}.$$

Then if

$$\max\{\rho_i : i = 1, ..., s\} < \min\{\gamma_i : i = 1, ..., s\},$$

then there exists an r-eigenform. On the other hand, if

$$\max\{\rho_i : i = 1, ..., s\} > \min\{\gamma_i : i = 1, ..., s\},$$

then there exist no r-eigenforms.